NOTES ON REAL-VALUED FUNCTIONS

NEIL STRICKLAND

In these notes, X is a topological space and $C(X)$ is the set of continuous real-valued functions $u: X \to \mathbb{R}$. We consider this as a metric space in the usual way.

Definition 0.0.1. Write $I = [0,1] \cap \mathbb{Q}$. A Urysohn filtration on X is a family

$$\underline{U} = (U_a)_{a \in I}$$

of open subsets of X such that

$$a < b \quad \Rightarrow \quad \overline{U_a} \subseteq U_b$$

Such a filtration is regular iff for all a we have

$$U_a = \bigcup_{b<a} U_b$$

Note that this implies $U_0 = \emptyset$. (These definitions are not standard).

Construction 0.0.2. *Given a Urysohn filtration $\underline{U}$ define $f_{\underline{U}}: X \to [0,1]$ by*

$$f_{\underline{U}}(x) = \inf\{a \in I \mid x \in U_a\}$$

This is to be interpreted as 1 if $x \notin U_a$ for any a.

Define also

$$U'_a = \bigcup_{b<a} U_b$$

Finally, given $g \in C(X, [0,1])$ write

$$V_a(g) = \{x \mid g(x) < a\} = g^{-1}(-\infty, a)$$

Theorem 0.0.3. (0) *$\underline{U}'$ is a regular Urysohn filtration.*

(1) $\underline{U}'' = \underline{U}'$
(2) *$\underline{V}(g)$ is a regular Urysohn filtration.*
(3) *$f_{\underline{U}}$ is continuous.*
(4) *$\underline{V}(f_{\underline{U}}) = \underline{U}'$.*
(5) $f_{\underline{V}(g)} = g$
(6) *There is a bijective correspondence between regular Urysohn filtrations on X and continuous functions $X \to [0,1]$, given by $\underline{U} \mapsto f_{\underline{U}}$ and $g \mapsto \underline{V}(g)$.*

Proof. In the following a, b and c are implicitly supposed to be elements of I, x is supposed to be a point of X and s to be an element of $[0,1]$. We shall repeatedly use without comment the fact that between any two distinct real numbers there lies a rational, and hence that

$$s = \inf\{a \mid s < a\}$$

where $\inf(\emptyset)$ is interpreted as 1.

(0) It is clear that $U'_a \subseteq U_a$. Suppose $a < b$, and write $c = (a+b)/2$ so $a < c < b$. As $\underline{U}$ is a Urysohn filtration, we have $\overline{U_a} \subseteq U_c$. Thus

$$\overline{U'_a} \subseteq \overline{U_a} \subseteq U_c \subseteq U'_b$$

This shows that $\underline{U}'$ is a Urysohn filtration. Moreover,

$$\bigcup_{b<a} U'_b = \bigcup_{c<b<a} U_c = \bigcup_{c<a} U_c = U'_a$$

which shows that $\underline{U}'$ is regular.

(1) This is immediate from the above.
(2) Suppose $g\colon X \to [0,1]$ is continuous and $a \in I$. Write

$$F_a(g) = \{x \mid g(x) \le a\} = g^{-1}(-\infty, a]$$

This is a closed subset of X, and $V_a(g) \subseteq F_a(g)$ so $\overline{V_a(g)} \subseteq F_a(g)$. Moreover, if $a < b$ then $F_a(g) \subseteq V_b(g)$ and thus $\overline{V_a(g)} \subseteq V_b(g)$. Thus $\underline{V}(g)$ is a Urysohn filtration.

Next note that if $x \in V_a(g)$ then $g(x) < a$ so there is a rational number $b \in I$ with $g(x) < b < a$ so $x \in V_b(g)$. Thus $V_a(g) = \bigcup_{b<a} V_b(g)$ and $\underline{V}(g)$ is regular.
(3) To show that $f = f_{\underline{U}}$ is continuous, we need only check that the preimages of the subbasic open sets $(-\infty, s)$ and (s, ∞) (for $s \in \mathbb{R}$) are open. For the first case, we have

$$\begin{aligned} f(x) < s &\Leftrightarrow \inf\{a \in I \mid x \in U_a\} < s \\ &\Leftrightarrow \exists a \in I \;\; a < s \text{ and } x \in U_a \\ &\Leftrightarrow x \in \bigcup_{a<s} U_a \end{aligned}$$

so $f^{-1}(-\infty, s) = \bigcup_{a<s} U_a$ which is open as required. We next need to show that $f^{-1}(s, \infty)$ is open, or equivalently that $f^{-1}(-\infty, s]$ is closed.

