AFFINE AND LINEAR OPERADS

N. P. STRICKLAND

Let K denote the Stasheff operad, so $K(n)$ is homeomorphic to a closed ball of dimension $n-2$. It would be nice to have a version of K in which the objects are polytopes, and the morphisms are inclusions of faces. It is not clear whether this is possible. (In the Loday model, the objects are polytopes but the morphisms are only piecewise linear.) This note shows that a certain class of attempts does not work.

Let $\mathcal{V}$ be the category of finite-dimensional real vector spaces and linear maps. Let $\mathcal{A}$ be the category with the same objects, but with all affine maps as morphisms, so $\mathcal{A}(V,W) \simeq \mathcal{V}(V,W) \times W$. There are evident functors $\mathcal{V} \to \mathcal{A} \to \mathcal{V}$, which are the identity on objects, and which compose to give the identity. Both these functors preserve products and thus send operads to operads. The hope is to exhibit K as a suboperad of an operad in $\mathcal{A}$. There is a very natural operad Γ in $\mathcal{V}$ that we will discuss below, for which $\dim(\Gamma(n)) = n-2$. It is easy to check that K cannot embed in Γ, but the hope was to change this by modifying the structure maps of Γ in a certain way, giving a new operad Γ' in $\mathcal{A}$. Our main result is that for all modifications in the class that we consider, the resulting operad Γ' is isomorphic to Γ, so our hope cannot be realised.

We now specify the version of the theory of operads that we will use. For us, an operad in a category $\mathcal{C}$ will consist of objects $E(A) \in \mathcal{C}$ for each nonempty finite ordered set A, together an element $1 \in E(1)$, and a map

$$\gamma_p \colon E(B) \times \prod_{b \in B} E(A_b) \to E(A)$$

for each ordered surjection $p \colon A \to B$, subject to some axioms. In particular, given ordered surjections $A \xrightarrow{p} B \xrightarrow{q} C$, the following diagram must commute:

$$\begin{array}{ccc}
E(C) \times \prod_c E(B_c) \times \prod_b E(A_b) & \xrightarrow{1 \times \prod_c \gamma_{p_c}} & E(C) \times \prod_c E(A_c) \\
{\scriptstyle \gamma_q \times 1} \downarrow & & \downarrow {\scriptstyle \gamma_{qp}} \\
E(B) \times \prod_b E(A_b) & \xrightarrow[\gamma_p]{} & E(A).
\end{array}$$

Here p_c is the map from $A_c = (qp)^{-1}\{c\}$ to $B_c = q^{-1}\{c\}$ obtained by restricting p. There is also a unit axiom, which we will not spell out.

Moreover, there is a hidden functorality condition: for an order-isomorphism $A \to A'$, we should have an isomorphism $E(A) \to E(A')$, and these should compose in the obvious way and be compatible with the maps γ_p. If $|A| = |A'|$ then there is a unique isomorphism $A \to A'$, and if $|A| \neq |A'|$ then there is no isomorphism. Because of this, the functorality condition is very mild. Similar issues will arise later in several places, but we shall not mention them explicitly.

Suppose we have an operad E in $\mathcal{V}$, with structure maps γ_p say. We would like to classify lifts of E to operads in $\mathcal{A}$.

Definition 0.1. We define vector spaces Θ and Φ as follows. An element of Θ consists of natural elements $\theta(A) \in E(A)$ for all A. An element of Φ consists of elements $\phi(p) \in E(A)$ for all $p \colon A \to B$, such that for all $A \xrightarrow{p} B \xrightarrow{q} C$ we have

$$\gamma_p(\phi(q), 0) + \phi(p) = \phi(qp) + \gamma_{qp}(0, (\phi(p_c))_{c \in C}).$$

We call this the *cocycle condition.*

We also define a map $\delta \colon \Theta \to \Phi$ by

$$(\delta\theta)(p) = \theta(A) - \gamma_p(\theta(B), (\theta(A_b))_{binB}).$$

Proposition 0.2. (a) *If $\phi \in \Phi$ and we put $\gamma'_p(x) = \gamma_p(x) + \phi(p)$, then the maps γ'_p make E into an operad in $\mathcal{A}$.*

