THE MODEL STRUCTURE FOR CHAIN COMPLEXES

N. P. STRICKLAND

Let Ch be the category of (possibly unbounded) chain complexes of abelian groups. Consider a map $f\colon A_* \to B_*$ in Ch. We say that:

- f is a cofibration iff it is injective, and $\mathrm{cok}(f)_n$ is a free abelian group for all n.
- f is a fibration iff it is surjective.
- f is a weak equivalence iff it is a quasiisomorphism, which means that $f_*\colon H_*A \to H_*B$ is an isomorphism.
- f is an acyclic cofibration iff it is a cofibration and a weak equivalence, or equivalently a monomorphism whose cokernel is acyclic and free in each degree.
- f is an acyclic fibration iff it is a fibration and a weak equivalence, or equivalently an epimorphism with acyclic kernel.

Theorem 1. *This defines a cofibrantly generated monoidal model structure on* Ch*.*

The rest of this note will constitute our proof, which is a little different from the other proofs in the literature. It has the advantage of being very concrete, explicit and functorial. On the other hand, it is not so conceptual, and it relies heavily on the fact that subgroups of free abelian groups are free, so it does not generalise to many rings other than $\mathbb{Z}$.

Definition 2. For any abelian group A, we let $I(A)$ be the kernel of the augmentation $\epsilon\colon \mathbb{Z}[A] \to \mathbb{Z}$. Let $\theta\colon \mathbb{Z}[A] \to A$ be the usual map given by $\theta(\sum_i n_i[a_i]) = \sum_i n_i a_i$, and put $I^2(A) = \ker(\theta\colon I(A) \to A)$. Note that I and I^2 give functors Ab $\to$ Ab that are nonadditive, but they do preserve zero morphisms, so they induce functors Ch $\to$ Ch. Note also that $I(A)$ is freely generated by elements $[a] - [0]$ for $a \in A \setminus 0$. As subgroups of free abelian groups are always free we see that $I^2(A)$ is also free, although in this case there is no obvious choice of basis.

Remark 3. It is well known and not hard to check that $I^2(A)$ is the square of the ideal $I(A)$ in the ring $\mathbb{Z}[A]$, but we will not need this.

Construction 4. Let $f\colon A_* \to B_*$ be a chain map. Define a graded group $W_* = W_*(f)$ by

$$W_* = A_* \oplus I(B_*) \oplus \Sigma^{-1} I(B_*).$$

We write a, β and β' for typical elements of the groups A_n, $I(B_n)$ and $I(B_{n+1})$. We define $d^W : W_n \to W_{n-1}$ by

$$d^W(a + \beta + \Sigma^{-1}\beta') = da + d\beta - \Sigma^{-1}(\beta + d\beta').$$

We also let $j\colon A_* \to W_*$ be the inclusion, and define $p\colon W_* \to B_*$ by $p(a + \beta + \Sigma^{-1}\beta') = f(a) + \theta(\beta)$.

Proposition 5. *W_* is a chain complex, and j and p are chain maps with $pj = f$. Moreover, j is an acyclic cofibration and p is a fibration.*

Proof. It is a straightforward computation to check that $d^W d^W = 0$, so W_* is a chain complex. It is visible that $d^W j = jd$, and it is again straightforward to check that $pd^W = dp$, so j and p are chain maps, and clearly $pj = f$. The map j is injective, and the cokernel is just the cone on $\Sigma^{-1} I(B_*)$, which is contractible and free in each degree; this means that j is an acyclic cofibration. Also, as θ is surjective we see that p is an epimorphism and thus a fibration. □

Construction 6. Let $f\colon A_* \to B_*$ be a chain map. Define a graded group $X_* = X_*(f)$ by

$$X_* = A_* \oplus \Sigma I(A_*) \oplus \Sigma^2 I^2(A_*) \oplus I(B_*) \oplus \Sigma I(B_*).$$

We write $a, \alpha', \alpha'', \beta$ and β' for typical elements of the groups A_n, $I(A_{n-1})$, $I^2(A_{n-2})$, $I(B_n)$ and $I^2(B_{n-1})$. We define $d^X : X_n \to X_{n-1}$ by

$$\begin{aligned}
d^X(a) &= da && \in A_{n-1} \\
d^X(\Sigma\alpha') &= \theta(\alpha') - \Sigma d\alpha' - f_*(\alpha') && \in A_{n-1} \oplus \Sigma I(A_{n-2}) \oplus I(B_{n-1}) \\
d^X(\Sigma^2\alpha'') &= \Sigma\alpha'' + \Sigma^2 d\alpha'' + \Sigma f_*\alpha'' && \in \Sigma I(A_{n-2}) \oplus \Sigma^2(I^2(A_{n-3})) \oplus \Sigma I^2(B_{n-2}) \\
d^X(\beta) &= d\beta && \in I(B_{n-1}) \\
d^X(\Sigma\beta') &= \beta' - \Sigma d\beta' && \in I(B_{n-1}) \oplus I^2(B_{n-2}).
\end{aligned}$$