$$\begin{aligned} f(x) \le s &\Leftrightarrow \inf\{a \in I \mid x \in U_a\} \le s \\ &\Leftrightarrow \forall b > s \; \exists a < b \;\; x \in U_a \\ &\Leftrightarrow \forall b > s \;\; x \in \overline{U_b} \\ &\Leftrightarrow x \in \bigcap_{b>s} \overline{U_b} \end{aligned}$$

The third equivalence requires a moment's thought; it is at this point that we use the condition $\overline{U_a} \subseteq U_b$. We conclude that $f^{-1}(-\infty, s] = \bigcap_{b>s} \overline{U_b}$ which is closed as required.
(4) We have just showed that

$$V_a(f_{\underline{U}}) = f_{\underline{U}}^{-1}(-\infty, a) = \bigcup_{b<a} U_b = U'_a$$

as required.
(5)

$$f_{\underline{V}(g)}(x) = \inf\{a \mid x \in V_a(g)\} = \inf\{a \mid g(x) < a\} = g(x)$$

(6) This is clear from the previous parts of the theorem.

□

Theorem 0.0.4 (Urysohn)**.** *If X is normal and Y_0 and Y_1 are disjoint closed subsets of X then there is a continuous function $f\colon X \to [0,1]$ such that $f = 0$ on Y_0 and $f = 1$ on Y_1.*

Proof. First, we choose an enumeration of I, in other words a sequence $(a_n)_{n\in\mathbb{N}}$ of points of I such that each $a \in I$ occurs as a_n for precisely one $n \in \mathbb{N}$. This is possible because I is countable. We can arrange it such that $a_0 = 0$ and $a_1 = 1$. We shall choose recursively open sets $U_k \subseteq X$ such that

(0) $Y_0 \subseteq U_k \subseteq Y_1^c$
(1) If $a_k < a_l$ then $\overline{U_k} \subseteq U_l$.

By normality, we can choose an open set U_0 with

$$Y_0 \subseteq U_0 \subseteq \overline{U_0} \subseteq Y_1^c$$

Similarly, we can choose U_1 with

$$\overline{U_0} \subseteq U_1 \subseteq \overline{U_1} \subseteq Y_1^c$$

Suppose that $n > 0$ and $U_0, \ldots U_n$ have been chosen. Let a_k be the largest of the values a_j for which $j \le n$ and $a_j < a_{n+1}$. Similarly, let a_l be the smallest of the values a_j for which $a_j > a_{n+1}$. Note that $a_0 < a_{n+1} < a_1$ so these definitions always make sense — we need not worry that there might not be any such j. Anyway, by assumption we have $\overline{U_k} \subseteq U_l$ so by normality we can find U_{n+1} with

$$\overline{U_k} \subseteq U_{n+1} \subseteq \overline{U_{n+1}} \subseteq U_l$$

It is easy to check that this satisfies the requirements. Finally, we define a Urysohn filtration by

$$V_a = U_k \text{ where } a = a_k$$

and a continuous function $f = f_{\underline{V}}$. It is easy to see that $f = 0$ on Y_0 and $f = 1$ on Y_1. □

Theorem 0.0.5 (Tietze)**.** *Suppose X is normal and $Y \subseteq X$ is closed. Suppose $f\colon Y \to [-1, 1]$ is continuous. Then there is a continuous function $g\colon X \to [-1, 1]$ with $g|_Y = f$.*

Lemma 0.0.6. *Suppose X is normal and $Y \subseteq X$ is closed. Suppose $f\colon Y \to [-r, r]$ is continuous. Then there is a continuous function $g\colon X \to [-r/3, r/3]$ with*

$$\|f - g\|_Y = \sup_{y \in Y} |f(y) - g(y)| \leq 2r/3$$

of lemma. Write

$$Y_- = f^{-1}[-r, -r/3] \subseteq Y$$
$$Y_0 = f^{-1}[-r/3, +r/3] \subseteq Y$$
$$Y_+ = f^{-1}[+r/3, +r] \subseteq Y$$

