NOTES ON POINT-SET TOPOLOGY

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1. Basic Concepts

Definition 1.1. A topology on a space X is a set τ of subsets of X satisfying:

T0 $\emptyset, X \in \tau$
T1 $(\forall i \in I \quad U_i \in \tau) \Rightarrow \bigcup_{i \in I} U_i \in \tau$
T2 $U, V \in \tau \Rightarrow U \cap V \in \tau$

We say that a set $U \subset X$ is τ-open (or just open) if $U \in \tau$.

Definition 1.2.

(0) A neighbourhood of a point $x \in X$ is an open set $U \subseteq X$ such that $x \in U$.
(1) An interior point of a set $Y \subseteq X$ is a point $x \in X$ such that Y contains a neighbourhood of x.
(2) $\overset{\circ}{Y} = \text{int}(Y)$ is the set of interior points of Y.
(3) A closure point of a set $Y \subseteq X$ is a point $x \in X$ such that every neighbourhood of x meets Y.
(4) $\overline{Y} = \text{cl}(Y)$ is the set of closure points of Y.
(5) A set $Y \subseteq X$ is closed iff its complement $Y^c = X \setminus Y$ is open

Proposition 1.3.

(0) *int*(Y) *is open; it is the union of the collection of all open sets* U *such that* $U \subseteq Y$.
(1) Y *is open iff int*$(Y) = Y$.
(2) *cl*(Y) *is closed; it is the intersection of the collection of all closed sets* F *such that* $Y \subseteq F$.

Proposition 1.4. *The set* ζ *of closed sets satisfies*

Z0 $\emptyset, X \in \zeta$
Z1 $(\forall i \in I \quad F_i \in \zeta) \Rightarrow \bigcap_{i \in I} F_i \in \zeta$
Z2 $F, G \in \zeta \Rightarrow F \cup G \in \zeta$

Definition 1.5. A topological basis on X is a collection β of subsets of X satisfying

B0 $\bigcup \beta = \bigcup_{U \in \beta} U = X$
B1 $(U, V \in \beta \text{ and } x \in U \cap V) \Rightarrow \exists W \in \beta \quad x \in W \subseteq U \cap V$

Definition 1.6. The topology generated by β is

$$\tau(\beta) = \{U \subseteq X \mid \forall x \in U \; \exists V \in \beta \quad x \in V \subseteq U\}$$

Proposition 1.7.

(0) $\tau(\beta)$ *is a topology on* X.
(1) $\beta \subseteq \tau(\beta)$.
(2) $\tau(\beta)$ *is the intersection of the family of those topologies* τ' *on* X *such that* $\beta \subseteq \tau'$.
(3) *A set* U *lies in* $\tau(\beta)$ *iff it is the union of some family of elements of* β.

Definition 1.8. β is a basis for a given topology τ iff $\tau = \tau(\beta)$.

Proposition 1.9. *This holds iff* $\beta \subseteq \tau$ *and*

$$\forall x \in U \in \tau \; \exists V \in \beta \quad x \in V \subseteq U$$

Definition 1.10. A topological subbasis on X is a collection σ of subsets of X satisfying the axiom:

S0 $\bigcup\sigma = \bigcup_{U\in\sigma} U = X$

Definition 1.11. The basis generated by a subbasis σ is

$$\beta(\sigma) = \{U_1 \cap \ldots \cap U_n \mid U_1, \ldots, U_n \in \sigma\}$$

The topology $\tau(\sigma)$ generated by σ is just

$$\tau(\sigma) = \tau(\beta(\sigma))$$

This is the intersection of the family of those topologies τ' on X such that $\sigma \subseteq \tau'$.

Definition 1.12. A pseudometric on a space X is a function $d\colon X \times X \to [0, \infty) \subset \mathbb{R}$ satisfying the following axioms:

M0 $d(x, x) = 0$
M1 $d(x, y) = d(y, x)$
M2 $d(x, z) \leq d(x, y) + d(y, z)$

It is a metric if it satisfies the additional axiom

M3: $d(x, y) = 0 \Rightarrow x = y$

Definition 1.13. If d is a pseudometric on X then

$$B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$$

$$\beta_d = \{B(x, \epsilon) \mid x \in X, \epsilon > 0\}$$

This is a basis for a topology τ_d on X, called the pseudometric topology.

2. Continuous Maps

Definition 2.1. If X and Y are topological spaces then a map $f\colon X \to Y$ is continuous iff

$$U \subseteq Y \text{ open } \Rightarrow f^{-1}(U) \subseteq X \text{ open}$$

Proposition 2.2. *Suppose X and Y are topological spaces, $f\colon X \to Y$ and σ is a subbasis for the topology on Y. Then f is continuous iff*

$$U \in \sigma \Rightarrow f^{-1}(U) \text{ open in } X$$

Proposition 2.3. *If X and Y are metric spaces, and we give them the metric topology, then $f\colon X \to Y$ is continuous iff for every $x \in X$ and $\epsilon > 0$ there exists $\delta > 0$ such that $d(x, x') < \delta$ implies $d(f(x), f(x')) < \epsilon$.*

Proposition 2.4. *If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous then so is $g \circ f\colon X \to Z$.*

Definition 2.5. A map $f\colon X \to Y$ is a homeomorphism iff it is bijective and both f and f^{-1} are continuous. Two spaces X and Y are homeomorphic iff there is a homeomorphism from one to the other.