(b) *This procedure gives all possible affine liftings of E.*
(c) *Suppose we have lifts E_0, E_1 given by elements $\phi_0, \phi_1 \in \Phi$, and we let M be the set of isomorphisms $E_0 \to E_1$ that cover the identity in $\mathcal{V}$. Then M bijects naturally with $\{\theta \in \Theta \mid \delta\theta = \phi_1 - \phi_0\}$.*

Proof. Write this out. □

We now consider a specific example. Let $\mathrm{Gaps}(A)$ denote the set of intervals of size 2 in A, so $|\mathrm{Gaps}(A)| = |A| - 1$. Given an ordered epimorphism $p\colon A \to B$, we define a map

$$\delta_p\colon \mathrm{Gaps}(A) \to \mathrm{Gaps}(B) \amalg \coprod_b \mathrm{Gaps}(A_b)$$

by

$$\delta_p(g) = \begin{cases} g \in \mathrm{Gaps}(A_b) & \text{if } p(g) = \{b\} \\ p(g) \in \mathrm{Gaps}(B) & \text{if } |p(g)| = 2. \end{cases}$$

One checks that this is a bijection. Now let $\mathcal{B}$ denote the opposite of the category of finite sets and bijections, considered as a symmetric monoidal category under disjoint union. One checks that the maps δ_b make Gaps into an operad in $\mathcal{B}$. Next, for any finite set X we put

$$W^*(X) = \{t\colon X \to \mathbb{R} \mid \sum_{x \in X} t(x) = 0\}.$$

We have natural maps $W^*(X) \times W^*(Y) \to W^*(X \amalg Y)$, making W^* into a lax monoidal functor $\mathcal{B} \to \mathcal{V}$. It follows that the vector spaces $\Gamma(A) = W^*(\mathrm{Gaps}(A))$ give an operad in $\mathcal{V}$. For $y \in \Gamma(B)$ and $x_b \in \Gamma(A_b)$ we have

$$\gamma_p(y, (x_b)_{b\in B})(g) = \begin{cases} x_b(g) & \text{if } p(g) = \{b\} \\ y(p(g)) & \text{if } |p(g)| = 2. \end{cases}$$

It is easy to see that $\Gamma(2)$ is a point, so Γ-algebras have a single binary operation. Moreover, one checks that this operation is associative, with the n-fold product corresponding to the zero element in $\Gamma(n)$. As the binary operation in the Stasheff operad is not associative, we see that the Stasheff operad does not embed in Γ. However, it might embed in some affine lift of Γ.

In the case $E = \Gamma$, an element $\theta \in \Theta$ consists of numbers $\theta(A, g)$ for $g \in \mathrm{Gaps}(A)$, such that $\sum_{g \in \mathrm{Gaps}(A)} \theta(A, g) = 0$. Similarly, an element $\phi \in \Phi$ is given by numbers $\phi(p, g)$ for each ordered surjection $p\colon A \to B$ and each $g \in \mathrm{Gaps}(A)$, with $\sum_g \phi(p, g) = 0$. These would have to satisfy the operad axiom mentioned previously. To make this more explicit, consider maps $A \xrightarrow{p} B \xrightarrow{q} C$ and a gap $g \in \mathrm{Gaps}(A)$. We distinguish three cases:

I $p(g) = \{b\}$, and $q(b) = c$
II $|p(g)| = 2$ and $qp(g) = \{c\}$
III $|p(g)| = 2$ and $|qp(g)| = 2$.

The cocycle condition translates as follows in the three cases:

I $\phi(p, g) = \phi(qp, g) + \phi(p_c, g)$
II $\phi(q, p(g)) + \phi(p, g) = \phi(qp, g) + \phi(p_c, g)$
III $\phi(q, p(g)) + \phi(p, g) = \phi(qp, g)$.

The map $\delta\colon \Theta \to \Phi$ is given by

$$(\delta\theta)(p, g) = \begin{cases} \theta(A, g) - \theta(A_b, g) & \text{if } p(g) = \{b\} \\ \theta(A, g) - \theta(B, p(g)) & \text{if } |p(g)| = 2. \end{cases}$$

Now let Θ^+ and Φ^+ be defined analogously to Θ and Φ, but without the conditions $\sum_g \theta(A, g) = 0$ and $\sum_g \phi(p, g) = 0$. Note that δ extends to give a map $\delta^+\colon \Theta^+ \to \Phi^+$ by the same formulae. We define maps $\pi, \pi'\colon \Phi^+ \to \Theta^+$ as follows. First, for any $g \in \mathrm{Gaps}(A)$, we let A/g be the quotient set in which the two elements of g are identified together. We give this the unique order for which the quotient map