These sets are clearly closed in Y, and Y is closed in X, so $Y_\pm$ are closed in X. As $Y_+ \cap Y_- = \emptyset$, Urysohn's theorem gives us a function $g\colon X \to [-r/3, r/3]$ with $g = -r/3$ on Y_- and $g = r/3$ on Y_+. By considering the cases $y \in Y_-$, $y \in Y_0$, and $y \in Y_+$ separately we see that $|f(y) - g(y)| \leq 2r/3$ for all $y \in Y$ and thus that $\|f - g\|_Y \leq 2r/3$. □

of theorem. We shall choose recursively continuous functions $g_k\colon X \to \mathbb{R}$ (for $k \geq 1$) with $\|g_k\| \leq (2/3)^k/2$ such that

$$\|f - \sum_{k=1}^{n} g_k\|_Y \leq (2/3)^n$$

Indeed, this works for $n = 0$ as $\|f\|_Y \leq 1$. Given g_k for $k \leq n$ we get g_{n+1} by applying the lemma to $f - \sum_1^n g_k$ with $r = (2/3)^n$. Finally, we set $g = \sum_1^\infty g_k$. The sum is uniformly convergent so g is continuous and $\|f - g\|_Y = 0$ so $g|_Y = f$ as required. □

Definition 0.0.7. A subset $A \subseteq C(X)$ is a subalgebra iff

- A0 Every constant function lies in A.
- A1 $u, v \in A \Rightarrow u + v \in A$.
- A2 $u, v \in A \Rightarrow uv \in A$.

Definition 0.0.8. Given $u, v \in C(X)$ define $u \vee v$ and $u \wedge v$ in $C(X)$ by

$$(u \vee v)(x) = \max(u(x), v(x))$$
$$(u \wedge v)(x) = \min(u(x), v(x))$$

A subset $A \subseteq C(X)$ is a sublattice iff

- L0 $u, v \in A \Rightarrow u \vee v \in A$
- L1 $u, v \in A \Rightarrow u \wedge v \in A$

Definition 0.0.9. A subset $A \subseteq C(X)$ is separating iff given distinct points x, y there exists $u \in A$ such that $u(x) \neq u(y)$.

Given a finite set $Y = \{y_1, \ldots y_n\} \subseteq X$ write $F(Y)$ for the set of all functions (possibly discontinuous) $u\colon Y \to \mathbb{R}$. Note that $F(Y) \simeq \mathbb{R}^n$. In most cases of interest X is Hausdorff so Y is discrete and $F(Y) = C(Y)$.

Definition 0.0.10. A subset $A \subseteq C(X)$ is interpolating iff for each finite $Y \subseteq X$ the restriction map $u \mapsto u|_Y$ is a surjection $A \to F(Y)$. Equivalently, given distinct points $y_1, \ldots y_n$ and real numbers $a_1, \ldots a_n$ there must exist a function $u \in A$ such that $u(y_k) = a_k$ for all k.

Note that if X is normal then $C(X)$ itself is separating by Urysohn's theorem.

Proposition 0.0.11. *The closure of a subalgebra is a subalgebra.*

Proof. Suppose $A \subseteq C(X)$ is a subalgebra. Clearly all constant functions lie in $\overline{A}$. One checks easily that the following functions are continuous:

$$\sigma\colon C(X)\times C(X)\to C(X) \qquad \sigma(u,v)=u+v$$

$$\mu\colon C(X)\times C(X)\to C(X) \qquad \mu(u,v)=uv$$

As A is an algebra, we have $\mu(A\times A)\subseteq A$ so

$$A\times A\subseteq \mu^{-1}(A)\subseteq \mu^{-1}(\overline{A})$$

As this last set is closed, we have

$$\overline{A}\times\overline{A}=\overline{A\times A}\subseteq \mu^{-1}(\overline{A})$$

so

$$\mu(\overline{A}\times\overline{A})\subseteq\overline{A}$$

Similarly, $\sigma(\overline{A}\times\overline{A})\subseteq\overline{A}$. It follows that $\overline{A}$ is a subalgebra as claimed. □

