EXAMPLES OF TOPOLOGICAL SPACES

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This is a list of examples of topological spaces. I am distributing it for a variety of reasons. First and foremost, I want to persuade you that there are good reasons to study topology; it is a powerful tool in almost every field of mathematics. I want also to drive home the disparate nature of the examples to which the theory applies. This means, on the one hand, that we achieve a great economy of effort, because we need only give one proof and it will apply in many contexts. On the other hand, we need to be careful and rigorous, because our arguments are supposed to be valid in situations far removed from our intuition.

Another reason for distributing these examples is to help you to understand the general theory. I will always try to give examples of abstract theorems, and there will be questions about examples on the problem sets (possibly drawn from this list) but it is always worthwhile to analyse further cases on your own initiative.

I have tried to include examples from a wide range of fields of mathematics. This means that there will probably be a number of examples which you do not have the neccessary background to understand. Do not worry about this. Nothing in this list will be examinable unless I actually lecture on it.

1. Euclidean Examples

The most basic example is the space $\mathbb{R}$ with the order topology. The open sets are the sets $U \subset \mathbb{R}$ such that every point in U lies in an open interval wholly contained in U; in symbols

$$x \in U \Rightarrow \exists a, b \in \mathbb{R} \quad x \in (a, b) \subseteq U$$

This topology is also defined by the metric

$$d(x, y) = |x - y|$$

A subset is compact if and only if it is bounded and closed. A subset A is connected if and only if it is convex, i.e.

$$a < b < c \text{ and } a, c \in A \Rightarrow b \in A$$

Almost as basic is the space $\mathbb{R}^n$ with the product topology. There are many different metrics which induce this topology. For example, we can consider three norms on $\mathbb{R}^n$:

$$\|\underline{v}\|_1 = \sum_k |v_k| \tag{1}$$

$$\|\underline{v}\|_2 = \sqrt{\sum_k |v_k|^2} \tag{2}$$

$$\|\underline{v}\|_\infty = \max_k |v_k| \tag{3}$$

We then define $d_1(\underline{u}, \underline{v}) = \|\underline{u} - \underline{v}\|_1$ and so on. This gives three different metrics d_1 , d_2 and d_∞. However, they all define the same topology.

In fact, it is an interesting theorem that every norm whatsoever induces the product topology. To explain a little: a function $\|\underline{v}\|$ of vectors $\underline{v}$ is a norm if:

$$\|\underline{v}\| \geq 0 \tag{4}$$

$$\|\underline{v}\| = 0 \Leftrightarrow \underline{v} = 0 \tag{5}$$

$$\|a\underline{v}\| = a\|\underline{v}\| \tag{6}$$

$$\|\underline{u} + \underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\| \tag{7}$$

Given a norm, we define a metric by $d(\underline{u},\underline{v}) = \|\underline{u}-\underline{v}\|$. This metric induces a topology, and the claim is that this is always the same as the product topology, no matter what norm we start with.

2. Examples from Functional Analysis

The examples in this section are all spaces of functions with various different topologies. They are important for analysing the convergence of Fourier series, the existence and uniqueness of solutions to differential equations, the spectral theory of operators in quantum mechanics, and many other things.

2.1. Continuous Functions.

$$C[0,1] = \{\text{continuous functions } f\colon [0,1]\to\mathbb{R}\}$$

This is a normed space with the following norm:

$$\|f\|_\infty = \sup\{f(x) \mid 0\le x\le 1\}$$

This is finite because a continuous real valued function on a compact space is bounded. From it we derive a metric:

$$d(f,g) = \|f-g\|$$

A sequence of functions $(f_n)_{n=0}^\infty$ converges to a function f with respect to this metric if and only if (in the usual language of real analysis) it converges uniformly. It follows (using the Weierstrass M-test) that $C[0,1]$ is complete as a metric space.

The polynomial functions from $[0,1]$ to $\mathbb{R}$ (such as $f(x) = 5x^2+6$) form a subspace $P[0,1]$ of $C[0,1]$. It is a dense subspace, by the Stone-Weierstrass theorem.

An interesting example of a continuous function from $C[0,1]$ to $\mathbb{R}$ is given by integration:

$$I\colon C[0,1]\to\mathbb{R} \qquad\qquad I(f) = \int_0^1 f(x)dx$$

Another is the evaluation function $\hat{a}$ for $a\in[0,1]$:

$$\hat{a}\colon C[0,1]\to\mathbb{R} \qquad\qquad \hat{a}(f) = f(a)$$

This idea of regarding $f(a)$ as a function of f rather than of a is certainly curious at first sight, but it turns out to be strikingly useful.

2.2. Square-Integrable Functions.

$$L^2(\mathbb{R}/\mathbb{Z}) = \{f\colon\mathbb{R}\to\mathbb{C} \text{ such that } f(x+1)=f(x) \text{ and } \int_0^1 |f(x)|^2dx<\infty\}$$

This space is the natural home of the theory of Fourier series. To make the definition of $L^2(\mathbb{R}/\mathbb{Z})$ precise, we need to mention that the integration sign means the Lebesgue integral, which is studied in courses on measure theory. However, this is merely a technicality; the Lebesgue integral agrees with any more elementary definition when the latter makes sense.