We also define maps $A_* \xrightarrow{i} X_* \xrightarrow{p} B_*$ by $i(a) = a$ and

$$p(a + \Sigma\alpha' + \Sigma^2\alpha'' + \beta + \Sigma\beta') = f(a) + \theta(b).$$

It is clear that i is injective, the cokernel of i is free in each degree, p is surjective, and $pi = f$.

Proposition 7. *(X_*, d^X) is a chain complex, and i and p are chain maps.*

Proof. This is a straightforward check of definitions. We have

$$\begin{aligned}
d^X d^X(a) &= d^2(a) = 0 \\
d^X d^X(\Sigma\alpha') &= d^X(\theta(\alpha') - \Sigma d\alpha' - f_*(\alpha')) \\
&= d\theta(\alpha') - (\theta(d\alpha') - \Sigma d^2\alpha' - f_*(d\alpha')) - df_*(\alpha') = 0 \\
d^X d^X(\Sigma^2\alpha'') &= d^X(\Sigma\alpha'' + \Sigma^2 d\alpha'' + \Sigma f_*(\alpha'')) \\
&= (\theta(\alpha'') - \Sigma d\alpha'' - f_*(\alpha'')) + (\Sigma d\alpha'' + \Sigma^2 d^2\alpha'' + \Sigma f_*(d\alpha'')) + (f_*(\alpha'') - \Sigma df_*(\alpha'')) \\
&= \theta(\alpha'') = 0 \\
d^X d^X(\beta) &= d^2\beta = 0 \\
d^X d^X(\Sigma\beta') &= d^X(\beta' - \Sigma d\beta') \\
&= d\beta' - (d\beta' - \Sigma d^2\beta') = 0.
\end{aligned}$$

This shows that X_* is a chain complex, and it is immediate that i is a chain map. We also have

$$\begin{aligned}
p(d^X(a)) &= f(da) = df(a) \\
p(d^X(\Sigma\alpha')) &= p(\theta(\alpha') - \Sigma d\alpha' - f_*(\alpha')) = f(\theta(\alpha')) - \theta(f_*(\alpha')) = 0 \\
p(d^X(\Sigma^2\alpha'')) &= p(\Sigma\alpha'' + \Sigma^2 d\alpha'' + \Sigma f_*(\alpha'')) = 0 \\
p(d^X(\beta)) &= p(d(\beta)) = \theta(d(\beta)) = d\theta(\beta) \\
p(d^X(\Sigma\beta')) &= p(\beta' - \Sigma d\beta') = \theta(\beta') = 0.
\end{aligned}$$

This is easily seen to agree with dp, so p is also a chain map. □

Proposition 8. *The map $p\colon X_* \to B_*$ is a quasiisomorphism (and thus an acyclic fibration).*

Proof. Put

$$\begin{aligned}
Z_* &= I^2(B_*) \oplus \Sigma I^2(B_*) \\
Y_* &= \Sigma I^2(A_*) \oplus \Sigma^2 I^2(A_*) \oplus Z_* \\
K_* &= \ker(p\colon X_* \to B_*).
\end{aligned}$$

Note that $Z_* \leq Y_* \leq K_*$, and these are inclusions of subcomplexes. As p is surjective, it will suffice to prove that K_* is acyclic. The complexes Z_* and Y_*/Z_* are easily seen to be contractible, so it will suffice to show that the complex $\overline{K}_* = K_*/Y_*$ is acyclic. Now put

$$\begin{aligned}
\overline{X}_n &= A_n \oplus \Sigma A_{n-1} \oplus B_n \\
d^{\overline{X}}(a + \Sigma a' + b) &= (da + a') - \Sigma da' + (db - f(a')) \\
\overline{p}(a + \Sigma a' + b) &= f(a) + b.
\end{aligned}$$