$r = r_{A,g} \colon A \to A/g$ is order-preserving. We also let $r' = r'_{A,g} \colon A \to g$ be the unique order-preserving retraction onto g. We then put

$$(\pi\phi)(A,g) = \phi(r,g)$$
$$(\pi'\phi)(A,g) = \phi(r',g).$$

We also define maps $\mathbb{R} \xrightarrow{\eta} \Theta^+ \xrightarrow{\epsilon} \mathbb{R}$ by $\eta(t)(A,g) = t$ and

$$\epsilon(\theta) = \theta(\{0,1\},\{0,1\}).$$

Note that $\epsilon\eta = 1$ and $\epsilon(\Theta) = 0$.

Lemma 0.3. *If $\phi \in \Phi^+$ and $p \colon A \to B$ and $g \in \mathrm{Gaps}(A)$, we have*

$$\phi(p,g) = \begin{cases} (\pi\phi)(A,g) - (\pi\phi)(A_b,g) & \text{if } p(g) = \{b\} \\ (\pi'\phi)(A,g) - (\pi'\phi)(B,p(g)) & \text{if } |p(g)| = 2. \end{cases}$$

Lemma 0.4. *In the first case, note that there is a unique map $\overline{p} \colon A/g \to B$ with $\overline{p}r = p$. The cocycle condition for this composite is covered by Case I: it says that $\phi(r,g) = \phi(p,g) + \phi(r_b,g)$, where r_b is a restriction of r which can be identified with the map $A_b \to A_b/g$. Thus $\phi(r,g) = (\pi\phi)(A,g)$ and $\phi(r_b,g) = (\pi\phi)(A_b,g)$, giving the first case of the lemma.*

For the second case, note that the composite $r'_{B,p(g)}p$ can be identified with $r'_{A,g}$. The cocycle condition for this composite is covered by Case III, and gives $(\pi'\phi)(B,p(g)) + \phi(p,g) = (\pi'\phi)(A,g)$, as required.

Lemma 0.5. *We have $\pi\delta = \pi'\delta = 1 - \eta\epsilon \colon \Theta \to \Theta$.*

Proof. Consider an element $\theta \in \Theta$, and put $\phi = \delta(\theta)$. Let b denote the image of g in A/g, so A_b is just the set g, so $\theta(A_b,g) = \epsilon(\theta)$. Using this, and the first clause in the definition of δ, we see that $\pi\delta(\theta)(A,g) = \phi(r_{A,g},g) = \theta(A,g) - \epsilon(\theta)$ as claimed.

Now instead consider $r' \colon A \to g$. Here the second clause in the definition of δ is relevant, giving $\pi'\delta(\theta)(A,g) = \phi(r',g) = \theta(A,g) - \theta(g,g) = \theta(A,g) - \epsilon(\theta)$, as required. □

Lemma 0.6. *$\epsilon\pi' = 0 = \epsilon\pi \colon \Phi^+ \to \mathbb{R}$, and $\delta\eta = 0 \colon \mathbb{R} \to \Phi^+$.*

Proof. Write 2 for $\{0,1\}$ (so $\mathrm{Gaps}(2) = \{2\}$), and p for the identity map of 2. From the definitions we have $\epsilon\pi'(\phi) = \phi(p,2)$. Taking $q = p$ and using case III of the cocycle identity, we see that $\phi(p,2) + \phi(p,2) = \phi(p,2)$, so $\phi(p,2) = 0$. Similarly, case I of the cocycle identity for $2 \to 1 \to 1$ gives $\epsilon\pi(\phi) = 0$. It is immediate from the definitions that $\delta\eta = 0$. □

Lemma 0.7. *The map $\pi' \colon \Phi^+ \to \Theta^+$ is injective.*

Proof. Suppose we have $\phi \in \Phi^+$ with $\pi'(\phi) = 0$. Put $\nu = \pi(\phi)$, and then

$$\nu[i:j] = \nu(\{0,1,\dots,i+j-1\},\{i-1,i\}).$$

We know from Lemma 0.3 that when $p \colon A \to B$ and $g \in \mathrm{Gaps}(A)$ we have

$$\phi(p,g) = \begin{cases} \nu(A,g) - \nu(A_b,g) & \text{if } p(g) = \{b\} \\ 0 & \text{if } |p(g)| = 2. \end{cases}$$

Moreover, every pair (A,g) is isomorphic to the pair involved in $\nu[i:j]$ for some i and j. It will therefore suffice to prove that $\nu[i:j] = 0$. Note that we have $\nu[1:1] = \epsilon\pi(\phi) = 0$ by Lemma 0.6.