Proposition 2.6. *The following maps are continuous (metric topologies used everywhere):*

(0) $\sigma\colon \mathbb{R}^2 \to \mathbb{R}$ $\quad \sigma(x, y) = x + y$
(1) $\mu\colon \mathbb{R}^2 \to \mathbb{R}$ $\quad \mu(x, y) = xy$
(2) $\nu\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ $\quad \nu(x) = 1/x$
(3) $\max, \min\colon \mathbb{R}^2 \to \mathbb{R}$
(4) $f\colon (-1, 1) \to \mathbb{R}$ $\quad f(x) = 2x/(1 - x^2)$
(5) $f^{-1}\colon \mathbb{R} \to (-1, 1)$ $\quad f^{-1}(y) = (\sqrt{y^2 + 1} - 1)/y$

3. Other Properties of Maps

Definition 3.1. We write $U \subseteq_O X$ to mean that U is an open subset of X. A map $f\colon X \to Y$ is

(0) continuous iff

$$V \subseteq_O Y \Rightarrow f^{-1}(V) \subseteq_O X$$

iff

$$G \subseteq_C Y \Rightarrow f^{-1}(G) \subseteq_C X$$

(1) open iff

$$U \subseteq_O X \Rightarrow f(U) \subseteq_O Y$$

(2) closed iff

$$F \subseteq_C X \Rightarrow f(F) \subseteq_C Y$$

(3) strongly continuous iff

$$V \subseteq_O Y \Leftrightarrow f^{-1}(V) \subseteq_O X$$

iff

$$G \subseteq_C Y \Leftrightarrow f^{-1}(G) \subseteq_C X$$

(4) a quotient map iff surjective and strongly continuous.
(5) an embedding iff injective and strongly continuous.
(6) proper iff continuous and

$$K \subseteq Y \text{ compact } \Rightarrow f^{-1}(K) \text{ compact}$$

4. Constructs

4.1. Subspaces.

Definition 4.1. Suppose τ is a topology on X and $Y \subseteq X$. The subspace topology on Y is

$$\tau|_Y = \{U \cap Y \mid U \in \tau\}$$

Write j (or j_Y or j_{YX}) for the inclusion map $Y \to X$.

Proposition 4.2. *If β is a basis for τ then*

$$\beta|_Y = \{V \cap Y \mid V \in \beta\}$$

is a basis for $\tau|_Y$; similarly for subbases.

Proposition 4.3. *If Y is open in X then*

$$U \textit{ open in } Y \Leftrightarrow (U \subseteq Y \textit{ and } U \textit{ open in } X)$$

If Y is closed in X then

$$F \textit{ closed in } Y \Leftrightarrow (F \subseteq Y \textit{ and } F \textit{ closed in } X)$$

Proposition 4.4. *If $Z \subseteq Y$ then*

$$cl_Y(Z) = cl_X(Z) \cap Y$$

Proposition 4.5 (Patching)**.** *Suppose $f\colon X \to Y$, that $X = \bigcup_I X_i$ and that each restriction $f|_{X_i}\colon X_i \to Y$ is continuous. If each X_i is open in X then f is continuous. Similarly, if each X_i is closed and I is finite then f is continuous.*

4.2. **Products.**

Definition 4.6. Suppose that I is a set, and for each $i \in I$ we have a space X_i with topology τ_i. The product space is

$$\prod_{i\in I} X_i = \{x \colon I \to \bigcup_{i\in I} X_i \mid \forall i \;\; x(i) \in X_i\}$$

In this context we will write x_i instead of $x(i)$. The projections are the maps

$$\pi_i \colon \prod_I X_i \to X_i \qquad\qquad \pi_i(x) = x_i$$

The following collection of sets is a subbasis for a topology, called the product topology:

$$\sigma = \{\pi_i^{-1}(U) \mid i \in I, U \in \tau_i\}$$

Definition 4.7. If $J \subseteq I$ then

$$\pi_J \colon \prod_I X_i \to \prod_J X_i$$

is defined by

$$\pi_J(x)_j = x_j \qquad\qquad (\text{ for } j \in J)$$

Proposition 4.8. *The following sets are bases for the product topology:*

$$\beta = \{\pi_J^{-1}(U) \mid J \subseteq I \textit{ is finite and } U \textit{ is open in } \prod_J X_i\}$$

$$\beta' = \{\pi_J^{-1}(\prod_{j\in J} U_j) \mid J \subseteq I \textit{ is finite and } U_j \textit{ is open in } X_j\}$$

Proposition 4.9. *If $Y_i \subseteq X_i$ for each $i \in I$ then the two topologies on $\prod_I Y_i$ (as a product of subspaces of the X_i or as a subspace of the product $\prod_I X_i$) are the same.*

Proposition 4.10. *If $Y_i \subseteq X_i$ for all i then*

$$\overline{\prod_I Y_i} = \prod_I \overline{Y}_i$$

4.3. **Quotients.**

Definition 4.11. A relation on a set X is a subset $R \subseteq X \times X$. In this context, we write xRy instead of $(x, y) \in R$.

Definition 4.12. A relation R is an equivalence relation if it satisfies the following axioms:

- E0 (reflexivity) xEx
- E1 (symmetry) $xEy \Leftrightarrow yEx$
- E2 (transitivity) (xEy and yEz) $\Rightarrow xEz$

Definition 4.13. If E is an equivalence relation on X then

$$[x] = \{x' \in X \mid xRx'\}$$

An equivalence class is a subset $y \subseteq X$ such that $y = [x]$ for some $x \in X$. The set of equivalence classes is written X/R. There is a surjective map

$$q \colon X \to X/R \qquad\qquad q(x) = [x]$$

Proposition 4.14.