As in the case of $C[0,1]$, we define a norm and thence a metric:

$$\|f\|_2 = \int_0^1 |f(x)|^2dx$$

$$d(f,g) = \|f-g\|_2$$

This is not quite a metric because it is possible to have $d(f,g)=0$ even when $f\neq g$. For example, we could have $g=0$ and

$$f(x) = \begin{cases} 1 & \text{if } x\in\mathbb{Z} \\ 0 & \text{otherwise}\end{cases}$$

This is again just an annoying technicality, which can be suppressed.

The basic examples of elements of $L^2(\mathbb{R}/\mathbb{Z})$ are the functions

$$e_n(x) = \exp(2n\pi ix)$$

and the basic example of a continuous function from $L^2(\mathbb{R}/\mathbb{Z})$ to $\mathbb{C}$ is the Fourier-coefficient function

$$C_n(f) = \int_0^1 f(x)\overline{e_n(x)}dx$$

The fundamental theorem about Fourier series is that for any $f \in L^2$,

$$f = \sum_{n\in\mathbb{Z}} C_n(f)e_n$$

where the sum converges with respect to the metric just described.

In fact, still more is true, as described in the next example.

2.3. **Square-Summable Sequences.**

$$L^2(\mathbb{Z}) = \{\text{series } \underline{c} = (c_n)_{n=-\infty}^{\infty} \text{ such that } \sum_n |c_n|^2 < \infty\}$$

$$\|\underline{c}\|_2 = \sum_n |c_n|^2$$

$$d(\underline{b}, \underline{c}) = \|\underline{b} - \underline{c}\|$$

There are continuous maps

$$L^2(\mathbb{Z}) \xrightarrow{F} L^2(\mathbb{R}/\mathbb{Z}) \xrightarrow{C} L^2(\mathbb{Z})$$

defined by

$$F(\underline{c}) = \sum_n c_n e_n$$

$$C(f) = (C_n(f))_{n=-\infty}^{\infty} = \left(\int_0^1 f(x)\overline{e_n(x)}dx\right)_{n=-\infty}^{\infty}$$

These are in fact mutually inverse isometric isomorphisms:

$$FC(f) = f \qquad CF(\underline{c}) = \underline{c}$$

$$d(C(f), C(g)) = d(f, g) \qquad d(F(\underline{b}), F(\underline{c})) = d(\underline{b}, \underline{c})$$

This means that the two L^2 spaces can be identified in a very strong sense.

2.4. **Smooth Functions.** The function spaces described above are good for studying things like integration, and differential equations can often be converted into integral equations by cunning means; but to study differentiation directly we need a different kind of space.

$$C^\infty(\mathbb{R}) = \{ \text{ infinitely differentiable functions } f\colon \mathbb{R} \to \mathbb{R}\}$$

Given a compact subset $K \subset \mathbb{R}$, we let $R_K(f)$ denote the restriction of f to K:

$$R_K\colon C^\infty(\mathbb{R}) \to C(K) \qquad R_K(f) = f|_K$$

We also write D for the function from $C^\infty(\mathbb{R})$ to itself sending a function to its derivative:

$$D\colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}) \qquad D(f) = f'$$

We give $C^\infty(\mathbb{R})$ the coarsest possible topology such that the maps R_K (for all compact sets K) and D are continuous. This topology is generated by a rather ugly metric, as follows:

$$P_n(f) = \min(1, \sup\{|f(x)| \quad \text{such that } \; -n \le x \le n\})$$

$$d(f) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} 2^{-m-n}P_n(D^m f)$$

$$d(f, g) = d(f - g)$$

Fortunately, one can usually use the characterisation of the topology in terms of R_K and D and ignore the metric.

Understanding this space and certain closely related spaces is the first step towards the theory of distributions, which is the proper home of the Dirac delta function and similar beasts.

3. Examples from Complex Analysis

3.1. The Riemann Sphere. The Riemann sphere $\mathbb{C}_\infty$ is the one-point compactification $\mathbb{C}\cup\{\infty\}$. The open subsets are the open sets in $\mathbb{C}$ together with the sets $U \cup \{\infty\}$ such that U is an open subset of $\mathbb{C}$ whose complement in $\mathbb{C}$ is compact.

If p and q are complex polynomials (not both zero) then the expression $r(z) = p(z)/q(z)$ can be made sense of as a continuous function from $\mathbb{C}_\infty$ to itself, although a certain amount of work needs to be done to justify this. This is much the most natural context in which to think about such functions.

3.2. Spaces of Analytic Functions. If U is a connected open subset of $\mathbb{C}$, we let $A(U)$ denote the space of analytic functions on U. If $K \subset U$ is compact and $f \in A(U)$ then we write

$$\|f\|_K = \max\{|f(z)| \text{ such that } z \in K\}$$

$$B(f, K, \epsilon) = \{g \in A(U) \text{ such that } \|f - g\|_K < \epsilon\}$$

The sets $B(f, K, \epsilon)$ form a basis for a topology on $A(U)$, called the topology of locally uniform convergence.