Given the above description of $\phi(A,g)$, case II of the cocycle identity simplifies as follows: if $|p(g)| = 2$ and $qp(g) = \{c\}$ then

$$\nu(A_c,g) - \nu(A,g) - \nu(B_c,p(g)) + \nu(B,p(g)) = 0.$$

(One also checks that cases I and III are automatically satisfied.)

Now consider a pullback diagram as follows:

$$\begin{array}{ccc} A' & \xrightarrow{u} & A \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow[v]{} & B \end{array}$$

Suppose that the horizontal maps are inclusions of intervals, and the vertical maps are order-preserving surjections, and that $g \in \mathrm{Gaps}(A') \subseteq \mathrm{Gaps}(A)$. By applying the cocycle identity to the maps $A \xrightarrow{p} B \xrightarrow{q} B/B'$, we find that

$$\nu(A', g) - \nu(A, g) - \nu(B', p(g)) + \nu(B, p(g)) = 0.$$

Now put $\lambda[i] = \nu[i : 1]$ and $\rho[j] = \nu[1 : j]$. We claim that $\nu[i : j] = \lambda[i] + \rho[j]$. To see this, consider the sets $A = \{0, \dots, i + j - 1\}$ and $I = \{0, \dots, i - 1\}$ and $J = \{i, \dots, i + j - 1\}$ and $g = \{i, i + 1\}$. Then put $B = A/I$ and $B' = (I \amalg \{i\})/I \simeq 2$, so $A' = I \amalg \{i\}$. The square now looks like

$$\begin{array}{ccc} I \amalg 1 & \xrightarrow{u} & I \amalg J \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ 1 \amalg 1 & \xrightarrow[v]{} & 1 \amalg J \end{array}$$

We then have $\nu(A, g) = \nu[i : j]$ and $\nu(A', g) = \lambda[i]$ and $\nu(B, p(g)) = \rho[j]$ and $\nu(B', p(g)) = \nu[1 : 1] = 0$. The cocycle identity therefore gives $\nu[i : j] = \lambda[i] + \rho[j]$ as claimed, and it will suffice to show that $\lambda = \rho = 0$. Note that $\lambda[1] = \rho[1] = \nu[1 : 1] = 0$.

Now instead consider the following square, in which the horizontal maps are the obvious inclusions and the vertical maps are the obvious quotients.

$$\begin{array}{ccc} J \amalg 1 & \xrightarrow{u} & I \amalg J \amalg 1 \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ J \amalg 1 & \xrightarrow[v]{} & 1 \amalg J \amalg 1 \end{array}$$

Take g to be the last gap in $I \amalg J \amalg 1$; the cocycle identity then gives $\lambda[j] - \lambda[i + j] - \lambda[j] + \lambda[i + 1] = 0$, or in other words $\lambda[i + j] = \lambda[i + 1]$. Here it is implicit that $i, j > 0$, so we conclude that $\lambda[k]$ is independent of k for $k > 1$.

Now instead consider the following square:

$$\begin{array}{ccc} J \amalg 1 & \xrightarrow{u} & I \amalg J \amalg 1 \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ 1 \amalg 1 & \xrightarrow[v]{} & I \amalg 1 \amalg 1 \end{array}$$

The cocycle identity then gives $\lambda[i+j] = \lambda[j] + \lambda[i+1]$, but $\lambda[k]$ is independent of k for $k > 1$, and it follows that $\lambda[k] = 0$ for $k > 1$. We saw earlier that $\lambda[1] = 0$, so $\lambda = 0$. By essentially the same argument, we also have $\rho = 0$. It follows that $\nu = 0$ and $\phi = 0$, as claimed. □

Lemma 0.8. *We have a short exact sequence* $\mathbb{R} \xrightarrow{\eta} \Theta^+ \xrightarrow{\delta} \Phi^+$*, which is split by* $\mathbb{R} \xleftarrow{\epsilon} \Theta^+ \xleftarrow{\pi'} \Phi^+$*.*

Proof. We have already seen that $\epsilon\eta = 1_{\mathbb{R}}$ and $\delta\eta = 0$ and $\epsilon\pi' = 0$ and $1_{\Theta^+} = \pi'\delta + \eta\epsilon$. All that is left is to check that $\delta\pi' = 1_{\Phi^+}$. To see this, compose the relation $1 = \pi'\delta + \eta\epsilon$ with π' and use $\epsilon\pi' = 0$ to get $\pi' = \pi'\delta\pi'$. As π' is injective, this gives $\delta\pi' = 1$ as required. □

Corollary 0.9. $\Phi = \delta(\Theta)$.

