FOCK SPACES

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Everything in this note is well-known to suitable experts; it is just a digestion for my own enlightenment.

1. Polarised universes

Let U be a complex universe. Given subuniverses $L, L' \leq U$ we write $L \sim L'$ iff $L/(L \cap L')$ and $L'/(L \cap L')$ are both finite-dimensional. This is easily seen to be an equivalence relation. We say that L is *standard* if both L and $L^\perp$ are infinite-dimensional, and $U = L \oplus L^\perp$. If L is standard and $L' \sim L$ then L' is also standard. A *polarisation* of U is an equivalence class of standard subuniverses. Let G be a polarisation.

Note that if $L, L' \in G$ then $L + L'$ and $L \cap L'$ also lie in G.

Definition 1.1. Given $M, N \in G$ with $M \leq N$, we put

$$G(M, N) = \{L \in G \mid M \leq L \leq N\}.$$

This is naturally identified with the Grassmannian of subspaces of N/M. The set G can thus be regarded as a filtered colimit of projective varieties.

Definition 1.2. Given $L, L' \in G$ we define $\dim(L, L') = \dim(L'/N) - \dim(L/N)$, for any $N \in G$ with $N \leq L \cap L'$. This is easily seen to be independent of N, and to satisfy $\dim(L, L) = 0$ and

$$\dim(L, L') + \dim(L', L'') = \dim(L, L'').$$

Definition 1.3. Given a vector space V, we write $\lambda^k V$ for the k'th exterior power. We also write $\lambda^W V = \lambda^{\dim(W)} V$, for any finite-dimensional vector space W. Finally, we write $\det(V) = \lambda^V V$. We note that when $W \leq V$ there is an isomorphism $\det(V) = \det(W) \otimes \det(V/W)$, which is natural in the pair (V, W).

Definition 1.4. Given $L, L' \in G$ we define

$$\det(L, L') = \det(L/(L \cap L'))^* \otimes \det(L'/(L \cap L')) = \operatorname{Hom}(\det(L/(L \cap L')), \det(L'/(L \cap L')))$$

(which is a one-dimensional complex vector space).

Proposition 1.5. *The set G can be made into a category, with $\det(L, L')$ as the morphisms from L to L'. Moreover, the composition map*

$$\det(L', L'') \otimes \det(L, L') \to \det(L, L'')$$

is an isomorphism.

Proof. First, for any $N \leq L \cap L'$ we put

$$\det(L, L'; N) = \operatorname{Hom}(\det(L/N), \det(L'/N)).$$

If $M \leq N$ then we have canonical isomorphisms $\det(L/M) = \det(L/N) \otimes \det(M/N)$ and $\det(L'/M) = \det(L'/N) \otimes \det(M/N)$. As $\det(M/N)$ is invertible, these induce an isomorphism $\det(L, L'; N) \simeq \det(L, L'; M)$. These isomorphisms compose in the obvious way. Thus, we can replace $\det(L, L')$ by $\det(L, L'; N)$ for any convenient N. Now take $N \leq L \cap L' \cap L''$, and put $Q = L/N$, $Q' = L'/N$ and $Q'' = L''/N$. We have

$$\begin{aligned}
\det(L', L'') \otimes \det(L, L') &= \det(Q'') \otimes \det(Q')^* \otimes \det(Q') \otimes \det(Q)^* \\
&= \det(Q'') \otimes \det(Q)^* \\
&= \det(L, L'').
\end{aligned}$$

This identification is easily seen to be independent of N, and to be associative. □

Definition 1.6. For any $L \in G$, we define the Fock space $F_*(L) = F_*(U, L)$ as follows. For any N, M with $N \leq L \leq M$, we put

$$F_d(L; N, M) = \det(L/N)^* \otimes \lambda^{d+L/N}(M/N) = \mathrm{Hom}(\det(L/N), \lambda^{d+L/N}(M/N)).$$

Now suppose we have $N' \leq N \leq L \leq M \leq M'$. On the one hand, we have $\det(L/N') = \det(L/N) \otimes \det(N/N')$. On the other hand, the ring structure of $\lambda^*(M'/N')$ gives a map

$$\mu\colon \det(N/N') \otimes \lambda^{d+L/N}\left(\frac{M}{N'}\right) = \lambda^{N/N'}\left(\frac{N}{N'}\right) \otimes \lambda^{d+L/N}\left(\frac{M}{N'}\right) \to \lambda^{d+L/N'}\left(\frac{M'}{N'}\right).$$