(0) *If $y, y' \in X/R$ then either $y = y'$ or $y \cap y' = \emptyset$.*
(1) *If $x \in X$ and $y \in X/R$ then $x \in y \Leftrightarrow y = [x]$.*
(2) *Each element $x \in X$ lies in precisely one equivalence class.*

Definition 4.15. If τ is a topology on X then the topology τ/R on X/R is given by

$$\tau/R = \{U \subseteq X/R \mid q^{-1}(U) \in \tau\}$$

4.4. **Disjoint Unions.**

Definition 4.16. Suppose that I is a set, and for each $i \in I$ we have a space X_i with topology τ_i. The disjoint union is

$$\coprod_{i \in I} X_i = \{(i, x) \mid x \in X_i\} \subseteq I \times \bigcup_I X_i$$

The inclusions are the maps

$$\iota_i \colon X_i \to \coprod_I X_i \qquad \iota_i(x) = (i, x)$$

The disjoint union topology is just

$$\tau = \{\coprod_I U_i \mid U_i \in \tau_i \text{ for all } i \in I\}$$

Proposition 4.17. *If the sets X_i are open disjoint subsets of some larger space X, then the map $(i, x) \mapsto x$ is a homeomorphism*

$$\coprod_I X_i \to \bigcup_I X_i \subseteq X$$

4.5. **Functorial Aspects.** In this subsection we keep the notation of the subsections above, and give all spaces the topologies constructed above.

Proposition 4.18. *The following maps are continuous:*

$$\begin{aligned} j \colon Y &\to X \\ \pi_i \colon \prod_I X_i &\to X_i \\ q \colon X &\to X/R \\ \iota_i \colon X_i &\to \coprod_I X_i \end{aligned}$$

Proposition 4.19. *A map $f \colon Z \to Y$ (where Y is a subspace of X given the subspace topology) is continuous iff $j \circ f \colon Z \to X$ is continuous.*

Proposition 4.20. *A map $f \colon Z \to \prod_I X_i$ is continuous iff all the composites $f_i = \pi_i \circ f \colon Z \to X_i$ are continuous.*

Proposition 4.21. *A map $g \colon X/R \to Z$ is continuous iff $g \circ q \colon X \to Z$ is continuous.*

Proposition 4.22. *A map $g \colon \coprod_I X_i \to Z$ is continuous iff all the composites $g_i = g \circ \iota_i$ are continuous.*

Proposition 4.23. *To give a continuous map $f \colon Z \to Y$ is the same as to give a continuous map $\tilde{f} = j \circ f \colon Z \to X$ such that $\tilde{f}(Z) \subseteq Y$.*

Proposition 4.24. *To give a continuous map $f \colon Z \to \prod_I X_i$ is the same as to give a family of continuous maps $f_i = \pi_i \circ f \colon Z \to X_i$.*

Proposition 4.25. *To give a continuous map $g \colon X/R \to Z$ is the same as to give a continuous map $\tilde{g} = g \circ q \colon X \to Z$ such that xRx' implies $\tilde{g}(x) = \tilde{g}(x')$.*

Proposition 4.26. *To give a continuous map $g \colon \coprod_I X_i \to Z$ is the same as to give a family of continuous maps $g_i = g \circ \iota_i \colon X_i \to Z$.*

5. The Hausdorff Property

Definition 5.1. A space X is Hausdorff iff for every pair $x, y \in X$ with $x \neq y$ there are open sets U, V such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$.

Proposition 5.2. *A pseudometric space is Hausdorff iff it is a metric space; in particular $\mathbb{R}$ is Hausdorff.*

Proposition 5.3. *Subspaces, products and disjoint unions of Hausdorff spaces are Hausdorff.*

Proposition 5.4. *A space X is Hausdorff iff the diagonal $\Delta = \{(x,x) \mid x \in X\}$ is a closed subset of $X \times X$.*

6. Connectedness

Definition 6.1. A separation of a space X is a pair of open and closed sets A,B such that $X = A \cup B$ and $A \cap B = \emptyset$ (so $B = X \setminus A$). A separation is trivial if $A = \emptyset$ or $B = \emptyset$. The space X is connected iff it is nonempty and has no nontrivial separation.

Proposition 6.2. *Suppose $A, B, Y \subseteq X$. Then A, B is a separation of Y (with its subspace topology) iff $Y = A \cup B$ and $\overline{A} \cap B = \emptyset = A \cap \overline{B}$. Here bars denote closure in X.*

Proposition 6.3. *If $Z \subseteq Y \subseteq \overline{Z} = cl_X(Z) \subseteq X$ and Z is connected then so is Y.*

Proposition 6.4. *If $(Y_i)_{i\in I}$ is a family of connected subsets of X and $Y_i \cap Y_j \neq \emptyset$ for all $i, j \in I$ then $Y = \bigcup_i Y_i$ is connected.*

Proposition 6.5. *Continuous images of connected sets are connected. That is, $f\colon X \to Y$ continuous and $Z \subseteq X$ connected implies $f(Z)$ connected (with its topology as a subspace of Y).*

Proposition 6.6. *Products and quotients of connected spaces are connected.*

Definition 6.7. Write $x \sim y$ iff there is a connected set $C \subseteq X$ such that $x, y \in C$. This is an equivalence relation, and the equivalence classes are called the components of X.

Proposition 6.8. *The components of X are maximal connected sets. In other words, if C is a component, $D \supseteq C$ and D is connected then $D = C$.*

Proposition 6.9. *The components of X are closed. If there are only finitely many of them, then they are also open.*

7. Local Connectedness

Definition 7.1. A space X is locally connected iff the connected open sets form a basis for the topology, equivalently iff there is a basis for the topology consisting of connected sets.

Proposition 7.2. *Open subspaces and finite products of locally connected spaces are locally connected.*

Proposition 7.3. $\mathbb{R}^n$ *is locally connected.*

Proposition 7.4. *If X is locally connected, then the components of X are open and closed.*

8. Path Connectedness

Definition 8.1. A path from x_0 to x_1 in a space X is a continuous map $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

Definition 8.2. A space X is path connected iff any two points in X are joined by a path in X.