Lemma 0.0.12. *There is a sequence $(p_n(x))_{n\in\mathbb{N}}$ of polynomial functions such that $p_n(x)\to|x|$ uniformly on the interval $[-1,1]$.*

Proof. First, note that the Taylor series for $\sqrt{1-x}$ is

$$\sqrt{1-x}=\sum_k a_k x^k$$

where (for $n\geq 2$)

$$\begin{aligned} a_n &= (-1)^n\binom{\frac{1}{2}}{n} \\ &= \frac{(-1)^n}{n!}\frac{1}{2}\frac{-1}{2}\frac{-3}{2}\cdots\frac{3-2n}{2} \\ &= \frac{-1}{2}\prod_{k=2}^{n}\frac{2k-3}{2k} \\ &= \frac{-1}{2}\prod_{k=2}^{n}\left(1-\frac{3}{2k}\right) \end{aligned}$$

In fact, this series converges uniformly on $[-1,1]$. To prove this, we need only show that

$$\sum_{n=0}^{\infty}|a_n|<\infty$$

Indeed, for any $t\geq 0$ we have $1-t\leq e^{-t}$ so

$$1-3/2k\leq\exp(-3/2k)$$

$$2|a_n|\leq\prod_{k=2}^{n}\exp(-3/2k)=\exp\left(-\sum_{2}^{n}3/2k\right)$$

However, by looking at the graph we find

$$\sum_{2}^{n}\frac{1}{k}\geq\int_{2}^{n+1}\frac{\mathrm{d}k}{k}=\log\left(\frac{n+1}{2}\right)$$

so

$$2|a_n|\leq\exp\left(\frac{-3}{2}\log\frac{n+1}{2}\right)=\left(\frac{n+1}{2}\right)^{-3/2}$$

$$\sum_{2}^{\infty}|a_n|\leq 2^{1/2}\sum_{n=2}^{\infty}(n+1)^{-3/2}\leq 2^{1/2}\int_{n=1}^{\infty}(n+1)^{-3/2}\mathrm{d}n=2$$

which is finite as required. It follows that the series converges uniformly to some continuous function. Standard arguments from complex analysis show that the limit must be the right one, viz. the nonnegative square root of $1 - x$. Finally, we write

$$r_n(x) = \sum_{k=0}^{n} a_k x^k$$
$$p_n(x) = r_n(1 - x^2)$$

It is now easy to check that these functions p_n are as required. □

Proposition 0.0.13. *A closed subalgebra is a sublattice.*

Proof. Suppose A is a closed subalgebra, and that $w \in A$ has $\|w\| \leq 1$. As A is a subalgebra and p_n is a polynomial, we have $p_n(w) \in A$. Moreover, for all x we have $w(x) \in [-1, 1]$ so

$$|p_n(w(x)) - |w(x)|| \leq \sup_{[-1,1]} \{|p_n(t) - |t||\} \to 0$$

so $p_n(w) \to |w|$ in $C(X)$. As A is closed, this implies $|w| \in A$. More generally, let us not assume that $\|w\| \leq 1$. Write $\alpha = \|w\|$. By the previous argument, $|w/\alpha| \in A$ but A is an algebra so $|w| = \alpha|w/\alpha| \in A$.

Finally, suppose $u, v \in A$. Write $w = u - v \in A$ so $|w| = |u - v| \in A$. One checks easily that

$$u \vee v = \tfrac{1}{2}(u + v + |u - v|) \in A$$
$$u \wedge v = \tfrac{1}{2}(u + v - |u - v|) \in A$$