Proof. Given $\phi \in \Phi$, put $\theta = \pi'(\phi) \in \Theta^+$, so $\phi = \delta(\theta)$. Put $\sigma(A) = \sum_{g \in \mathrm{Gaps}(A)} \theta(A, g)$. It will suffice to show that $\theta \in \Theta$, or in other words that $\sigma = 0$. Note that $\sigma(1) = 0$ (because $\mathrm{Gaps}(1) = \emptyset$) and also $\sigma(2) = \theta(2, 2) = \epsilon(\theta) = \epsilon(\pi'(\phi)) = 0$.

From the definition of δ, we have

$$\phi(p, g) = \begin{cases} \theta(A, g) - \theta(A_b, g) & \text{if } p(g) = \{b\} \\ \theta(A, g) - \theta(B, p(g)) & \text{if } |p(g)| = 2. \end{cases}$$

Note also that $\phi \in \Phi \subset \Phi^+$, so $\sum_{g\in \mathrm{Gaps}(A)} \phi(p,g) = 0$. Putting this together, and using the bijection $\mathrm{Gaps}(A) = \mathrm{Gaps}(B) \amalg \coprod_b \mathrm{Gaps}(A_b)$, we get

$$\sigma(A) = \sigma(B) + \sum_b \sigma(A_b).$$

In particular, if all the sets A_b have size at most two, we see that $\sigma(A) = \sigma(B)$. Using this, we can easily prove by induction that $\sigma(A) = 0$ for all A. □

Corollary 0.9 tells us that any affine shift of the operad Γ is isomorphic to Γ. We can also prove a little more. Let us say that a *multiplicative shift* of Γ is an operad in $\mathcal{V}$ whose underlying spaces are the same as for Γ, with structure maps of the form

$$\gamma'_p(y; (x_b)_{b\in B})(A, g) = \mu(p, g)\gamma_p(y; (x_b)_{b\in B})(A, g)$$

for some system of numbers $\mu(p, g) > 0$. For any such rescaling, we see that the numbers $\phi(p, g) = \log(\mu(p, g))$ define a point in Φ^+, which therefore has the form $\delta(\theta)$ for some $\theta \in \Theta^+$. The numbers $\exp(\theta(A, g))$ then give rise to an isomorphism between the old operad structure and the new one.

An attempt at a projective version

Now put $\widetilde{\Gamma}(A) = \mathrm{Map}(\Gamma(A), \mathbb{R})$, so $\Gamma(A) = \widetilde{\Gamma}(A)/\mathbb{R}$. The spaces $\widetilde{\Gamma}(A)$ then form an operad in $\mathcal{V}$, for which the structure maps are isomorphisms. For any nonzero vector space V we put $P^+V = (V \setminus \{0\})/\mathbb{R}^+$. We will also put $P^+0 = 1$.

There are no natural maps $P^+V \times P^+W \to P^+(V \oplus W)$, and because of this, the spaces $P^+\widetilde{\Gamma}(A)$ do not naturally form an operad.

Next, consider a space V with a splitting $V = \bigoplus_i V_i$, in which some of the V_i may be zero. Let $P^+(V_1, \dots, V_r)$ denote the subspace of $P^+(V)$ consisting of classes $[x_1 : \dots : x_r]$ such that for all i with $V_i \neq 0$ we have $x_i \neq 0$. We then have projections

$$\pi_i \colon P^+(V_1, \dots, V_r) \to P^+(V_i)$$

for $r = 1, \dots, r$. Given subspaces $K_i \subseteq P^+(V_i)$, a *projective product* of these subspaces means a subspace $K \subseteq P^+(V_1, \dots, V_r)$ such that maps $\pi_i \colon K \to P^+(V_i)$ combine to give a bijection $K \to \prod_i K_i$.