Proposition 8.3. *Products and continuous images of path connected spaces are path connected.*

Proposition 8.4. *A path connected space is connected.*

Definition 8.5. Write $x \sim y$ iff there is a path in X from x to y. This is an equivalence relation, and the equivalence classes are called the path components of X.

Definition 8.6. A space X is locally path connected iff the path connected open sets form a basis for the topology.

Proposition 8.7. *Open subspaces and finite products of locally path connected spaces are locally path connected.*

Proposition 8.8. *If X is locally path connected, then the path components are open and closed. If also X is connected then X is path connected.*

9. Compactness

Definition 9.1. An open cover of a subspace Y in a space X is a family $(U_i)_{i\in I}$ such that $Y \subseteq \bigcup_I U_i = X$. A finite subcover is a subfamily $(U_i)_{i\in J}$ for some finite subset J of I, such that $Y \subseteq \bigcup_{i\in J} U_i$. An open cover of X means an open cover of X in X.

Definition 9.2. A space X is compact iff every open cover of X has a finite subcover.

Proposition 9.3. *A subspace $Y \subseteq X$ is compact in the subspace topology iff every open cover of Y in X has a finite subcover.*

Definition 9.4. A collection $\mathcal{F} = (F_i)_{i\in I}$ of subsets of a space X has the finite intersection property (FIP) iff $\bigcap_{i\in J} F_i \neq \emptyset$ for every finite $J \subseteq I$.

Proposition 9.5. *A space X is compact iff every family $(F_i)_{i\in I}$ of closed sets with FIP has $\bigcap_{i\in I} F_i \neq \emptyset$.*

Proposition 9.6 (Alexander's Lemma)**.** *Suppose that σ is a subbasis for the topology on X, and that every open covering $X = \bigcup_{i\in I} U_i$ by subbasic open sets (i.e. $U_i \in \sigma$ for all $i \in I$) has a finite subcover. Then X is compact.*

Proposition 9.7. *A closed subspace of a compact space is compact. A compact subspace of a Hausdorff space is closed.*

Proposition 9.8. *A finite union of compact subspaces is compact.*

Proposition 9.9. *A continuous image of a compact space is compact.*

Proposition 9.10. *The product of a family of compact spaces is compact.*

Proposition 9.11. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

Proposition 9.12. *A subspace of $\mathbb{R}^n$ is compact iff bounded and closed.*

Lemma 9.13 (Tube Lemma)**.** *If $U \subseteq X \times Y$ is open and $Z \subseteq Y$ is compact and $\{x\} \times Z \subseteq U$ then there is a neighbourhood V of x such that $V \times Y \subseteq U$.*

10. Ultrafilters

Definition 10.1. A set $\mathcal{F}$ of subsets of X has the finite intersection property iff for each finite list $S_1, \dots S_n$ with each $S_k \in X$ we have $S_1 \cap \dots S_n \neq \emptyset$. Such a set $\mathcal{F}$ will be called a family with the finite intersection property or an FFIP or a filter subbase.

Definition 10.2. An ultrafilter (or UF) on X is a set $\mathcal{W}$ of subsets of X which is a maximal FFIP. In other words:

U0 $\mathcal{W}$ has the FIP.
U1 If $\mathcal{W} \subseteq \mathcal{W}'$ and $\mathcal{W}'$ has FIP then $\mathcal{W}' = \mathcal{W}$.

Proposition 10.3. *Let $\mathcal{W}$ be an ultrafilter.*

(0) *If $S \in \mathcal{W}$ and $T \supseteq S$ then $T \in \mathcal{W}$.*
(1) *If $S_1, \dots, S_n$ are in $\mathcal{W}$ then $S_1 \cap \dots S_n \in \mathcal{W}$.*
(2) *If $S \subseteq X$ then either $S \in \mathcal{W}$ or $S^c \in \mathcal{W}$ (but not both).*
(3) *If $T \subseteq X$ and $T \cap S \neq \emptyset$ for every $S \in \mathcal{W}$ then $T \in \mathcal{W}$.*
(4) *If $S \cup T \in \mathcal{W}$ then $S \in \mathcal{W}$ or $T \in \mathcal{W}$.*
(5) *$X \in \mathcal{W}$*

Proposition 10.4. *Suppose $\mathcal{W}$ has FIP. Then $\mathcal{W}$ is an ultrafilter iff for each $S \subseteq X$ we have $S \in \mathcal{W}$ or $S^c \in \mathcal{W}$.*

Definition 10.5. A chain of FFIP's is a set $\mathcal{L}$ of FFIP's on X such that whenever $\mathcal{F}, \mathcal{G} \in \mathcal{L}$ we have either $\mathcal{F} \subseteq \mathcal{G}$ or $\mathcal{G} \subseteq \mathcal{F}$. In other words, $\mathcal{L}$ is linearly ordered by inclusion.

Proposition 10.6. *If $\mathcal{L}$ is a chain of FFIP's on X then the set*

$$\mathcal{F} = \bigcup_{\mathcal{G}\in\mathcal{L}} \mathcal{G} = \{S \subseteq X \mid \exists \mathcal{G} \in \mathcal{L} \quad S \in \mathcal{G}\}$$

has FIP.

Recall that

$$\mathcal{N}_x = \{ \text{ neighbourhoods of } x\} = \{ \text{ open sets } U \mid x \in U\}$$

Suppose that σ is a subbasis for the topology on X. Write

$$\mathcal{N}'_x = \{ \text{ subbasic neighbourhoods of } x\} = \{U \in \sigma \mid x \in U\}$$

Definition 10.7. An ultrafilter $\mathcal{W}$ converges to $x \in X$ if any of the following equivalent conditions hold:

- C0 $x \in \bigcap_{S\in\mathcal{W}} \overline{S}$
- C1 $\mathcal{N}_x \subseteq \mathcal{W}$
- C2 $\mathcal{N}'_x \subseteq \mathcal{W}$
- C3 For all $S \in \mathcal{W}$ and $U \in \mathcal{N}'_x$ we have $S \cap U \neq \emptyset$.