This topology has remarkably good properties, much stronger than the corresponding ones for the space of merely continuous functions on U. Firstly, it follows from the Cauchy integral formulae that the differentiation function is continuous:

$$D: A(U) \to A(U) \qquad D(f) = f'$$

If Γ is a simple closed contour whose interior is contained in U and F is the set $\{f \in A(U) \mid f \text{ has no zeros on } \Gamma\}$ then we can define a function

$$v_\Gamma : F \to \mathbb{N}$$

$$v_\Gamma(f) = \text{ number of zeros of } f \text{ inside } \Gamma$$

Here zeros are counted by multiplicity in the usual way, so that $f(z) = (z-1)^2$ counts as having two zeros at $z = 1$. This function turns out to be continuous, and thus (as $\mathbb{N}$ is discrete) constant on the connected components of F.

Using this, we can prove another rather interesting theorem. Let G be the set of injective analytic functions $f: U \to \mathbb{C}$, so $G \subset A(U)$. The theorem is that the closure is given by

$$\overline{G} = G \cup \{ \text{ constant functions } \}$$

Still more interesting is the following theorem of Montel. Let us say that a set $F \subset A(U)$ is locally bounded if for every compact set $K \subset U$ there is a constant M with $\|f\|_K \leq M$ for every $f \in F$. Montel's theorem states that F is compact if and only if it is locally bounded and closed.

The power of the above two results is revealed by the fact that Riemann mapping theorem is a relatively simple consequence. This theorem states that any simply connected proper open subset of $\mathbb{C}$ (no matter how wild its boundary) is conformally equivalent to the unit disc.

4. Examples from Differential Geometry and Algebraic Topology

The main interest of most of the following examples is their global topology, in other words, what sort of holes they have in them and how the holes twist around each other and so on. This course will lay important foundations for the study of such questions, and if we have time towards the end we will address a few of the simpler ones. However, to understand such things fully we would need the apparatus of algebraic topology; while this is particularly profound and beautiful, it will have to wait until future courses. Nonetheless, we can at least take a quick look at some of the phenomena which occur.

Disclaimer: my enthusiasm for this section has of course nothing whatever to do with the subject of my research, honest.

4.1. **Spheres.**

$$S^n = \text{ the } n\text{-sphere } = \{\underline{x} = (x_0, \dots x_n) \in \mathbb{R}^{n+1} \mid \sum x_k^2 = 1\}$$

While these spaces are rather simple, they are in an important sense the building blocks from which most other spaces of interest are constructed. It turns out that to understand the process of construction, one has to study the continuous maps from one sphere to another. An interesting example is the Hopf map:

$$\eta \colon S^3 \to S^2$$

To define it, we think of S^3 as a subset of $\mathbb{R}^4 = \mathbb{C}^2$:

$$S^3 = \{(z, w) \in \mathbb{C}^2 \text{ such that } |z|^2 + |w|^2 = 1\}$$

On the other hand, we think of S^2 (which is an ordinary sphere, like the surface of a basketball) as the Riemann sphere $\mathbb{C} \cup \infty$. The Hopf map is then just division:

$$\eta(z, w) = z/w \in \mathbb{C} \cup \infty$$

One interesting property is that the inverse image of any point in S^2 is a circle in S^3. Any two such circles are linked, like the links in a chain.

4.2. **The Projective Plane.** Our next example is the real projective plane:

$$\mathbb{R}P^2 = \{ \text{ lines through the origin in } \mathbb{R}^3\}$$

Any such line crosses the unit sphere S^2 in two opposite points. Using this we can identify $\mathbb{R}P^2$ with the space of opposite pairs of points, that is

$$\mathbb{R}P^2 = S^2/\sim \qquad\qquad \underline{x} \sim \underline{y} \text{ iff } \underline{x} = \pm\underline{y}$$

We give $\mathbb{R}P^2$ the quotient topology coming from this identification. This makes it a compact, connected, Hausdorff space.

Here is an example of a useful geometric construction involving this space. Suppose X is a nice smooth surface in $\mathbb{R}^3$. For any point $x \in X$, there are two unit normal vectors to X at x, say $\underline{n}$ and $-\underline{n}$. It is not always possible to designate one of these as the positive normal in a way which is consistent over the whole surface. Nonetheless, both unit normals define the same point in $\mathbb{R}P^2$, so we get an unambiguous map

$$g \colon X \to \mathbb{R}P^2$$

This is called the Gauss map.