Thus A is a sublattice. □

Proposition 0.0.14. *A separating subalgebra is interpolating.*

Proof. Let $A \subseteq C(X)$ be a separating subalgebra, and Y a finite subset of X. Suppose y, z are two distinct points in Y. As A is separating, there is a function $u_{yz} \in A$ with $u(y) \neq u(z)$. Write

$$v_{yz}(x) = \frac{v(x) - v(z)}{v(y) - v(z)}$$

As A is a subalgebra, $v_{yz} \in A$. Clearly $v_{yz}(y) = 1$ and $v_{yz}(z) = 0$. Now write

$$v_y(x) = \prod_{z \in Y, z \neq y} v_{yz}(x)$$

Again, $v_y \in A$ because A is a subalgebra. Clearly $v_y(y) = 1$ and $v_y(z) = 0$ if z is any other point in Y. Finally, suppose $f\colon Y \to \mathbb{R}$ is an arbitary function. Define

$$v = \sum_{y \in Y} f(y) v_y$$

This is an element of A and for $z \in Y$ we have

$$v(z) = \sum_{y \in Y} f(y) v_y(z) = f(z)$$

(the terms in the sum for which $y \neq z$ are zero). Thus $v|_Y = f$. This shows that A is interpolating. □

Proposition 0.0.15. *If X is compact then an interpolating sublattice is dense in $C(X)$.*

Proof. Suppose $A \subseteq C(X)$ is an interpolating sublattice. Suppose $f \in C(X)$ and $\epsilon > 0$. We are required to find $h \in A$ such that $\|f - h\| < \epsilon$, or equivalently $f - \epsilon < h < f + \epsilon$.

For each $g \in A$ write

$$U(g) = \{x \mid g(x) > f(x) - \epsilon\}$$
$$V(g) = \{x \mid g(x) < f(x) + \epsilon\}$$

These sets are clearly open. Fix $x \in X$ and write

$$\mathcal{U}(x) = \{U(g) \mid g \in A \text{ and } g(x) = f(x)\}$$

I claim this is an open cover of X. Indeed, suppose $y \in X$. By the interpolation property there is a function $g \in A$ with $g|_{\{x,y\}} = f|_{\{x,y\}}$, in other words $g(x) = f(x)$ and $g(y) = f(y) > f(y) - \epsilon$ so $y \in U(g) \in \mathcal{U}(x)$. This proves the claim.

There is thus a finite subcover $X = U(g_1) \cup \ldots U(g_n)$ with $g_k \in A$ and $g_k(x) = f(x)$. Write

$$g = g_1 \vee \ldots \vee g_n$$

This is in A as A is a sublattice. It is easy to see that $g > f - \epsilon$ and $g(x) = f(x)$ (so $x \in V(g)$).

Now write

$$\mathcal{V} = \{V(g) \mid g \in A \text{ and } g > f - \epsilon\}$$

If g is constructed as above then $x \in V(g) \in \mathcal{V}$, so $\mathcal{V}$ is an open cover of X. We thus have a finite subcover $X = V(g_1) \cup \ldots V(g_m)$ with $g_k \in A$ and $g_k > f - \epsilon$ Write

$$h = g_1 \wedge \ldots g_m$$

This again lies in A and $f - \epsilon < h < f + \epsilon$ as required. □

Theorem 0.0.16 (Stone-Weierstraß). *If X is compact then a separating subalgebra of $C(X)$ is dense.*

Proof. Let A be a separating subalgebra. Then $\overline{A}$ is a subalgebra by proposition **??**. It is thus a sublattice by proposition **??**, and it is interpolating by proposition **??**. It is thus dense by proposition **??**. As it is closed and dense, we see that $\overline{A} = C(X)$. In other words, A itself is dense. □

Definition 0.0.17. A subset $A \subseteq C(X)$ is equicontinuous iff given $x \in X$ and $\epsilon > 0$ there is a neighbourhood $U \in \mathcal{N}_x$ such that for any $u \in A$ and $y \in U$ we have $|u(x) - u(y)| < \epsilon$.