If so, we write $\mathcal{W} \to x$ and say that x is a limit of $\mathcal{W}$.

Theorem 10.8. *The space X is Hausdorff iff every ultrafilter converges to at most one point.*

Theorem 10.9. *The following are equivalent:*

(0) *X is compact.*
(1) *Every covering of X by subbasic open sets has a finite subcover.*
(2) *Every ultrafilter on X has a limit.*

Theorem 10.10 (Tychonov)**.** *Suppose $(X_i)_{i\in I}$ is a family of compact spaces. Then $X = \prod_I X_i$ is compact.*

11. Compactness and Completeness in Metric Spaces

Definition 11.1. A sequence $\underline{x} = (x_n)_{n\in\mathbb{N}}$ is Cauchy iff

$$\forall \epsilon > 0 \quad \exists N \quad \forall k, l \geq N \quad d(x_k, x_l) < \epsilon$$

iff $\epsilon_N \to 0$ as $N \to \infty$, where

$$\epsilon_N = \sup_{k,l\geq N} d(x_k, x_l)$$

Note that convergent sequences are Cauchy. Note also that $\epsilon_{N+1} \leq \epsilon_N$.

Definition 11.2. A metric space X is complete iff every Cauchy sequence in X converges.

Proposition 11.3. *$\mathbb{R}^n$ is complete. The space $C(X)$ of bounded continuous real valued functions on a topological space X is complete under the metric*

$$d(f, g) = \|f - g\|_\infty = \sup\{|f(x) - g(x)| \mid x \in X\}$$

Proposition 11.4. *A subspace Y of a complete metric X space (considered as a metric space in the obvious way) is complete iff closed in X.*

Definition 11.5. The diameter of a subset A of a metric space is

$$\mathrm{diam}(A) = \sup\{d(x, y) \mid x, y \in A\} \qquad (\mathrm{diam}(\emptyset) = -\infty)$$

The set A is bounded iff $\mathrm{diam}(A) < \infty$.

Definition 11.6. An ϵ-net for X is a finite subset $F \subseteq X$ such that

$$X \subseteq \bigcup_{x\in F} B(x, \epsilon)$$

X is totally bounded iff it has an ϵ-net for every $\epsilon > 0$.

Proposition 11.7. *A subset of a totally bounded set is totally bounded. The closure of a totally bounded set is totally bounded. A subset $Y \subseteq X$ is totally bounded iff there is a finite subset $F \subseteq X$ such that*

$$Y \subseteq \bigcup_{x \in F} B(x, \epsilon)$$

Proposition 11.8 (Cantor)**.** *If X is complete and $Y_n \subseteq X$ is closed and $Y_{n+1} \subseteq Y_n$ and $diam(Y_n) \to 0$ then there is a point $y \in X$ such that*

$$\bigcap_n Y_n = \{y\}$$

Proposition 11.9. *X is compact iff it is complete and totally bounded.*

Definition 11.10. A Lebesgue number for an open covering $\mathcal{U}$ of X is a number $\epsilon > 0$ such that for each $x \in X$ there is a set $U \in \mathcal{U}$ such that $B(x, \epsilon) \subseteq U$.

Proposition 11.11. *In a compact metric space, every open cover has a Lebesgue number.*

Definition 11.12. A subset $Y \subseteq X$ is nowhere dense if $\overline{Y}$ has empty interior. A Baire space is a topological space X in which every countable intersection of dense open sets is dense, or equivalently, every countable union of nowhere dense sets has empty interior.

Proposition 11.13 (Baire)**.** *A complete metric space is Baire. A compact Hausdorff space is Baire.*

Definition 11.14. An isometry (or isometric embedding) is a map $f \colon X \to Y$ such that $d(f(x), f(y)) = d(x, y)$. An isometric isomorphism is a bijective isometry. A completion of X is a complete metric space Y together with an isometry $i \colon X \to Y$ such that $i(X)$ is dense in Y.

Note that an isometry is automatically continuous and injective. A bijective isometry is a homeomorphism and the inverse is an isometry. If $f \colon X \to Y$ is an isometry, then the induced map $f \colon X \to f(X)$ (where $f(X) \subseteq Y$ is given the subspace topology) is a homeomorphism, hence th use of the term "embedding".

Construction 11.15. Let X be a metric space. Write

$$CS(X) = \{ \text{ Cauchy sequences } \underline{x} = (x_n)_{n \in \mathbb{N}} \text{ in } X \ \}$$

For $\underline{x}, \underline{y} \in CS(X)$ write

$$d(\underline{x}, \underline{y}) = \lim_{n \to \infty} d(x_n, y_n)$$

$$\underline{x} \sim \underline{y} \qquad \text{iff} \qquad d(\underline{x}, \underline{y}) = 0$$

$$\tilde{X} = CS(X)/\sim$$

$$d([\underline{x}], [\underline{y}]) = d(\underline{x}, \underline{y})$$

Define $i \colon X \to \tilde{X}$ by

$$i(x) = [(x, x, x, \dots)]$$

Proposition 11.16. *Everything above makes sense, and $i \colon X \to \tilde{X}$ becomes a completion of X.*

