

SPECTRA OF UNITS

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1. THE DEFINING ADJUNCTION

If R is a ring spectrum, then we let $GL_1(R)$ denote the preimage of $(\pi_0 R)^\times$ under the evident map $\Omega^\infty R \rightarrow \pi_0 R$. If R has an E_∞ structure then $GL_1(R)$ is a grouplike E_∞ space and thus an infinite loop space, associated to a connective spectrum that we call $gl_1(R)$, the *spectrum of units* for R . We also write $bgl_1(R)$ for $\Sigma gl_1(R)$.

Conversely, if X is an arbitrary spectrum then the additive structure makes $\Omega^\infty X$ into an E_∞ space and so makes $\Sigma_+^\infty \Omega^\infty X$ into an E_∞ ring spectrum. It is formally reasonable to expect an adjunction

$$\text{Spectra}(X, gl_1(R)) \simeq (E_\infty \text{Rings})(\Sigma_+^\infty \Omega^\infty X, R).$$

Although this idea familiar to many people, we do not know of any rigorous theorem along these lines. One would expect that any such theorem would involve a number of technicalities about model structures, comparison of operads and so on. In this note we will explain a conjecture about how this should work out. We will use the framework described in [1].

Let \mathcal{M} be the EKMM category of S -modules, and write \wedge for the natural smash product in \mathcal{M} . Let \mathcal{C} denote the category of commutative monoids in \mathcal{M} . Let S_c^k be the cofibrant k -sphere in \mathcal{M} .

Conjecture 1.1. There are functors $T: \mathcal{M} \rightarrow \mathcal{C}$ and $gl_1: \mathcal{C} \rightarrow \mathcal{M}$ such that

- (a) T is left adjoint to gl_1 , and the adjunction is enriched and passes to homotopy.
- (b) TS^0 is obtained from the free commutative ring PS^0 by inverting the tautological element $x \in \pi_0(PS^0)$.
- (c) There is a natural map $\alpha_X: TX \rightarrow \Sigma_+^\infty \mathcal{M}(S_c^0, X)$ in \mathcal{M} , which is a weak equivalence for all X . The composite

$$x^{-1}PS^0 = \Sigma^\infty(\coprod_n B\Sigma_n)^+ \rightarrow TS^0 \xrightarrow{\alpha_{S^0}} \Sigma_+^\infty QS^0$$

is the usual equivalence.

For the rest of this note, we will assume this conjecture.

Note that if $k < 0$ then $\mathcal{M}(S_c^0, S_c^{-k})$ is contractible, so TS_c^{-k} is weakly equivalent to S^0 in \mathcal{C} , so $\pi_k gl_1(R) = \bar{h}\mathcal{C}(TS_c^{-k}, R) = 0$. Thus, the spectrum $gl_1(R)$ is (-1) -connected for all R . Moreover, we have

$$\Omega_-^\infty gl_1(R) = \mathcal{M}(S_c^0, gl_1(R)) = \mathcal{C}(T(S_c^0), R) = \mathcal{C}(x^{-1}PS^0, R) \subset \mathcal{C}(PS^0, R) = \Omega_-^\infty R.$$

It is not hard to see that the relevant subspace of $\Omega_-^\infty R$ is just $GL_1(R)$ as defined earlier, and so $\pi_0 gl_1(R) = (\pi_0 R)^\times$.

Note that the functor $R \mapsto \bar{h}\mathcal{C}(TX, R) = [X, gl_1(R)]$ takes values in the category of abelian groups, so TX is naturally a cogroup object in $\bar{h}\mathcal{C}$, with structure maps

$$S^0 \xleftarrow{\epsilon} TX \xrightarrow{\psi} TX \wedge TX$$

say. These should be compatible with the collapse and diagonal maps of the space $\Omega^\infty X$ under α_X .

The topological compatibility of our adjunction means that $gl_1(F(K_+, R)) = F(K_+, gl_1(R))$ for any space K . It follows that

$$[K_+, gl_1(R)] = \pi_0 gl_1(F(K_+, R)) = (R^0 K)^\times.$$

2. THOM SPECTRA

Suppose we have a spectrum X and a map $\zeta : \Omega X = \Sigma^{-1}X \rightarrow \mathrm{gl}_1(S)$, or equivalently a map $X \rightarrow \mathrm{bgl}_1(S)$, or a map $\zeta^\# : T\Omega X \rightarrow S$ in \mathcal{C} . We define the corresponding *Thom spectrum* $T\zeta$ to be the homotopy pushout in \mathcal{C} of the maps

$$S^0 \xleftarrow{\varepsilon} T\Omega X \xrightarrow{\zeta^\#} S^0.$$

We find that $\mathcal{C}(T\zeta, R)$ is the space of paths from 0 to $\mathrm{gl}_1(\eta) \circ \zeta$ in $\mathcal{M}(\Omega X, \mathrm{gl}_1(R))$. (I learned this point of view from Mike Hopkins; I do not know whether there is any earlier history.)