Using the map

$$(a+\Sigma\alpha'+\Sigma^2\alpha''+\beta+\Sigma\beta')\mapsto(a+\Sigma\theta(\alpha')+\theta(\beta))$$

we can identify $\overline{X}_*$ with X_*/Y_*, and thus $\overline{K}_*$ with the kernel of $\overline{p}$. Let C_* be the cone on A_*, so $C_*=A_*\oplus\Sigma A_*$ with $d^C(a+\Sigma a')=da+a'-\Sigma da$. Define $j\colon C_*\to\overline{X}_*$ by $j(a+\Sigma a')=a+\Sigma a'-f(a)$. This is easily seen to be a chain map and a kernel for $\overline{p}$, so $\overline{K}_*$ is isomorphic to C_* and is contractible. □

Definition 9. In the case $A_*=0$ we write $\Gamma(B)_*$ for X_*. Thus $\Gamma(B)_n=I(B_n)\oplus I^2(B_{n-1})$ with $d(\beta+\Sigma\beta')=d\beta+\beta'-\Sigma d\beta'$, and $\Gamma(B)_*$ is free in each degree, and we have a surjective quasiisomorphism $p\colon\Gamma(B)_*\to B_*$ given by $p(\beta+\Sigma\beta')=\theta(\beta)$.

Remark 10. Another way to think about X_* is as follows. Write j for the inclusion $I^2\to I$. We have maps of chain complexes

$$I^2(A_*)\xrightarrow{\left[\begin{smallmatrix}j\\f_*\end{smallmatrix}\right]}I(A_*)\oplus I^2(B_*)\xrightarrow{\left[\begin{smallmatrix}\theta&0\\-f_*&j\end{smallmatrix}\right]}A_*\oplus I(B_*).$$

The composite of these maps is zero, so we can regard the above diagram as a double complex. The totalisation of this double complex is X_*.

We now start to discuss lifting properties. It will be convenient to introduce some test objects:

Definition 11. Let M be an abelian group. We write Σ^nM for the complex consisting of a copy of M in degree n. We also write C^nM for the complex consisting of two copies of M in degrees n and $n+1$, with the differential between them being the identity.

We have

$$\begin{aligned}\mathrm{Ch}(A_*,C^nM)&\simeq\mathrm{Hom}(A_n,M)\\ \mathrm{Ch}(A_*,\Sigma.M)&\simeq\mathrm{Hom}(A_n/dA_{n+1},M).\end{aligned}$$

There is a short exact sequence $\Sigma^nM\to C^nM\to\Sigma^{n+1}M$; applying $\mathrm{Ch}(A_*,-)$ to this gives the left exact sequence

$$\mathrm{Hom}(A_n/dA_{n+1},M)\xrightarrow{\pi^*}\mathrm{Hom}(A_n,M)\xrightarrow{d^*}\mathrm{Hom}(A_{n+1}/dA_{n+2},M).$$

Lemma 12. *Let A_* be a chain complex that is free in each degree. Then there exists a splitting $A_*=Y_*\oplus Z_*$ of graded groups and injective maps $d'\colon Y_n\to Z_{n-1}$ such that the differential is given by $d(y+z)=d'(y)$. Moreover, A_* is acyclic iff d' is an isomorphism iff A_* is contractible.*

Proof. Put $Z_n=\ker(d\colon A_n\to A_{n-1})$ and $B_n=\mathrm{image}(d\colon A_{n+1}\to A_n)$, so $B_n\leq Z_n\leq A_n$ and therefore Z_n and B_n are free. We can therefore split the epimorphism $d\colon A_n\to B_{n-1}$ and thus choose a subgroup $Y_n\leq A_n$ such that $d\colon Y_n\to B_{n-1}$ is an isomorphism. If $a\in A_n$ we see that there is a unique $y\in Y_n$ with $dy=da$, which means that the element $z=a-y$ lies in Z_n. Using this we see that $A_n=Y_n\oplus Z_n$, and it is clear that the differential has the stated form. Now H_*A is the cokernel of d' so A_* is acyclic iff d' is an isomorphism, in which case A_* is isomorphic to the cone on Z_* and is contractible. □

Proposition 13. *The functor $\mathrm{Ch}(A_*,-)$ converts acyclic fibrations to epimorphisms iff A_* is free in each degree.*

Proof. Suppose that $\mathrm{Ch}(A_*,-)$ converts acyclic fibrations to epimorphisms. For any surjection $f\colon M\to N$ of abelian groups we have an acyclic fibration $C^n(f)\colon C^nM\to C^nN$; using these as test objects, we see that each A_n is projective and thus free.