4.3. **Configuration Spaces.** Our next example is called the unordered configuration space of k points in $\mathbb{C}$:

$$B_k = \{ \text{ finite sets } S \subset \mathbb{C} \text{ with } k \text{ elements } \}$$

We can describe a topology on this space in two different ways (they turn out to be the same topology). One way is to consider the ordered configuration space

$$F_k = \{\underline{z} = (z_1, \dots z_k) \in \mathbb{C}^k \mid z_i \neq z_j \text{ when } i \neq j\}$$

There is a surjective map from F_k to B_k which sends the ordered k-tuple $(z_1, \dots z_k)$ to the unordered set $\{z_1, \dots z_k\}$. Each point in B_k comes from $k!$ points in F_k, corresponding to the different orders which could be imposed. For example, the six preimages of the point

$$\{i, \pi, e\} \in B_3$$

are the following six points in F_3:

$$(i, \pi, e)\ (i, e, \pi)\ (\pi, i, e)\ (\pi, e, i)\ (e, i, \pi)\ (e, \pi, i)$$

We can thus topologise B_k as a quotient space of F_k. In fact, F_k is a covering space of B_k; I hope to discuss covering spaces towards the end of the course.

Another approach to the topology on B_k is as follows. Take $k = 3$ for simplicity. Given a set

$$S = \{u, v, w\} \subset \mathbb{C}$$

consider the polynomial

$$p_S(t) = (t-u)(t-v)(t-w) = t^3 + at^2 + bt + c$$

The numbers

$$a = -(u+v+w) \qquad b = uv + uw + vw \qquad c = -uvw$$

depend only the set S and not on the order in which I listed the elements. We thus get a well defined map

$$g\colon B_3 \to \mathbb{C}^3 \qquad g(S) = (a,b,c)$$

This is injective, because S is precisely the set of roots of p_S and so is determined uniquely by (a,b,c). The image of g can be shown to be an open set in $\mathbb{C}^3$. We can use this to define a topology on B_3, in which the open sets are precisely the sets $g^*(U)$ where U is open in $\mathbb{C}^3$. As stated previously, this is the same as the quotient topology coming from F_3.

The space F_3 is actually quite simple; you can check that the map

$$f(u,v,w) = (u, v-u, (w-u)/(v-u))$$

gives a homeomorphism

$$f\colon F_3 \to \mathbb{C} \times \mathbb{C} \setminus \{0\} \times \mathbb{C} \setminus \{0,1\}$$

However, the spaces F_k for $k > 3$ are rather complicated, and the spaces B_k are still worse. They are rumoured to have an important relationship with the physics of string theory, which is one good reason to study them.

You can show that B_k and F_k are connected, Hausdorff, locally compact and metrisable topological manifolds, but that they are not compact.

4.4. **Loop Spaces.** Next, we consider loop spaces on spheres:

$$\Lambda^n S^m = \{ \text{ continuous maps } \lambda\colon S^n \to S^m\}$$

We give this the compact-open topology, which is defined as follows. Given a compact set $K \subset S^n$ and an open set $U \subset S^m$ we write

$$W(K,U) = \{\lambda \in \Lambda^n S^m \mid \lambda(K) \subset U\}$$

These sets $W(K,U)$ form a subbasis for the compact-open topology, so the open sets in this topology are precisely the arbitary unions of finite intersections of sets of the form $W(K,U)$. The important point about the compact open topology is that the following "evaluation" map is continuous:

$$\mathrm{ev}\colon S^n \times \Lambda^n S^m \to S^m \qquad \mathrm{ev}(x,\lambda) = \lambda(x)$$

Here is an interesting map $\sigma\colon S^1 \to \Lambda^1 S^2$. Think of S^2 as the globe. A point in $\Lambda^1 S^2$ is a function from S^1 to the globe, that is, a parameterised loop on the globe. The great circles which start at the North pole, run down to a point x on the equator, then down to the South pole and back up the other side, form a family of such loops. There is one such loop λ_x for each point x on the equator. On the other hand, we can also identify the equator with S^1. We obtain a map σ sending $x \in S^1 = \text{ equator }$ to the loop λ_x. You can show that this is continuous.

A very hard, but very important, problem is to understand the connected components of $\Lambda^n S^m$ when $n \geq m$. A great deal of partial information is known, but the general case remains intractable. The simplest case is that of $\Lambda^1 S^1$, which is the space of continuous maps from the circle to itself. It is simplest here to think of S^1 as the unit circle in the complex plane. Let $C_0(S^1,\mathbb{R})$ be the subspace of $C(S^1,\mathbb{R})$ consisting of continuous functions $f\colon S^1 \to \mathbb{R}$ such that $f(1) = 0$. There is a continuous function

$$\gamma\colon \mathbb{Z} \times S^1 \times C_0(S^1,\mathbb{R}) \to \Lambda^1 S^1$$

given by

$$\gamma(n,w,f)(z) = z^n w \exp(if(z))$$

which turns out to be a homeomorphism. As $S^1 \times C_0(S^1,\mathbb{R})$ is connected (why ?) this shows that the set of components of $\Lambda^1 S^1$ bijects naturally with $\mathbb{Z}$.

4.5. **Matrix Groups.** The last example in this section is the orthogonal group:

$$O_3 = \{3 \times 3 \text{ matrices such that } A^T = A^{-1}\}$$

(here A^T denotes the transposed matrix).