Example 0.10. Suppose that each K_i is a single point, represented by a ray $R_i \subset V_i \setminus \{0\}$. If we choose $v_i \in R_i$ for all i and then put $v = \sum_i v_i$ and $K = \mathbb{R}^+ v \in S(V)$, then K is a projective product of the K_i. Every projective product arises in this way for some choice of the vectors $v_i \in R_i$. We can thus choose many different projective products.

Definition 0.11. By a *projective operad in* $\widetilde{\Gamma}$ we mean a system of subsets $K(A) \subseteq S\widetilde{\Gamma}(A)$ and subsets $K(p) \subseteq K(A)$ for all $p \colon A \to B$, such that the subspace

$$S(\gamma_p)^{-1}(K(p)) \subseteq S\left(\widetilde{\Gamma}(B) \oplus \bigoplus_b \widetilde{\Gamma}(A_b)\right)$$

is a projective product of the spaces $K(B)$ and $K(A_b)$ (for $b \in B$). If this holds, the the operad structure maps for $\widetilde{\Gamma}$ give rise to maps

$$K(B) \times \prod_b K(A_p) \xrightarrow{\pi^{-1}} K(p) \xrightarrow{\mathrm{inc}} K(A).$$

We further assume that these maps make K into an operad.

Definition 0.12. Now note that $S(\widetilde{\Gamma}(2))$ consists of two points, which we call μ^+ and μ^-. Let K be a projective operad in $\widetilde{\Gamma}$, such that $K(2) = \{\mu^+\}$. Given a binary Stasheff tree $\mathcal{T} \in \mathcal{K}(A)$, we can now define $\mu_{\mathcal{T}} \in K(A)$ in an evident way. Explicitly, let A_0 and A_1 be the two children of the root in $\mathcal{T}$, and define $p \colon A \to 2$ by $p(a) = i$ for $a \in A_i$, and let $\mathcal{T}_i$ be the tree on A_i induced by $\mathcal{T}$; then $\mu_{\mathcal{T}}$ is defined recursively as $\gamma_p(\mu^+; \mu_{\mathcal{T}_0}, \mu_{\mathcal{T}_1})$. We say that K is a *projective Stasheff operad* if

(a) $K(A)$ is the convex hull of the points $\mu_{\mathcal{T}}$, and this is a polytope of dimension $|A| - 2$.
(b) The points of $K(A)$ consist of functions on $\mathrm{Gaps}(A)$ that are everywhere strictly positive.

(c) $K(p)$ is the convex hull of the points $\mu_{\mathcal{T}}$ for all trees $\mathcal{T}$ such that $\mathcal{F}_b \subseteq \mathcal{T}$, where

$$\mathcal{F}_b = \{A_b \mid b \in B \text{ and } |A_b| > 1\}.$$

Moreover, this is a face of codimension $|\mathcal{F}_b|$ in $K(A)$.

We believe that there is no projective Stasheff operad.

Suppose that there is. Define trees $\mathcal{L}, \mathcal{R} \in \mathcal{K}(3)$ by $\mathcal{L} = \{\{0,1\},\{0,1,2\}\}$ and $\mathcal{R} = \{\{1,2\},\{0,1,2\}\}$.

$\mathcal{L}$ $\mathcal{R}$

Put $\lambda = \mu_{\mathcal{L}}$ and $\rho = \mu_{\mathcal{R}}$, so $\lambda, \rho \in S(\text{Map}(\text{Gaps}(3), \mathbb{R})) = S(\mathbb{R}^2)$. There are thus unique numbers $u, v > 0$ such that $\lambda = [1 : u]$ and $\rho = [1 : v]$. We next argue that all other points $\mu_{\mathcal{T}}$ (for $\mathcal{T} \in \mathcal{K}(A)$ with $|A| > 3$) are determined by λ and ρ.

