NOTES ON ORDINALS

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(0) There is a class of objects called ordinals, of which the first few are

$$0, 1, 2, \ldots, \omega, \omega+1, \omega+2, \ldots, 2\omega, \ldots, \omega^2, \ldots, \omega^\omega, \ldots$$

(1) There are too many ordinals for the class of ordinals to be a set; it is "roughly the same size" as the class of *all* sets.

(2) Algebraic operations with ordinals (e.g. ω^2) must be treated with caution. For example, $1+\omega = \omega \neq \omega+1$. The reason for this phenomenon is actually quite easy to understand, but I shall not go into it here.

(3) There is a linear order relation on ordinals. In other words, for any pair of ordinals α and β precisely one of the alternatives $\alpha < \beta$, $\alpha = \beta$ and $\alpha > \beta$ is true.

(4) The ordinals are well-ordered by this relation — any nonempty collection S of ordinals has a least element α, so $\alpha \in S$ and $\alpha \leq \beta$ for any $\beta \in S$.

(5) For any ordinal κ, the collection $S(\kappa)$ of ordinals $\alpha < \kappa$ is a set.

(6) For any set X there is an ordinal κ and a bijection $S(\kappa)@>>>X$, so $X = \{x_\alpha \mid \alpha < \kappa\}$ say.

(7) For any set X there is an ordinal λ so large that there is no injective map $S(\lambda)@>>>X$.

(8) An ordinal α is a successor ordinal iff $\alpha = \beta + 1$ for some β iff there is no ordinal γ with $\beta < \gamma < \alpha$. A limit ordinal is an ordinal (such as ω) which is not a successor.

(9) Transfinite induction over ordinals is valid. Suppose we have a statement $P(\alpha)$ about ordinals α, and we can show that $P(\alpha)$ is true whenever $P(\beta)$ is true for all $\beta < \alpha$. Then $P(\alpha)$ is true for all α. Indeed, consider the collection S of ordinals for which P is false. If S were nonempty, it would have a least element α. This would mean that $P(\beta)$ holds for all $\beta < \alpha$, leading swiftly to a contradiction.

(10) Transfinite recursion is valid. We can define a function f of ordinals by specifying $f(\alpha)$ in terms of the values $f(\beta)$ for $\beta < \alpha$.