This is topologised as a subspace of $\mathbb{R}^9$. You can show that it is compact. It is also a group under matrix multiplication. You can show that the group operations are given by continuous maps:

$$\mu\colon O_3 \times O_3 \to O_3 \qquad \mu(A, B) = AB$$

$$\chi\colon O_3 \to O_3 \qquad \chi(A) = A^{-1}$$

Such a matrix has determinant ± 1. As the determinant gives a continuous map

$$\det\colon \{3 \times 3 \text{ matrices}\} \to \mathbb{R}$$

we see that O_3 is disconnected. It falls into two parts:

$$SO_3 = O_3^+ = \{A \mid \det(A) = 1\} \qquad O_3^- = \{A \mid \det(A) = -1\}$$

It can be shown that O_3^+ is the space of rotation matrices.

Suppose that L is a line through the origin in $\mathbb{R}^3$. Let P be the plane orthogonal to L. There is a map ρ_L from $\mathbb{R}^3$ to itself, sending a vector $v \in \mathbb{R}^3$ to its mirror image after reflection in P. If the two unit vectors in L are n and $-n$, then you can check that the formula is

$$\rho_L(v) = v - 2(n.v)n = A_L v$$

Here A_L is the matrix whose (i, j) entry is $\delta_{ij} - 2n_i n_j$, and δ_{ij} is the Kronecker symbol:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Using this, we can see that there is a continuous map $R\colon \mathbb{R}P^2 \to O_3^-$ sending L to A_L.

Here is another interesting map. A matrix $A \in O_3$ satisfies $\|Av\| = \|v\|$ for every vector $v \in \mathbb{R}^3$, so the action of A gives a continuous map $\alpha_A\colon S^2 \to S^2$. We thus get a map

$$\alpha\colon O_3 \to \Lambda^2 S^2 \qquad \alpha(A) = \alpha_A$$

You can check that this map is again continuous.

5. Fractal Examples

5.1. **The Cantor Set.** The simplest example of a fractal is the Cantor set. We define

$$U_l = \bigcup_{k=-3^{l-1}}^{3^{l-1}} ((3k - \tfrac{1}{2})3^{-l}, (3k + \tfrac{1}{2})3^{-l}) \subset [-\tfrac{1}{2}, +\tfrac{1}{2}]$$

The Cantor set is then

$$X = [-\tfrac{1}{2}, +\tfrac{1}{2}] \setminus \bigcup_{l \geq 0} U_l$$

Another description is as follows: we start with the interval $[-\frac{1}{2}, +\frac{1}{2}]$ and remove the middle third $(-\frac{1}{6}, +\frac{1}{6})$ to leave two closed intervals $[-\frac{1}{2}, -\frac{1}{6}]$ and $[\frac{1}{6}, \frac{1}{2}]$. We remove the middle thirds of each of these to get four closed intervals of length $1/9$, and so on. What we get in the limit is the Cantor set again. At the n'th stage we have 2^n closed intervals, each of length 3^{-n}, so the total length is $(2/3)^n$. There is a well behaved concept of the "total length" of a subset of the real line (called Lebesgue measure), which works even for curious sets like the Cantor set; from the above we can see that the Lebesgue measure of X must be zero.

It is easy to see that X is compact and Hausdorff. It is also totally disconnected: you can show that the connected components of X are points. It is also perfect: every point $a \in X$ lies in the closure of $X \setminus \{a\}$. All these properties are quite typical of fractals.

The Cantor set is actually homeomorphic to an infinite Cartesian product of copies of the two point discrete space $\{-1, 1\}$. Indeed, you can show that the map

$$f\colon \{-1,1\}^{\mathbb{Z}_+} \to X \qquad\qquad f(\underline{a}) = \sum_{k>0} a_k 3^{-k}$$

is a homeomorphism.

Another typical fractal property is self-similarity: every neighbourhood of every point in X contains a homeomorphic copy of the whole set. To be more specific, consider a basic neighbourhood $(a-\epsilon, a+\epsilon)$ of a point $a \in X$. For large l we have $3^{-l} < \epsilon$. It follows from the homeomorphism in the last paragraph that there is a unique integer n such that n is not divisible by 3 and $|3^l a - n| \leq \frac{1}{2}$. It then follows that the map

$$g\colon X \to X \cap (a-\epsilon, a+\epsilon)$$
$$g(b) = n + 3^{-l} b$$

gives a homeomorphism between X and a small neighbourhood of a in X.

Such *exact* self-similarity is not actually very typical. In more complicated cases, there is approximate self-similarity. To make this precise, we need to say what it means for two sets to be almost the same, in other words, to impose a topology on a suitable collection of subsets of a given space X. We shall consider several such topologies elsewhere in these notes.