Let I be the ideal in $\lambda^*(M/N')$ generated by $N/N' \leq \lambda^1(M/N')$. Then $\det(N/N')I = 0$ and $\lambda^*(M/N')/I = \lambda^*(M/N)$. Our map μ thus induces a map

$$\overline{\mu}\colon \det(N/N') \otimes \lambda^{d+L/N}\left(\frac{M}{N}\right) \to \lambda^{L/N'}\left(\frac{M'}{N'}\right),$$

and thus a map

$$\begin{aligned}
F_d(L; N, M) &= \mathrm{Hom}(\det(L/N), \lambda^{L/N}(M/N)) \\
&\simeq \mathrm{Hom}(\det(N/N') \otimes \det(L/N), \det(N/N') \otimes \lambda^{L/N}(M/N)) \\
&\simeq \mathrm{Hom}(\det(L/N'), \det(N/N') \otimes \lambda^{L/N}(M/N)) \\
&\xrightarrow{\overline{\mu}_*} \mathrm{Hom}(\det(L/N'), \lambda^{L/N'}(M'/N')) \\
&= F(L; N', M').
\end{aligned}$$

It is easy to see that these maps are injective, and that they compose together in the obvious way. We can thus define

$$F_*(L) = \varinjlim_{N,M} F_*(L; N, M).$$

Proposition 1.7. *There are natural isomorphisms*

$$F_*(L') = \det(L', L) \otimes \Sigma^{\dim(L',L)} F_*(L)$$

for all $L', L \in G$.

Proof. Put $m = \dim(L', L)$. It will suffice to give compatible isomorphisms $F_d(L'; N, M) \simeq \det(L', L) \otimes F_{d-e}(L; N, M)$ for all N, M with $N \leq L \cap L'$ and $M \geq L + L'$. Put $n = \dim(L/N)$, so $\dim(L'/N) = n - e$. We then have $\det(L', L) = \det(L'/N)^* \otimes \det(L/N)$, so

$$\begin{aligned}
F_d(L'; N, M) &= \det(L'/N)^* \otimes \lambda^{d+n-e}(M/N) \\
&= \det(L, L') \otimes \det(L/N)^* \otimes \lambda^{d+n-e}(M/N) \\
&= \det(L, L') \otimes F_{d-e}(L; N, M)
\end{aligned}$$

as required. ∎

Definition 1.8. Given $L \in G$, we put $G_0(L) = \{L' \in G \mid \dim(L, L') = 0\}$. Given N, M with $N \leq L \leq M$ we put

$$G_0(L; N, M) = G_0(L) \cap G(N, M) = \{L' \mid N \leq L' \leq M \text{ and } \dim(L, L') = 0\}.$$

We also let $D(L)$ denote the line bundle over $G_0(L)$ with fibre $\det(L', L)$ at L'

Proposition 1.9. *There is a natural isomorphism*

$$\Gamma(G_0(L; N, M); D(L)) = F_0(L; N, M)^*$$

(where $\Gamma(-,-)$ denotes the space of algebraic sections).

Proof. Put $d = \dim(L/N)$, so $\dim(L'/N) = d$ for $L' \in G_0(L)$. Let T be the bundle over $G(N, M)$ with fibre L'/N at L'. The restriction of $D(L)$ to $G_0(L; N, M)$ is $\det(L/N) \otimes \det(T)^*$, so

$$\Gamma(G_0(L; N, M); D(L)) = \det(L/N) \otimes \Gamma(G_0(L; N, M); \det(T)^*).$$

On the other hand, we have

$$F_0(L; N, M)^* = \det(L/N) \otimes \lambda^d(T)^*.$$

The claim now follows from Lemma 1.10 below. □

Lemma 1.10. *Let V be a finite-dimensional complex vector space, and let T be the tautological bundle over* $\mathrm{Grass}_k(V)$ *(the Grassmannian variety of subspaces of V). Then* $\Gamma(\mathrm{Grass}(V); \det(T)^*) = \lambda^k(V)^*$.

Proof. Suppose we have an element $\phi \in \lambda^k(V)^*$. For $W \in \mathrm{Grass}_k(V)$ we let $\sigma(\phi)_W$ denote the restriction of ϕ to $\det(W) = \lambda^k W \leq \lambda^k V$, so $\sigma(\phi)_W$ is an element of $\det(W)^*$, which is the fbre of the bundle $\det(T)^*$ at the point W. Thus, we can regard $\sigma(\phi)$ as a section of $\lambda^k(T)^*$, which is easily seen to be algebraic. Thus, we have a map

$$\sigma \colon \lambda^k(V)^* \to \Gamma(\mathrm{Grass}_k(V); \lambda^k(T)^*).$$

If $k = \dim(V)$ then $\mathrm{Grass}_k(V) = \{V\}$ and σ is obviously bijective. We therefore suppose that $k < \dim(V)$.