Conversely, suppose that A_n is free for all n, so we can split A_* as $Y_*\oplus Z_*$ as in Lemma 12. We first claim that if K_* is an acyclic complex and $k\colon A_*\to K_*$ is a chain map then k is nullhomotopic. To see this, put $ZK_n=\ker(K_n\xrightarrow{d}K_{n-1})$; as K_* is acyclic, the map $d\colon K_{n+1}\to ZK_n$ is surjective. Now k gives a map $Z_n\to ZK_n$ and Z_n is free so we can choose a lift $t\colon Z_n\to K_{n+1}$ with $dt=k$. We now have a homomorphism $Y_n\to ZK_n$ given by $y\mapsto k(y)-t(d'y)$, so we can choose a lift $s\colon Y_n\to K_{n+1}$ with $ds(y)=k(y)-t(d'y)$. We then put $r(y+z)=s(y)+t(z)$ and observe that $dr+rd=k$ as required.

Now consider an acyclic fibration $q\colon L_*\to M_*$, so q is surjective and the kernel K_* is acyclic. Let $j\colon K_*\to L_*$ be the inclusion. Suppose we have a chain map $g\colon A_*\to M_*$. As A_* is degreewise free and q is surjective we can choose a map $h'\colon A_*\to L_*$ of graded groups with $qh'=g$. For $a\in A_n$ put

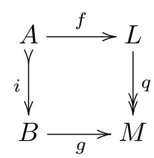
$k(a) = \Sigma(dh'(a) - h'(da)) \in (\Sigma K)_n$. We have $dk(a) = \Sigma dh'd(a) = k(da)$, so k is a chain map $A_* \to \Sigma K_*$. By the previous paragraph we can choose maps $r_n\colon A_n \to (\Sigma K)_{n+1}$ with $dr + rd = k$. We then have $r(a) = \Sigma r'(a)$ say, and $-dr' + r'd = \Sigma^{-1}k$. It follows that the map $h = h' + r'\colon A_* \to L_*$ is a chain map with $qh = g$, as required. □

Proposition 14. *The functor* $\mathrm{Ch}(A_*, -)$ *converts all fibrations to epimorphisms iff* A_* *is contractible and free in each degree.*

Proof. First suppose that $\mathrm{Ch}(A_*, -)$ sends fibrations to epimorphisms. By the previous proposition, A_* is free in each degree. By considering the fibration $C^nM \to \Sigma^{n+1}M$ we see that the map $d^*\colon \mathrm{Hom}(A_n, M) \to \mathrm{Hom}(A_{n+1}/dA_{n+2}, M)$ is surjective for all M, so the map $d\colon A_{n+1}/dA_{n+2} \to A_n$ must be a split monomorphism, which implies that A_* is acyclic. Using Lemma 12 we deduce that A_* is in fact contractible.

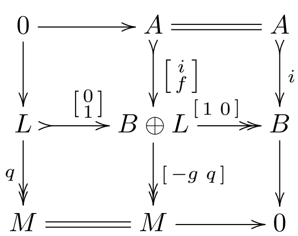
Conversely, suppose we start with the assumption that A_* is degreewise free and contractible. We then have $A_* = Y_* \oplus Z_*$, with the differential given by an isomorphism $Y_n \to Z_{n-1}$. This gives an isomorphism $\mathrm{Ch}(A_*, L_*) = \mathrm{Ab}_*(Y_*, L_*)$ and Y_* is projective in Ab_* so this functor preserves epimorphisms, as required. □

Proposition 15. *Let* $\mathcal{A}$ *be an abelian category, and let* $A \xrightarrow{i} B \xrightarrow{p} C$ *and* $K \xrightarrow{j} L \xrightarrow{q} M$ *be short exact sequences. For any diagram as shown,*



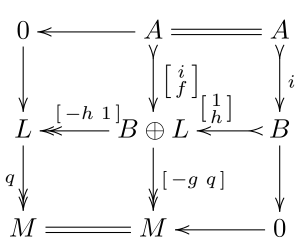
we let $H(f,g)$ *denote the set of maps* $h\colon B \to L$ *such that* $qh = g$ *and* $hi = f$. *Then there is a naturally defined extension* $K \to T(f,g) \to C$ *such that splittings of* $T(f,g)$ *biject with* $H(f,g)$. *In particular, if* $\mathrm{Ext}^1_{\mathcal{A}}(C,K) = 0$, *then* $H(f,g)$ *is always nonempty, so* i *has the left lifting property with respect to* q.

Proof. Consider the following diagram:



Each column can be regarded as a complex, so the whole diagram is a short exact sequence of complexes, leading to a long exact sequence of homology groups. This long exact sequence has only three nonzero terms, so it gives a short exact sequence of the form $K \xrightarrow{k} T \xrightarrow{r} C$, where $T = T(f,g)$ is the unique homology group of the middle column.