Theorem 0.0.18 (Arzela-Ascoli). *Suppose X is compact Hausdorff. A subset $A \subseteq C(X)$ is compact iff it is bounded, closed and equicontinuous.*

First, we prove a lemma:

Lemma 0.0.19. *The evaluation functions*

$$ev\colon C(X) \times X \to \mathbb{R} \qquad ev(u, x) = u(x)$$

and

$$ev_x\colon C(X) \to \mathbb{R} \qquad ev_x(u) = u(x)$$

are continuous.

of lemma. Suppose $\mathrm{ev}(u, x) = u(x) = s$ and $\epsilon > 0$. We have to find a neighbourhood U of (u, x) in $C(X) \times X$ such that

$$|\mathrm{ev}(v, y) - s| = |v(y) - s| = |v(y) - u(x)| < \epsilon$$

when $(v, y) \in U$. First, as u is continuous, there is a neighbourhood W of x in X such that $|u(y) - u(x)| < \epsilon/2$ when $y \in W$. Thus, if $\|v - u\| < \epsilon/2$ then

$$|v(y) - u(x)| \leq |v(y) - u(y)| + |u(y) - u(x)| < \|v - u\| + \epsilon/2 < \epsilon$$

so we can take $V = B(u, \epsilon/2) \times W$. This shows that ev is continuous; the proof for ev_x is similar but easier. □

We also recall a standard fact (page 168 of the book):

Lemma 0.0.20. *If Y is compact and $U \subseteq Y \times X$ is open and $Y \times \{x\} \subseteq U$ then there is a neighbourhood V of x in X such that $Y \times V \subseteq U$.*

of theorem. First, suppose A is compact. By standard results valid in any metric space, it is bounded and closed, so we need only show that A is equicontinuous. Suppose $x \in X$ and $\epsilon > 0$. We can define a map

$$\phi\colon A \times X \to \mathbb{R} \qquad \phi(u, y) = u(y) - u(x) = \mathrm{ev}(u, y) - \mathrm{ev}_x(u)$$

This is continuous by the first lemma. We define an open subset $U \subseteq A \times X$ by

$$U = \{(u, y) \in A \times X \mid |u(y) - u(x)| < \epsilon\} = \phi^{-1}(-\epsilon, \epsilon)$$

This is open and contains $A \times \{x\}$. By the second lemma, there is a neighbourhood V of x such that $A \times V \subseteq U$, so for $u \in A$ and $y \in V$ we have $|u(y) - u(x)| < \epsilon$. This shows that A is equicontinuous.

Conversely, suppose A is closed, bounded and equicontinuous. We know that $C(X)$ is a complete metric space, and that a subspace of a complete metric space is compact iff totally bounded and closed, so we need only show that A is totally bounded. Suppose $\epsilon > 0$. From the definition of equcontinuity and the compactness of X we see that there is a finite set $Y \subseteq X$ and neighbourhoods $U_y \in \mathcal{N}_y$ for $y \in Y$ such that

$$X = \bigcup_{y \in Y} U_y$$

and

$$u \in A,\ x \in U_y \quad \Rightarrow \quad |u(x) - u(y)| < \epsilon/3$$

Next, consider

$$F(Y) = \{ \text{ all functions } v \colon Y \to \mathbb{R}\}$$

Note that if Y has n points then $F(Y) \simeq \mathbb{R}^n$. It follows that a bounded subset of $F(Y)$ is totally bounded — this can be seen either directly (by writing $\mathbb{R}^n$ as a union of small open boxes) or via the characterisation of compact subsets of $\mathbb{R}^n$.

Write

$$A|_Y = \{u|_Y \mid u \in A\}$$

This set is bounded (as A is) and thus totally bounded, so we can find a finite set $B \subseteq A$ such that $B|_Y$ is an $\epsilon/4$-net in $A|_Y$. The claim is that B is an ϵ-net in A. Indeed, suppose $u \in A$. As $B|_Y$ is an $\epsilon/4$-net in $A|_Y$, there is an element $v \in B$ with $\|u|_Y - v|_Y\| < \epsilon/4$. Suppose $x \in X$. Then $x \in U_y$ for some $y \in Y$. Thus

$$|u(x) - v(x)| < |u(x) - u(y)| + |u(y) - v(y)| + |v(y) - v(x)|$$

As $u, v \in A$ and $x \in U_y$ we have

$$|u(x) - u(y)|, |v(x) - v(y)| < \epsilon/4$$

As $\|u|_Y - v|_Y\| < \epsilon/4$, we have

$$|u(y) - v(y)| < \epsilon/4$$

Thus

$$|u(x) - v(x)| < 3\epsilon/4$$

This holds for all x so $\|u - v\| \leq 3\epsilon/4 < \epsilon$. Thus B is an ϵ-net as required. □