5.2. **Attractors for Newton's Method.** Consider what happens if we try to look for complex roots of the equation $z^3 - 1 = 0$ using Newton's method. We start with some initial guess z_0, and recursively define

$$z_{n+1} = z_n - (z_n^3 - 1)/3z_n^2$$

If $z_n = 0$ we take $z_m = \infty$ for all $m > n$. The hope is that this sequence will converge to some number z which is a root of the equation. Of course, we know what the roots are — there are three of them:

$$z = 1 \qquad\qquad z = \omega = e^{2\pi i/3} \qquad\qquad z = \overline{\omega} = e^{-2\pi i/3}$$

If our initial guess z_0 is close to ω (say) then the sequence (z_k) will converge rapidly to ω. However, if z_0 is intermediate between two of the roots, then the sequence need not converge at all. If it does converge then the limit depends in a very sensitive and intricate way on the precise position of z_0.

Possibly the simplest example of a point for which the sequence does not converge is as follows:

$$\rho = 10^{-1/6}$$
$$\theta = \cos^{-1}(-1/\sqrt{5/32})/3$$
$$\alpha = \rho \exp(i\theta)$$

If $z_0 = \alpha$ then the sequence is just

$$\alpha, \overline{\alpha}, \alpha, \overline{\alpha}, \ldots$$

Before proceeding further, we change notation a little. We write

$$g(z) = z - (z^3 - 1)/3z^2 = \frac{2z^3 + 1}{3z^2}$$
$$g^{(3)}(z) = g(g(g(z))) \qquad\qquad g^{(4)}(z) = g(g(g(g(z)))) \text{ etc.}$$

These functions are called the iterates of g. In the old notation, we have

$$z_n = g^{(n)}(z_0)$$

We can divide the complex plane into four parts, as follows:

$$F_1 = \{w \mid g^{(n)}(w) \to 1 \text{ as } n \to \infty\}$$
$$F_\omega = \{w \mid g^{(n)}(w) \to \omega \text{ as } n \to \infty\}$$
$$F_{\overline{\omega}} = \{w \mid g^{(n)}(w) \to \overline{\omega} \text{ as } n \to \infty\}$$
$$F = F_1 \cup F_\omega \cup F_{\overline{\omega}}$$
$$J = (\mathbb{C} \cup \{\infty\}) \setminus F \quad \subset \mathbb{C} \cup \{\infty\}$$

It is not difficult to write a program to plot these sets and colour them in four different colours. You will find that they are extremely intricate.

The sets J and F are called the Julia set and the Fatou set respectively. The Julia set is uncountable, closed, and perfect, and has empty interior. The Fatou set is open and has infinitely many connected components. The proofs of these facts involve subtle arguments in both complex analysis and general topology.

If we write bU for the boundary of U, then we have the following very curious fact:

$$bF_1 = bF_\omega = bF_{\overline{\omega}} = J$$

You should try to find another example of three disjoint open sets with the same boundary, to see quite how curious this fact is.

There is an interesting characterisation of F in terms of the topology of spaces of analytic functions. For any open set $V \subset \mathbb{C}$ we write

$$G_V = \{g^{(n)}|_V \text{ such that } n \in \mathbb{N}\}$$

We regard this as a subspace of the space of continuous maps from V to the Riemann sphere $\mathbb{C} \cup \{\infty\}$, endowed with the compact-open topology. It turns out that F is the largest set V such that G_V has compact closure. This is the appropriate definition of the Fatou set for a more general rational function g.

5.3. **The Mandelbrot Set.** Given a complex number c, define $q_c(z) = z^2 + c$ and

$$f_n(c) = q_c^{(n)}(0) = q_c(q_c(q_c \dots (0) \dots))$$

where q_c is applied n times. In other words:

$$f_0(c) = 0$$

$$f_{n+1}(c) = q_c(f_n(c)) = f_n(c)^2 + c$$

The Mandelbrot set M is defined as

$$M = \{c \in \mathbb{C} \mid |f_n(c)| \leq 2 \text{ for all } n\}$$

Many of you have probably seen pictures of this set - there are a number of computer programs available to plot it, and indeed you can quite easily write such a program yourself. It has a very intricate fractal boundary. Many small parts of the set contain approximate copies of the whole set — in other words the Mandelbrot set is aproximately self-similar.

A number of interesting topological properties of M are known. Firstly, it is compact — this is easy to see. Secondly, it is connected. This is at first sight implausible — if you look at pictures of M, you will see many small "islands" well separated from the main body of the set. However, on closer inspection there appear to be thin tendrils linking the islands to the centre. The proof that M is connected is quite formidable, involving very powerful methods from complex analysis. However, it is also true that the *complement* of the Mandelbrot set is connected, and this is comparatively straightforward to prove. This has a natural geometric interpretation. As M is compact, the complement M^c has a single unbounded component. If M had any holes in it (e.g. if M were something like the closed annulus $1 \leq |z| \leq 2$) then M^c would also have a bounded component, and so would be disconnected. Thus the fact that M^c is connected just means that M has no holes.

Another interesting fact is that M is the closure of its interior. It is an open question whether M is locally connected — many other things would follow if it were.