Proposition 11.17. *If $f \colon X \to Y$ is an isometry and Y is a complete metric space then there is a unique isometry $\tilde{f} \colon \tilde{X} \to Y$ such that $\tilde{f} \circ i = f$. It is given by*

$$\tilde{f}([\underline{x}]) = \lim_{n \to \infty} f(x_n)$$

12. Separation Axioms

Definition 12.1. A space X is

(0) T_0 iff given $x \neq y$ in X either $\exists U \in \mathcal{N}_x \quad y \notin U$ or $\exists V \in \mathcal{N}_y \quad x \notin V$
(1) T_1 iff given $x \neq y$ in X then $\exists U \in \mathcal{N}_x \quad y \notin U$
(2) T_2 (or Hausdorff) iff given $x \neq y$ in X there exist $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ such that $U \cap V = \emptyset$.
(3) Regular iff T_1 and given closed $Y \subset X$ and $z \notin Y$ there exist disjoint open sets U and V such that $Y \subseteq U$ and $z \in V$
(4) Normal iff T_1 and given disjoint closed sets $Y, Z \subset X$ there exist disjoint open sets U and V such that $Y \subseteq U$ and $Z \subseteq V$.

Proposition 12.2. *X is T_0 iff $x \neq y \Rightarrow \overline{\{x\}} \neq \overline{\{y\}}$.*

Proposition 12.3. *X is T_1 iff $\{x\}$ is closed for each $x \in X$.*

Proposition 12.4. *X is regular iff it is T_1 and given $U \in \mathcal{N}_x$ there exists $V \in \mathcal{N}_x$ such that*

$$x \in V \subseteq \overline{V} \subseteq U$$

Proposition 12.5. *X is normal iff it is T_1 and given $F \subseteq U$ with F closed and U open there exists an open set V with*

$$F \subseteq V \subseteq \overline{V} \subseteq U$$

Proposition 12.6. *Normal $\Rightarrow$ Regular $\Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$*

Proposition 12.7. *A metric space is normal.*

Lemma 12.8. *If X is Hausdorff, $Y \subseteq X$ is compact and $x \notin Y$ then there is a neighbourhood U of x and an open set $V \supseteq Y$ such that $U \cap V = \emptyset$. It follows that $x \notin \overline{V}$ and $\overline{U} \cap Y = \emptyset$.*

Proposition 12.9. *A compact Hausdorff space is normal.*

13. Real Valued Functions

In this section, X is a topological space and $C(X)$ is the set of continuous real-valued functions $u \colon X \to \mathbb{R}$. We consider this as a metric space in the usual way.

Definition 13.1. Write $I = [0,1] \cap \mathbb{Q}$. A Urysohn filtration on X is a family

$$\underline{U} = (U_a)_{a \in I}$$

of open subsets of X such that

$$a < b \quad \Rightarrow \quad \overline{U_a} \subseteq U_b$$

Such a filtration is regular iff for all a we have

$$U_a = \bigcup_{b<a} U_b$$

Note that this implies $U_0 = \emptyset$. (These definitions are not standard).

Construction 13.2. Given a Urysohn filtration $\underline{U}$ define $f_{\underline{U}} \colon X \to [0,1]$ by

$$f_{\underline{U}}(x) = \inf\{a \in I \mid x \in U_a\}$$

This is to be interpreted as 1 if $x \notin U_a$ for any a.

Define also

$$U'_a = \bigcup_{b<a} U_b$$

Finally, given $g \in C(X, [0,1])$ write

$$V_a(g) = \{x \mid g(x) < a\} = g^{-1}(-\infty, a)$$

Theorem 13.3.

(0) *$\underline{U}'$ is a regular Urysohn filtration.*
(1) $\underline{U}'' = \underline{U}'$

(2) $\underline{V}(g)$ *is a regular Urysohn filtration.*
(3) $f_{\underline{U}}$ *is continuous.*
(4) $\underline{V}(f_{\underline{U}}) = \underline{U}'$.
(5) $f_{\underline{V}(g)} = g$
(6) *There is a bijective correspondence between regular Urysohn filtrations on X and continuous functions $X \to [0,1]$, given by $\underline{U} \mapsto f_{\underline{U}}$ and $g \mapsto \underline{V}(g)$.*

Theorem 13.4 (Urysohn)**.** *If X is normal and Y_0 and Y_1 are disjoint closed subsets of X then there is a continuous function $f\colon X \to [0,1]$ such that $f = 0$ on Y_0 and $f = 1$ on Y_1.*

Theorem 13.5 (Tietze)**.** *Suppose X is normal and $Y \subseteq X$ is closed. Suppose $f\colon Y \to [-1,1]$ is continuous. Then there is a continuous function $g\colon X \to [-1,1]$ with $g|_Y = f$.*

Lemma 13.6. *There is a sequence $(p_n(x))_{n\in\mathbb{N}}$ of polynomial functions such that $p_n(x) \to |x|$ uniformly on the interval $[-1,1]$.*

Definition 13.7. A subset $A \subseteq C(X)$ is a subalgebra iff

A0 Every constant function lies in A.
A1 $u, v \in A \Rightarrow u + v \in A$.
A2 $u, v \in A \Rightarrow uv \in A$.

Definition 13.8. Given $u, v \in C(X)$ define $u \vee v$ and $u \wedge v$ in $C(X)$ by

$$(u \vee v)(x) = \max(u(x), v(x))$$

$$(u \wedge v)(x) = \min(u(x), v(x))$$

A subset $A \subseteq C(X)$ is a sublattice iff

L0 $u, v \in A \Rightarrow u \vee v \in A$
L1 $u, v \in A \Rightarrow u \wedge v \in A$

Definition 13.9. A subset $A \subseteq C(X)$ is separating iff given distinct points x, y there exists $u \in A$ such that $u(x) \neq u(y)$.

Given a finite set $Y = \{y_1, \dots y_n\} \subseteq X$ write $F(Y)$ for the set of all functions (possibly discontinuous) $u\colon Y \to \mathbb{R}$. Note that $F(Y) \simeq \mathbb{R}^n$. In most cases of interest X is Hausdorff so Y is discrete and $F(Y) = C(Y)$.