Now suppose we have $s \in \Gamma(\mathrm{Grass}_k(V); \lambda^k(T))^*$. Let X be the set of linearly independent lists $\underline{v} = (v_1, \dots, v_k)$ in V^k. Given $\underline{v} \in X$, we define

$$W = \mathrm{span}(\underline{v}) \in \mathrm{Grass}_k(V)$$
$$\tau(s)(\underline{v}) = s_W(v_1 \wedge \ldots \wedge v_k).$$

One checks that $V^k \setminus X$ has codimension $n - k + 1 \geq 2$ in V^k. As $\tau(s)$ is a rational function that is regular away from a closed subvariety of codimension at least two, it extends uniquely as a globally defined polynomial function. We also see from the definition that

$$\tau(s)(\lambda_1 v_1, \dots, \lambda_k v_k) = \left(\prod_i \lambda_i\right) \tau(s)(v_1, \dots, v_k)$$
$$\tau(s)(v_{\pi(1)}, \dots, v_{\pi(k)}) = \mathrm{sgn}(\pi)\tau(s)(v_1, \dots, v_k),$$

showing that $\tau(s)$ is alternating and multilinear. It can thus be regarded as an element of $(\lambda^k V)^*$. It is easy to see that the maps σ and τ are mutually inverse isomorphisms. □

2. Rings

Put $A = \mathbb{C}[z]$ and $K = \mathbb{C}[z^{\pm 1}]$. If $f = \sum_n a_n z^n \in K$, we put $\overline{f} = \sum_n \overline{a}_n z^{-n}$. We say that $f \in K$ is *real* if $f = \overline{f}$, and *positive* iff $f(z) \in [0, \infty)$ for all $z \in S^1$. Using the formula

$$a_k = \frac{1}{2\pi i} \int_{S^1} f(z) z^{-1-k} dz$$
$$= \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} f(e^{i\theta}) d\theta,$$

we see that positive functions are always real. It is also easy to see that $f\overline{f}$ is always positive, for any $f \in K$.

We also write

$$(f, g) = f\overline{g} \in K$$
$$\tau(f) = a_0 = (2\pi i)^{-1} \int f(z) z^{-1} dz$$
$$\langle f, g \rangle = \tau((f, g)) = \sum_k a_k \overline{b}_{-k}$$

(where $f = \sum a_k z^k$ and $g = \sum b_k z^k$).

3. Modules

Now let U be a free module of rank d over K, equipped with a sesquilinear form $(,)\colon U \otimes_{\mathbb{R}} U \to K$ satisfying $(fu, gv) = f\overline{g}(u,v)$. We assume that there exists a basis $\{e_i\}$ for U over K with $(e_i, e_j) = \delta_{ij}$. We then put $\langle u, v\rangle = \tau((u,v))$, which gives an inner product on U, making it a complex universe.

Lemma 3.1. *If $L \leq P$ is an A-submodule, then the following are equivalent.*

(a) *L is finitely generated over A, and P/L is z-torsion.*
(b) *L is free of rank d over A.* □

Definition 3.2. An *A-lattice* in P is an A-submodule satisfying the above conditions. It is easy to see that every lattice is a standard subuniverse, and that all lattices are equivalent under the relation introduced in Section 1.

We let G be the polarisation defined by any A-lattice. We then write G^A for the set of A-lattices. We also note that if $L, L' \in G^A$ then $L \cap L'$ and $L + L'$ are also in G^A.

We now consider a finite-dimensional universe V, and put $U = K \otimes V$, with $(f \otimes v, g \otimes w) = f\overline{g}\langle v, w\rangle$. We then put $L = A \otimes V$ and

$$FF_*(V) = F_*(U, L) = F_*(K \otimes V, A \otimes V).$$

This is clearly functorial for isomorphisms of V's, and so in particular it has an action of S^1 coming from the action on V by multiplication. We let $FF_{d,n}(V)$ be the part where $\lambda \in S^1$ acts as multiplication by λ^n. This allows us to define the graded character

$$\chi_V(s,t) = \sum_{d,n} s^d (-t)^n \dim(FF_{d,n}(L)).$$