Suppose we have an interval $L \in \mathcal{T}$ with $L \neq A$, and let $p \colon A \to A/L$ be the projection. Put $\mathcal{U} = \{B \in \mathcal{T} \mid B \subseteq L\} \in \mathcal{K}(L)$ and $\mathcal{V} = \{q(B) \mid B \in \mathcal{T},\ B \not\subseteq L\} \in \mathcal{K}(K/L)$, and for $a \in A \setminus L$ let 1_a denote the element of $K(\{a\})$. From the definitions we have

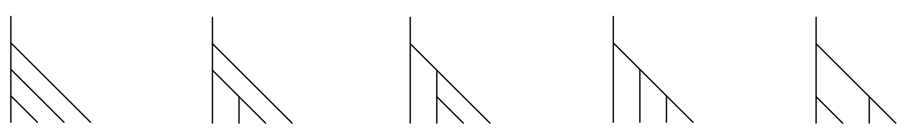
$$\mu_{\mathcal{T}} = \gamma_p(\mu_{\mathcal{V}}; \mu_{\mathcal{U}}, (1_a)_{a \in A \setminus L}).$$

We say that $\mathcal{T}$ splits at L into $\mathcal{U}$ and $\mathcal{V}$.

Next note that p gives a bijection $\text{Gaps}(A) = \text{Gaps}(L) \amalg \text{Gaps}(A/L)$ and thus a splitting $\widetilde{\Gamma}(A) = \widetilde{\Gamma}(L) \oplus \widetilde{\Gamma}(A/L)$, with respect to which we can represent $\mu_{\mathcal{T}}$ as $(\alpha_{\mathcal{T},L}, \beta_{\mathcal{T},L})$ say. By Definition 0.11, $\alpha_{\mathcal{T},L}$ must be a positive multiple of $\mu_{\mathcal{U}}$, and $\beta_{\mathcal{T},L}$ must be a positive multiple of $\mu_{\mathcal{V}}$. Together, these two facts constrain $\mu_{\mathcal{T}}$ to lie $S(W_{\mathcal{T},L})$, for a certain two-dimensional space $W_{\mathcal{T},L} \subset \widetilde{\Gamma}(A)$. More precisely, let $f \colon \text{Gaps}(A) \to \mathbb{R}^+$ represent $\mu_{\mathcal{T}}$. The above tells us $f(g)/f(g')$ whenever $g, g' \in \text{Gaps}(L)$ or $g, g' \in \text{Gaps}(A/L)$, so we can determine $\mu_{\mathcal{T}} = \mathbb{R}^+ f$ if we can find $f(g)/f(g')$ for a single pair g, g' with $g \in \text{Gaps}(L)$ and $g' \in \text{Gaps}(A/L)$. This can be done by choosing any other interval $L' \in \mathcal{T} \setminus \{A, L\}$ and considering $W_{\mathcal{T},L'}$. (Such an L' exists for $|A| > 3$.)

The above shows that given λ and ρ, there is at most one consistent choice for $\mu_{\mathcal{T}}$ for all $\mathcal{T}$. In the case $|A| = 4$ there are precisly two intervals $L \in \mathcal{T}$ with $L \neq A$, and using this we see that there is in fact a unique consistent choice. The formulae are as follows:

T_0 T_1 T_2 T_3 T_4



$[1 : u : u^2]$ $[1 : v : u]$ $[u : v : uv]$ $[1 : v : v^2]$ $[1 : u : uv]$

For example, consider the tree $\mathcal{T}_2 = \{L, L', A\}$, where $L = \{1,2\}$ and $L' = \{1,2,3\}$. The corresponding point $z = \mu_{\mathcal{T}_2}$ must have the form $[w : x : y]$ for some $w, x, y > 0$. If we split $\mathcal{T}$ at L, we find that $\text{Gaps}(L) = \{\{1,2\}\}$ and $\text{Gaps}(A/L) = \{\{0,1\},\{2,3\}\}$. The relevant tree $\mathcal{U}$ is the unique binary tree on L, whereas $\mathcal{V}$ is isomorphic to the tree $\mathcal{R}$ on $\{0,1,2\}$ so $\mu_{\mathcal{V}} = [1 : v]$. It follows that $[w : y] = [1 : v]$, so $y = vw$. Now instead split at L', so $\text{Gaps}(L') = \{\{1,2\},\{2,3\}\}$ and $\mathcal{U}$ is isomorphic to $\mathcal{L}$, so $\mu_{\mathcal{U}} = [1 : u]$. We find that $[x : y] = [1 : u]$, so $y = ux$. It follows that $[x : y : z] = [v : u : uv]$ as claimed. The other vectors can be calculated in the same way.

Part of the assumptions for a Stasheff operad is that the points $z_i = \mu_{\mathcal{T}_i}$ form a genuine pentagon, with sides joining z_i to z_{i+1} (where i is considered modulo 5). In particular, this means that z_{i-1}, z_i and z_{i+1} should be linearly independent. If $u = 1$ then this fails for $i = 1$, and if $v = 1$ it fails for $i = 2$. We may thus assume that $u, v \neq 1$.