Definition 13.10. A subset $A \subseteq C(X)$ is interpolating iff for each finite $Y \subseteq X$ the restriction map $u \mapsto u|_Y$ is a surjection $A \to F(Y)$. Equivalently, given distinct points $y_1, \dots y_n$ and real numbers $a_1, \dots a_n$ there must exist a function $u \in A$ such that $u(y_k) = a_k$ for all k.

Note that if X is normal then $C(X)$ itself is separating by Urysohn's theorem.

Proposition 13.11. *The closure of a subalgebra is a subalgebra.*

Proposition 13.12. *A closed subalgebra is a sublattice.*

Proposition 13.13. *A separating subalgebra is interpolating.*

Proposition 13.14. *If X is compact then an interpolating sublattice is dense in $C(X)$.*

Theorem 13.15 (Stone-Weierstraß)**.** *If X is compact then a separating subalgebra of $C(X)$ is dense.*

Definition 13.16. A subset $F \subseteq C(X)$ is equicontinuous iff given $x \in X$ and $\epsilon > 0$ there is a neighbourhood $U \in \mathcal{N}_x$ such that for any $u \in F$ and $y \in U$ we have $|u(x) - u(y)| < \epsilon$.

Theorem 13.17 (Arzela-Ascoli)**.** *Suppose X is compact Hausdorff. A subset $F \subseteq C(X)$ has compact closure iff it is bounded and equicontinuous.*

14. Local Compactness

Definition 14.1. A subspace $Y \subseteq X$ is precompact in X iff $\overline{Y}$ is compact.

Definition 14.2. A space X is locally compact iff every point has a precompact neighbourhood iff the precompact open sets cover X.

Lemma 14.3. *Suppose X is Hausdorff, $Y \subseteq X$ is compact and $x \notin Y$. Then there is a neighbourhood V of x such that $\overline{V} \cap Y = \emptyset$.*

Proposition 14.4. *Suppose X is Hausdorff. Then X is locally compact then for each $x \in X$ and $U \in \mathcal{N}_x$ there is a precompact neighbourhood V of x with $\overline{V} \subseteq U$.*

Corollary 14.5. *If X is locally compact Hausdorff then X is regular and the precompact open sets form a basis.*

Construction 14.6. Suppose X is locally compact Hausdorff. Let ∞ be some object not in X and write $X_\infty = X \cup \{\infty\}$. We declare a subset $U \subseteq X_\infty$ to be open iff

(1) $U \cap X$ is open in X.
(2) If $\infty \in U$ then $X \setminus U$ is compact.

This defines a topology on X_∞, making it a compact Hausdorff space. X is an open subset of X_∞, and the subspace topology on X is the same as the original topology.

Proposition 14.7. *A map $f \colon X \to Y$ of locally compact Hausdorff spaces is proper iff the map evident map $f_\infty \colon X_\infty \to Y_\infty$ sending ∞ to ∞ is continuous.*

15. Countability Axioms

Definition 15.1. A set X is countable iff there is an injective map $f \colon X \to \mathbb{N}$ iff $X = \emptyset$ or there is a surjective map $g \colon \mathbb{N} \to X$ (so $X = \{g(0), g(1), \dots\}$).

Proposition 15.2. *$\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$ are countable but $\mathbb{R}$ and $\mathcal{P}(\mathbb{N}) = \{$ subsets of $\mathbb{N}\}$ are not.*

Proposition 15.3.

(0) *If $f \colon X \to Y$ is injective and Y is countable then so is X.*
(1) *If $g \colon X \to Y$ is surjective and X is countable then so is Y.*
(2) *If I is finite and each X_i is countable then so is $\prod_I X_i$.*
(3) *If I is countable and each X_i is countable then so is $\bigcup_I X_i$.*
(4) *If X is countable then $\mathcal{P}_f(X) = \{$ finite subsets of $X\}$ is countable.*
(5) *If X is countable and infinite then there is a bijection $f \colon \mathbb{N} \to X$.*

Definition 15.4. Let X be a space and $x \in X$. A neighbourhood basis at x is set β of neighbourhoods of x such that for any neighbourhood U of x there is a set $V \in \beta$ such that $V \subseteq U$.

Definition 15.5.

(0) X is separable iff it has a countable dense subset.
(1) X is C_1 or first countable iff each point $x \in X$ has a countable neighbourhood basis.
(2) X is C_2 or second countable iff there is a countable basis for the topology on X.

Proposition 15.6. *A metric space is C_1.*

Definition 15.7. A set $Y \subseteq X$ is sequentially closed iff for each convergent sequence $y_n \to y$ with $y_n \in Y$ we have $y \in Y$. The sequential closure of a subset $Y \subseteq X$ is the set of points $x \in X$ such that there exists a sequence (y_n) in Y such that $y_n \to x$.

Proposition 15.8. *In any space, closed sets are sequentially closed and the sequential closure is contained in the ordinary closure. In a C_1 space, sequentially closed sets are closed and the sequential closure is equal to the ordinary closure.*

Definition 15.9. A map $f \colon X \to Y$ is sequentially continuous iff

$$x_n \to x \quad \Rightarrow \quad f(x_n) \to f(x)$$

Proposition 15.10. *If $f \colon X \to Y$ is continuous then it is sequentially continuous. If $g \colon X \to Y$ is sequentially continuous and X is C_1 then g is continuous.*

16. Paracompactness and Partitions of Unity

Definition 16.1. A collection $\mathcal{A} = \{A_i\}_{i\in I}$ of subsets of X is point-finite iff for each x the set $\{i \mid x \in A_i\}$ is finite. The collection $\mathcal{A}$ is locally finite iff each x has a neighbourhood N such that $\{i \mid N \cap A_i \neq \emptyset\}$ is finite.