In particular, for the case $\zeta = 0$ we have

$$\mathcal{C}(T(X \xrightarrow{0} \mathrm{bgl}_1(S)), R) = \Omega\mathcal{M}(\Omega X, \mathrm{gl}_1(R)) = \mathcal{M}(X, \mathrm{gl}_1(R)) = \mathcal{C}(TX, R),$$

so

$$T(X \xrightarrow{0} \mathrm{bgl}_1(R)) = TX \simeq \Sigma_+^\infty \Omega_-^\infty X.$$

Now suppose we have another map $\xi : Y \rightarrow \mathrm{bgl}_1(S)$. From the above description we see that

$$T\zeta \wedge T\xi = T(X \vee Y \xrightarrow{(\zeta, \xi)} \mathrm{bgl}_1(S)).$$

In particular, we can take $Y = X$ and $\zeta = \xi$, and then use a shearing isomorphism to get

$$T\zeta \wedge T\zeta \simeq T\zeta \wedge \Sigma_+^\infty \Omega_-^\infty X.$$

It follows that for every naive ring spectrum R with a map $T\zeta \rightarrow R$, we have a natural isomorphism $R_*T\zeta \simeq R_*\Sigma_+^\infty \Omega_-^\infty X$ of R_* -algebras.

Similarly, we can use the maps

$$X \xrightarrow{(1,1)} X \vee X \xrightarrow{(\zeta, 0)} \mathrm{bgl}_1(S)$$

to get a map $T\zeta \rightarrow T\zeta \wedge TX$, giving a coaction of the cogroup TX on $T\zeta$.

3. EQUIVARIANT VERSIONS

We can enrich our original conjecture as follows. Let \mathcal{L} be the category of infinite universes.

Conjecture 3.1. There are functors $T : \mathcal{L} \times \mathcal{M} \rightarrow \mathcal{C}$ and $\mathrm{gl}_1 : \mathcal{L} \times \mathcal{C} \rightarrow \mathcal{M}$ such that

- (a) For each universe \mathbb{U} , the functor $T(\mathbb{U}, -)$ is left adjoint to $\mathrm{gl}_1(\mathbb{U}, -)$, and the adjunction is enriched, natural in \mathbb{U} , and passes to homotopy.
- (b) There is a natural map $T(\mathbb{U}, X) \rightarrow \Sigma_+^\infty \mathcal{M}(S(\mathbb{U}), X)$, which is a weak equivalence for all \mathbb{U} and X .
- (c) There is a natural equivalence $T(\mathbb{R}, X) \simeq TX$, where TX is as considered previously.

Now let G be a finite group, and let $G\mathcal{M}$ be the category of objects in \mathcal{M} with an action of G , and so on. Let $\mathbb{U} = \mathbb{C}[G]^\infty$ be the standard complete G -universe. We give $G\mathcal{M}$ the model structure with $S(\mathbb{U}_G)$ as the basic cell, so the associated homotopy category is the usual category of genuine G -spectra. We will write $\lambda^G : G\mathcal{M} \rightarrow \mathcal{M}$ for the Lewis-May fixed point functor, which in our framework is given by $\lambda^G X = F_G(S(\mathbb{U}), X)$. We will also let ϕ^G denote the geometric fixed point functor, given on cofibrant objects by $\phi^G(X) = \lambda^G(\widetilde{EG} \wedge X)$.

For any $X \in G\mathcal{M}$ we combine the G -actions on X and \mathbb{U} to get an action on $T(\mathbb{U}, X)$. Similarly, for $R \in G\mathcal{C}$ we get an action on $\mathrm{gl}_1(\mathbb{U}, R)$. The homeomorphism

$$\mathcal{M}(X, \mathrm{gl}_1(\mathbb{U}, R)) \simeq \mathcal{C}(T(\mathbb{U}, X), R)$$

is then G -equivariant, so we get an adjunction

$$G\mathcal{M}(X, \mathrm{gl}_1(\mathbb{U}, R)) \simeq G\mathcal{C}(T(\mathbb{U}, X), R).$$

Using this, we can make all our constructions equivariant.