6. Examples from Algebraic Geometry

Frequently, when one is faced a geometric problem involving sets of points in $\mathbb{R}^n$, the sets in question are defined by polynomial equations, and any relevant functions between them are also given by polynomials. A situation like this can be analysed using the usual topology on $\mathbb{R}^n$ — the sets are then closed and the functions are continuous. However, polynomials are far more rigid than arbitary continuous functions, so we could hope to replace the usual topology with a coarser topology which would give more information. We will explore this idea in this section, but we will use $\mathbb{C}$ rather than $\mathbb{R}$ for technical reasons.

6.1. **The Zariski Topology on $\mathbb{C}^n$.** We write $\mathbb{C}[z_1, \dots z_n]$ for the set of polynomial functions from $\mathbb{C}^n$ to $\mathbb{C}$, for example the function

$$f(z_1, z_2, z_3) = z_1^4 + iz_3$$

is an element of $\mathbb{C}[z_1, z_2, z_3]$. Given such a function f, we write

$$D(f) = \{\underline{z} \mid f(\underline{z}) \neq 0\}$$

It is easy to see that

$$D(f) \cap D(g) = D(fg)$$

These sets $D(f)$ form a basis for a new topology on $\mathbb{C}^n$, called the Zariski topology. Of course, the sets $D(f)$ are open in the usual topology, which implies that the Zariski topology is coarser than the usual one.

The closed sets for the Zariski topology are all of the form

$$V(f_1, \dots f_m) = \{\underline{z} \in \mathbb{C}^n \mid f_1(\underline{z}) = \dots f_m(\underline{z}) = 0\}$$

In principle, it seems that we ought to also allow sets like $V(f_1, f_2, \dots)$ with infinitely many f's. However, it is a consequence of the important Hilbert Basis Theorem that any such set can be rewritten as $V(g_1, \dots g_m)$ for some finite list of polynomials $\{g_1, \dots g_m\}$.

Let us write X_n for the space $\mathbb{C}^n$ equipped with the Zariski topology. It is *not* the same, incidentally, as the product topology on $\mathbb{C}^n = \mathbb{C} \times \dots \mathbb{C}$ derived from the Zariski topology on each factor.

The space X_n has a number of properties which are strikingly different from those of the spaces considered previously. Firstly, it is not Hausdorff. Indeed, any two non-empty open sets intersect non-trivially, or in other words, every non-empty open set is dense. In fact, it is this example and related ones which provide the main reason for bothering to study non-Hausdorff spaces.

However, every set consisting of a single point is closed, so that X_n does satisfy the separation axiom T_1.

The next curious property of X_n is that very many subspaces are compact. In particular, every open subspace is compact. This is very unlike the situation with Hausdorff spaces, in which every compact set is closed.

6.2. **Prime Spectra of Rings.** Our next example requires some rather more sophisticated algebra. Consider a commutative ring A. We let $X = \operatorname{spec}(A)$ denote the set of prime ideals in A. Given any ideal $\mathfrak{a} \leq A$, we define

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{spec}(A) \mid \mathfrak{a} \leq \mathfrak{p}\}$$

These subsets of $\operatorname{spec}(A)$ satisfy

$$V(0) = \operatorname{spec}(A) \qquad\qquad V(A) = \emptyset$$

$$V(\Sigma_i \mathfrak{a}_i) = \bigcap_i V(\mathfrak{a}_i)$$

$$V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$$

The sets $D(\mathfrak{a}) = \operatorname{spec}(A) \setminus V(\mathfrak{a})$ are the open sets for a topology on $X = \operatorname{spec}(A)$, which we again refer to as the Zariski topology. It is also good to consider the subset $\max(A)$, consisting of the maximal ideals of A — we give this the subspace topology.

In particular, we can consider the case $A = \mathbb{C}[z_1, \dots z_m]$. For any point $\underline{z} \in \mathbb{C}^n$ there is a surjective evaluation homomorphism

$$\hat{\underline{z}} \colon A \to \mathbb{C} \qquad\qquad \hat{\underline{z}}(f) = f(z)$$

The kernel of this is a maximal ideal:

$$\mathfrak{m}_{\underline{z}} = \ker(\hat{\underline{z}} \colon A \to \mathbb{C}) = \{f \in A \mid f(\underline{z}) = 0\}$$

In fact, the map

$$\mathbb{C}^n \to \max(A) \qquad\qquad \underline{z} \mapsto \mathfrak{m}_{\underline{z}}$$

is a homeomorphism if we give the left hand side the Zariski topology, as in the last section.

The space $\mathrm{spec}(A)$ is always compact. The closed points correspond to maximal ideals of A, so the space $\max(A)$ is always T_1. The larger space $\mathrm{spec}(A)$ is almost never T_1, however. For example, if $A = \mathbb{Z}$ then there is a point in $\mathrm{spec}(A)$ corresponding to the prime ideal $\{0\}$, and the closure of this point is the whole space. On the other hand, the weaker axiom T_0 is always satisfied. This and related examples are the main reason for studying spaces which are not T_1.