Definition 16.2. Suppose $\mathcal{U} = \{U_i\}_{i\in I}$ and $\mathcal{V} = \{V_j\}_{j\in J}$ are open covers of X. We say that $\mathcal{V}$ is a refinement of $\mathcal{U}$ iff for each $j \in J$ there exists $i \in I$ such that $V_j \subseteq U_i$.

Definition 16.3. A space X is paracompact iff it is Hausdorff and every open cover has a locally finite refinement.

Proposition 16.4. *A compact Hausdorff space is paracompact.*

Proposition 16.5. *Suppose that X is Hausdorff and can be written as $X = \bigcup_n X_n$ where X_n is open, $\overline{X_n}$ is compact and $\overline{X_n} \subseteq X_{n+1}$. Then X is paracompact.*

Proposition 16.6. *A locally compact Hausdorff second countable space is paracompact.*

Proposition 16.7. *A paracompact space is normal.*

Definition 16.8. If $\phi\colon X \to \mathbb{R}$ is continuous, the support of ϕ is

$$\operatorname{supp}(\phi) = \overline{\{x \mid \phi(x) \neq 0\}}$$

Proposition 16.9. *If the collection $\{supp(\phi_i)\}_{i\in I}$ is locally finite then $\phi = \sum_i \phi_i$ is well-defined and continuous. Moreover*

$$supp(\phi) = \bigcup_i supp(\phi_i)$$

Definition 16.10. A partition of unity on X is a collection $\{\phi_i\}_{i\in I}$ of continuous functions $\phi_i\colon X \to [0,1]$ such that $\sum_i \phi_i = 1$ and the collection of supports is locally finite. Such a partition of unity is subordinate to a cover $\mathcal{U} = \{U_j\}_{j\in J}$ iff for all i there exists j such that $\operatorname{supp}(\phi_i) \subseteq U_j$.

Lemma 16.11 (Shrinking Lemma). *If X is normal and $\mathcal{U} = \{U_i\}_{i\in I}$ is a locally finite open cover then there is an open cover $\mathcal{V} = \{V_i\}_{i\in I}$ such that $\overline{V_i} \subseteq U_i$.*

Proposition 16.12. *If X is paracompact then every open cover has a subordinate partition of unity.*

17. Function Spaces

Definition 17.1.

$$C(X,Y) = \{ \text{ continuous maps } u\colon X \to Y\}$$

If $K \subseteq X$ is compact Hausdorff and $U \subseteq Y$ is open then we write

$$S(K,U) = \{u \in C(X,Y) \mid u(K) \subseteq U\}$$

The compact-open topology on $C(X,Y)$ is the topology generated by the subbasis

$$\sigma = \{S(K,U) \mid K \subseteq X \text{ compact },\ U \subseteq Y \text{ open }\}$$

Proposition 17.2. *If X is compact Hausdorff and Y is a metric space then the compact-open topology is the same as that derived from the metric*

$$d(u,v) = \max\{d(u(x), v(x)) \mid x \in X\}$$

Definition 17.3.

(0) $\operatorname{in}\colon X \to C(Y, X\times Y)$ is defined by

$$\operatorname{in}(x)(y) = (x,y)$$

(1) $\operatorname{ev}\colon C(Y,Z)\times Y \to Z$ is defined by

$$\operatorname{ev}(v,y) = v(y)$$

(2) If $u\colon X\times Y\to Z$ is continuous, then $u^{\#}\colon X\to C(Y,Z)$ is defined by
$$u^{\#}(x)(y)=u(x,y)$$
(3) If $v\colon X\to C(Y,Z)$ is continuous then $v_{\#}\colon X\times Y\to Z$ is defined by
$$v_{\#}(x,y)=v(x)(y)$$
(4) If $f\colon X\to Y$ is continuous, then $f^*\colon C(Y,Z)\to C(X,Z)$ is defined by
$$f^*(v)=v\circ f$$
$$f^*(Y\xrightarrow{v}Z)=(X\xrightarrow{f}Y\xrightarrow{v}Z)$$
(5) If $g\colon Y\to Z$ is continuous, then $g_*\colon C(X,Y)\to C(X,Z)$ is defined by
$$g_*(u)=g\circ u$$
$$g_*(X\xrightarrow{u}Y)=(X\xrightarrow{u}Y\xrightarrow{g}Z)$$
(6) $\mathrm{comp}\colon C(X,Y)\times C(Y,Z)\to C(X,Z)$ is defined by
$$\mathrm{comp}(u,v)=v\circ u$$
$$\mathrm{comp}(X\xrightarrow{u}Y,Y\xrightarrow{v}Z)=(X\xrightarrow{u}Y\xrightarrow{v}Z)$$

Proposition 17.4. *The functions $in(x)\colon Y\to X\times Y$ and $v_{\#}(x)\colon Y\to Z$ are continuous, as was implicitly assumed in the notation used above.*

Proposition 17.5.
$$v_{\#}=(X\times Y\xrightarrow{v\times id_Y}C(Y,Z)\times Y\xrightarrow{ev}Z)$$
$$u^{\#}=(X\xrightarrow{in}C(Y,X\times Y)\xrightarrow{u_*}C(Y,Z))$$

Proposition 17.6. *If Y is Hausdorff and $Y'\subseteq Y$ is compact then the restriction of the evaluation map*
$$ev\colon C(Y,Z)\times Y'\to Z$$
is continuous.

Proposition 17.7. *The functions in, $u^{\#}$, f^* and g_* are continuous. Moreover, if Y is locally compact then the maps ev, $v_{\#}$ and comp are continuous.*