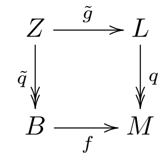
If $h \in H(f,g)$, then consider the following diagram:



The columns are the same complexes as before, and the horizontal maps give a splitting of our previous short exact sequence of complexes, and so induce a splitting of the homology group T.

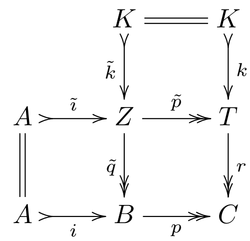
For the opposite correspondence, we need more information about T. Let Z be the corresponding group of cycles, which is the kernel of the map $(-g,q)\colon B \oplus L \to M$, or in other words, the pullback of q and g.

We name the maps in the pullback square as follows:



Thus $\tilde{g}$ and $\tilde{q}$ are just the projections $B \oplus L \to L$ and $B \oplus L \to B$, restricted to Z.

The differential in our complex is the map $\tilde{\imath} := (i, f) \colon A \to Z$, so T is by definition the cokernel of $\tilde{\imath}$; we write $\tilde{p} \colon Z \to T$ for the quotient map. We write $\tilde{k} := (0, j) \colon K \to Z$. One checks that the following diagram commutes:



We also see (by inspection of definitions and diagram chasing) that all rows and columns are exact, and that the bottom right square is a pullback.

Now suppose we are given a splitting of the sequence $K \xrightarrow{k} T \xrightarrow{r} C$, given by a map $n \colon C \to T$ with $rn = 1$. By the pullback property, there is a unique map $\tilde{n} \colon B \to Z$ with $\tilde{q}\tilde{n} = 1_B$ and $\tilde{p}\tilde{n} = np \colon B \to T$. We claim that the map $h := \tilde{g}\tilde{n} \colon B \to L$ lies in $H(f, g)$. Indeed, we first have $qh = q\tilde{g}\tilde{n} = g\tilde{q}\tilde{n} = g$. Next, one checks that $\tilde{p}(\tilde{\imath} - \tilde{n}i) = 0$ and $\tilde{q}(\tilde{\imath} - \tilde{n}i) = 0$ so the pullback property tells us that $\tilde{n}i = \tilde{\imath} \colon A \to Z$. This gives $hi = \tilde{g}\tilde{\imath} = f$ as required.

We leave it to the reader to check that the above constructions are mutually inverse. □

Proposition 16. *A map $i \colon A_* \to B_*$ has the left lifting property with respect to acyclic fibrations iff i is a cofibration.*

Proof. First suppose that i has the lifting property. By considering the acyclic fibration $C^n M \to 0$ we see that for every map $u \colon A_n \to M$ there exists $v \colon B_n \to M$ with $vi = u$. By taking u to be the identity map, we see that i is a degreewise split monomorphism, with cokernel C_* say. The map $0 \to C_*$ is a pushout of i, so it has left lifting for acyclic fibrations, which means precisely that $\mathrm{Ch}(C_*, -)$ sends acyclic fibrations to epimorphisms. Thus, Proposition 13 tells us that C_* is degreewise free, so i is a cofibration.

Conversely, suppose that i is a cofibration, with cokernel C_* say. Let $q \colon L_* \to M_*$ be an acyclic fibration, so q is surjective and the kernel K_* is acyclic. By Proposition 15, it will suffice to show that every short exact sequence $K_* \to T_* \to C_*$ is split, but this follows directly from Proposition 13. □

Proposition 17. *A map $i \colon A_* \to B_*$ has the left lifting property with respect to all fibrations iff i is an acyclic cofibration.*

Proof. First suppose that i has the lifting property. The previous result tells us that i is a cofibration, with cokernel C_* say. The map $0 \to C_*$ is a pushout of i and so again has the lifting property, which means (by Proposition 14) that C_* is contractible, so i is an acyclic cofibration.

Conversely, suppose that i is an acyclic cofibration, and again write C_* for the cokernel. We see from Proposition 14 that every short exact sequence $K_* \to T_* \to C_*$ is split, and it follows using Proposition 15 that i has left lifting for all fibrations. □

Corollary 18. Ch *is a model category.*

Proof. It is clear that Ch has finite limits and colimits, that the classes of fibrations, cofibrations and weak equivalences are closed under retracts, and that the weak equivalences satisfy the two-out-of-three condition. Functorial factorisations are provided by constructions 4 and 6. The lifting axioms are satisfied by Proposition 16 and 17. □