However, it is not easy to understand the equivariant homotopy type of the space

$$GL_1(R) = \mathcal{M}(S(\mathbb{U}), \mathrm{gl}_1(\mathbb{U}, R)) = \mathcal{C}(T(\mathbb{U}, S(\mathbb{U})), R).$$

In the case $G = 1$ we know that $T(\mathbb{U}, S(\mathbb{U})) = T(S_c^0)$ (which is equivalent to $\Sigma_+^\infty QS^0$) is obtained from the free object PS_c^0 by inverting a single homotopy element, which implies that $GL_1(R)$ is just the union of the

invertible components in $\Omega^\infty R = \mathcal{M}(S_c^0, R) = \mathcal{C}(PS_c^0, R)$. The analogue for general G is more complicated. To explain it, we introduce several categories associated to G .

- bG is the category with one object, whose endomorphism monoid is G . This has $BbG = BG$.
- For any set T , we let eT be the category with object set T , and a single morphism between any pair of objects. Any bijection $T \rightarrow U$ gives an isomorphism $eT \rightarrow eU$ of categories. In particular, G acts on itself by left multiplication, and so acts on eG by automorphisms of categories. We find that $BeG = EG$ as G -spaces. We also find that $bG = (eG)/G$ as categories, and that $BbG = (BeG)/G$ as spaces.
- We let \mathcal{F} be the category of finite sets and bijections, so $B\mathcal{F} \simeq \coprod_n B\Sigma_n$. We thus have $\Sigma_+^\infty B\mathcal{F} = PS^0$. As \mathcal{F} is a symmetric monoidal category, it has a K -theory spectrum $K(\mathcal{F})$, which in this case is just S^0 .
- We write sG for the category whose objects are pairs (X, u) where X is a finite set, and $u: X \rightarrow G$. The morphisms from (X, u) to (Y, v) are the bijections $X \rightarrow Y$ (independent of u and v). The group G acts on sG by isomorphisms, by the rule $g.(X, u) = (X, gu)$, where $(gu)(x) = g.u(x)$. This is equivariantly equivalent to the free symmetric monoidal category generated by eG . We have $BsG = D(EG)$ (where D is the total extended power functor), and the group completion of this is $Q(EG_+)$. It is natural to think of $K(sG)$ as being closely related to the cofibrant G -sphere $S(\mathbb{U}_G)$, although we do not know of a way to make this precise. (It would not be strong enough to say that $K(sG)$ is weakly equivalent to $S(\mathbb{U}_G)$ in the usual model structure on $G\mathcal{M}$, essentially because the non-cofibrant unit sphere S^0 (with trivial action) is already weakly equivalent to $S(\mathbb{U}_G)$.) As part of the evidence, recall that

$$\Omega^\infty S(\mathbb{U}_G) = \mathcal{M}(S(\mathbb{R}^\infty), S(\mathbb{U}_G)) = Q\mathcal{L}(\mathbb{U}_G, \mathbb{R}^\infty)_+,$$

and $\mathcal{L}(\mathbb{U}_G, \mathbb{R}^\infty)$ is a model for EG , so $\Omega^\infty S(\mathbb{U}_G) = Q(EG_+) = \Omega^\infty K(sG)$ as G -spaces.

- We write $qG = [eG, \mathcal{F}] = \text{SymMon}(sG, \mathcal{F})$. If we had a sufficiently good K -theory functor, we would expect a natural map $B\text{SymMon}(\mathcal{A}, \mathcal{B}) \rightarrow \mathcal{M}(K(\mathcal{A}), K(\mathcal{B}))$. In particular, we would get an equivariant map

$$BqG \rightarrow \mathcal{M}(K(sG), K(\mathcal{F})) \simeq \mathcal{M}(S(\mathbb{U}_G), S(\mathbb{R}^\infty)) = Q_G S^0.$$

I think it should be possible to construct this by more pedestrian methods (eg configuration spaces).