For $|A| > 4$, it could in principle happen that the constraints on $\mu_{\mathcal{T}}$ were inconsistent, so that $\mu_{\mathcal{T}}$ would not exist. We have checked that this problem does not occur for $|A| = 5$, but we do not have a general argument. However, a different problem arises. Given an interval $L \subset A$ with $|L| > 1$, consider the set V_L

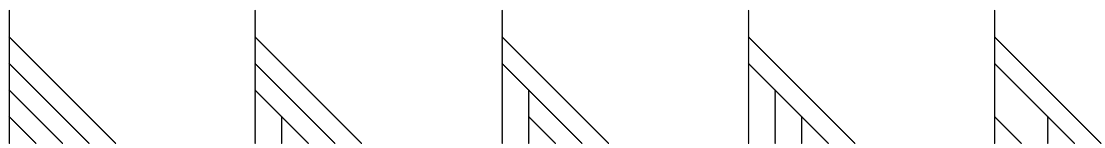
consisting of the points $\mu_{\mathcal{T}}$ for all binary Stasheff trees on A containing L. The convex hull of V_L is supposed to be the facet $K(p)$ of $K(A)$, where $p\colon A \to A/L$ is the projection. In particular, the span of V_L should be a hyperplane in $\widetilde{\Gamma}A$.

This works automatically when $|A| \leq 4$. The case $|A| = 1$ is covered by special cases of the definitions and is trivial. In the case $|A| = 2$ there are no L's and the condition is vacuously satisfied. In the case $|A| = 3$, each V_L is a singleton and thus spans a one-dimensional subspace of the two-dimensional space $\widetilde{\Gamma}(A)$, as required. In the case $|A| = 4$, one checks that each set V_L has size two, and thus spans at most a two-dimensional subspace of the three-dimensional space $\widetilde{\Gamma}(A)$.

When $|A| = 5$, things become more complex. The sets V_L have size 4 or 5 (depending on L) so there would have to be nontrivial linear relations for V_L to be contained in a hyperplane. We will show that no such relations can hold. Unfortunately, we have only a computational proof of this fact.

Consider the set $A = \{0, 1, 2, 3, 4\}$ and the interval $L = \{0, 1, 2, 3\}$. For $i = 0, \dots, 4$ put $\mathcal{T}_i' = \mathcal{T}_i \cup \{A\}$, where $\mathcal{T}_i$ is the tree on L described earlier. Put $v_i = \mu_{\mathcal{T}_i}$ and $v_i' = \mu_{\mathcal{T}_i'}$. If $v_i' = [w : x : y : z]$, we see by splitting at L that $v_i = [w : x : z]$. If $\{0, 1, 2\} \in \mathcal{T}_i$ then we can split there to find that $[y : z] = [1 : u]$. If instead $\{1, 2, 3\} \in \mathcal{T}_i$ then we can split there to see that $[x : z] = [1 : u]$. This just leaves the case of T_4', which can be split at $\{0, 1\}$ to give $[x, y, z] = [1 : v : u]$. Putting this together, we get the following:

T_0' $\quad$ T_1' $\quad$ T_2' $\quad$ T_3' $\quad$ T_4'



$[1 : u : u^2 : u^3]$ $\quad$ $[1 : v : u : u^2]$ $\quad$ $[u : v : uv : u^2]$ $\quad$ $[1 : v : v^2 : u]$ $\quad$ $[1 : u : uv : u^2]$

Now let M be the matrix whose rows are the vectors v_i':

$$M = \begin{bmatrix} 1 & u & u^2 & u^3 \\ 1 & v & u & u^2 \\ u & v & uv & u^2 \\ 1 & v & v^2 & u \\ 1 & u & uv & u^2 \end{bmatrix}.$$

This is required to have rank three, so all 4×4 minors must vanish. If we omit the last row and take the determinant, we get $u^2 v(v-1)(u-1)(uv-1)$. We saw earlier that $u \neq 1 \neq v$, and also $u, v > 0$, so we must have $v = 1/u$. Now instead omit the first row and take the determinant to get $(1-u)^3/u$, but we have seen that this is nonzero. Thus, there is no projective Stasheff operad.

References