7. Examples from Algebraic Number Theory

Number theory is to a large extent the study of Diophantine equations, that is, polynomial equations whose solutions are required to be integers. The most famous example is of course the Fermat equation

$$x^n + y^n = z^n$$

It was finally proved in June 1993 by Andrew Wiles that there are no non-zero integer solutions to this when $n > 2$. This had been stated by Fermat over 300 years previously, but although he claimed to have a proof, Fermat did not write it down — it is generally believed that he must have been mistaken.

Anyway, one might ask how it is possible to attack such problems. One method is to work modulo m for some convenient integer m. For example, consider the equation $x^2 + x + 1 = 0$. By considering the two cases in which x is even or odd, we see that the left hand side is always odd and the right hand side is zero so there can be no integer solutions. We can rephrase this argument: there are no solutions mod 2, and hence none integrally.

For a slightly different example, consider the equation $x^2 + 2 = 0$. This does have a solution mod 2 (we can take $x = 0$) but it has no solution mod 4.

This all leads up to the idea that we should look for solutions modulo various large numbers m (better still, numbers m with many factors) and view these as "approximate solutions" to the original equation. It actually turns out to be technically convenient to focus on one prime p at a time, and consider solutions modulo p^m for large m. This is the core idea behind the constructions of this section.

7.1. **The p-adic Metric.** Let p be a prime number. For any non-zero integer n, we can repeatedly divide by p until this is no longer possible, and thus write $n = p^v m$ for uniquely determined integers m and $v \geq 0$. We then write

$$|n|_p = p^{-v}$$

$$|0|_p = 0$$

$$d_p(l, n) = |l - n|_p$$

We find that d_p is a metric on $\mathbb{Z}$, called the p-adic metric. It is very different from the usual metric. For example, as $k \to \infty$ the numbers p^k converge to zero p-adically but diverge to infinity in the usual sense.

Here is an analogy which may make this idea seem more natural. A formal power series is a formal expression $\sum_{k=0}^{\infty} a_k x^k$ with the a_k being real numbers and x a symbol. There is no requirement of convergence. The set of formal power series forms a ring in an obvious way - it is called $\mathbb{R}[\![x]\!]$. It is usual to think of two power series f and g as being close to each other if they agree to a high order, in other words, if $f - g$ is divisible by a high power of x. This is analogous to the p-adic topology, in which n and m are close if $n - m$ is divisible by a high power of p.

The space $\mathbb{Z}$ with this metric has few good properties. Things improve greatly if we consider the completion of $\mathbb{Z}$ with respect to d_p, which is called the space of p-adic integers, denoted $\mathbb{Z}_p$. This space is compact, Hausdorff, and totally disconnected.

7.2. **The p-adic Rationals.** We can generalise the above slightly. If $0 \neq a \in \mathbb{Q}$ then we can write $a = p^v b/c$ where b and c are integers not divisible by p, and then write $|a|_p = p^{-v}$. This gives a metric on $\mathbb{Q}$. The completion is $\mathbb{Q}_p$, the field of p-adic rationals. It is locally compact, Hausdorff and totally disconnected. All algebraic operations are continuous. There is a good notion of the Fourier transform of a suitable function $f\colon \mathbb{Q}_p \to \mathbb{C}$, which is extremely important in modern number theory.

7.3. **The Ring of Adèles.** It turns out to be convenient to regard ∞ as a prime and write $|a|_\infty$ for the usual absolute value, so $\mathbb{Q}_\infty = \mathbb{R}$. Write $\mathcal{C}$ for the set of finite primes, and $\mathcal{C}_\infty = \mathcal{C} \cup \{\infty\}$. We then have a topological ring:

$$\prod_{p \in \mathcal{C}_\infty} \mathbb{Q}_p = \mathbb{R} \times \prod_{\mathcal{C}} \mathbb{Q}_p$$

An element $\underline{a} = (a_p)$ of this product is said to be an adèle if $a_p \in \mathbb{Z}_p \subset \mathbb{Q}_p$ for all but finitely many finite primes $p \in \mathcal{C}$. The set of adèles forms a ring $\mathbb{A}$. You can check that it is locally compact, although the infinite product ring is not. If $a \in \mathbb{Q}$ then there is an adèle $\hat{a}$ with $\hat{a}_p = a$ for all p (why is this an adèle ?). This identifies $\mathbb{Q}$ with a subspace of $\mathbb{A}$. It is discrete in the subspace topology. We can form the quotient additive group:

$$\mathbb{A}/\mathbb{Q} = \mathbb{A}/\sim \qquad\qquad a \sim b \text{ iff } a - b \in \mathbb{Q}$$

it is an important fact that this is compact in the quotient topology.

On the other hand, we can consider the analogous construction involving only the finite primes:

$$\mathbb{A}' = \{(a_p) \in \prod_{\mathcal{C}} \mathbb{Q}_p \mid a_p \in \mathbb{Z}_p \text{ for almost all } p\}$$

It is another important fact that $\mathbb{Q}$ is dense in $\mathbb{A}'$. This is closely related to the Chinese Remainder Theorem.