- We write $G\mathcal{F} = [bG, \mathcal{F}]$ for the category of finite G -sets and isomorphisms, and GT for the subcategory of transitive G -sets. One can check that $G\mathcal{F}$ is the free symmetric monoidal category generated by GT , so $BG\mathcal{F} = D(BGT)$ and $K(G\mathcal{F}) = \Sigma_+^\infty BGT$. The tom Dieck splitting says that $\lambda^G S^0$ can also be identified with $\Sigma_+^\infty BGT$, so $K(G\mathcal{F}) = \lambda^G S^0$. We also have $(qG)^G = [(eG)/G, \mathcal{F}] = [bG, \mathcal{F}] = G\mathcal{F}$, so $(BqG)^G = BG\mathcal{F}$. More generally, for any $H \leq G$ one checks that eG is H -equivariantly equivalent to eH , and so $(qG)^H = H\mathcal{F}$ and $(BqG)^H = BH\mathcal{F}$.
- We also write fG for the subcategory of free G -sets in $G\mathcal{F}$. We have a functor from bG to fG , sending the unique object in bG to $G \in \text{obj}(fG)$, and sending the morphism $g \in bG$ to the morphism $(x \mapsto xg^{-1}) \in fG(G, G)$. This extends to an equivalence from the free symmetric monoidal category on bG to fG . It follows that $BfG = D(BG)$ and $K(fG) = \Sigma_+^\infty BG$.

Now let $A_+G = \pi_0(G\mathcal{F}) = \pi_0 BG\mathcal{F} = \pi_0^G BqG$ be the Burnside semiring for G . The group completion of this is the Burnside ring AG , which can also be described as $\pi_0^G(Q_G S^0)$. Let U be the set of subsets of G , with its evident action of G , and note that this contains a copy of G/H for each subgroup H . For any $x \in AG$ we have $x + n[U] \in A_+G$ for $n \gg 0$, so AG is the colimit of the sequence

$$A_+G \xrightarrow{+[U]} A_+G \xrightarrow{+[U]} A_+G \xrightarrow{+[U]} \dots$$

This comes from a sequence

$$BqG \rightarrow BqG \rightarrow BqG \rightarrow \dots$$

of G -spaces, and I think one can show that the homotopy colimit is $Q_G S^0$. Now let u be the element of $\mathbb{Z}[A_+G]$ corresponding to $[U]$, or its image under the natural stabilisation map

$$\mathbb{Z}[A_+G] = \mathbb{Z}\{\pi_0^G BqG\} \rightarrow \pi_0^G \Sigma_+^\infty BqG.$$

We deduce that $\Sigma_+^\infty Q_G S^0 = (\Sigma_+^\infty BqG)[u^{-1}]$ as ring spectra. It is not hard to deduce that this can be regarded as an equivalence of strict modules over T , where T is a strictly commutative ring equivalent to $\Sigma_+^\infty BqG$. We can then form the Bousfield localisation of T in the category of commutative T -algebras, with respect to the module $T[u^{-1}]$; this gives us a commutative T -algebra that is equivalent to $T[u^{-1}]$ as an T -module, and is equivalent to $\Sigma_+^\infty Q_G S^0$ as an T -algebra. It follows that $GC(\Sigma_+^\infty Q_G S^0, R)$ is the union of certain components in the space $GC(\Sigma_+^\infty BqG, R)$, for any $R \in GC$. If we replace R by $F(G_+, R)$, we deduce that the G -space $GL_1(R) = \mathcal{C}(\Sigma_+^\infty Q_G S^0, R)$ is the union of certain components in the space $\mathcal{C}(\Sigma_+^\infty BqG, R)$.

It is illuminating to consider the image of the equivalence $\Sigma_+^\infty Q_G S^0 = u^{-1}\Sigma_+^\infty BqG$ under various functors. Firstly, we have the geometric fixed point functors ϕ^H , which satisfy $\phi^H \Sigma_+^\infty X = \Sigma_+^\infty X^H$. In particular, we have

$$\phi^H \Sigma_+^\infty BqG = \Sigma_+^\infty (BqG)^H = \Sigma_+^\infty BH\mathcal{F},$$

so

$$\phi^H u^{-1}\Sigma_+^\infty BqG = u^{-1}\Sigma_+^\infty BH\mathcal{F} = \Sigma_+^\infty (Q_H S^0)^H.$$

We also know that $Q_G S^0$ is H -equivariantly equivalent to $Q_H S^0$, so

$$\Sigma_+^\infty (Q_H S^0)^H = \phi^H \Sigma_+^\infty Q_G S^0,$$

as expected.

It is also interesting to consider $\pi_0^G \Sigma_+^\infty BqG$ and $\pi_0^G \Sigma_+^\infty Q_G S^0$. For this, we need still more categories.

Definition 3.2. \mathcal{F}_2 is the category whose objects are maps of finite sets, and whose morphisms are commutative squares in which the horizontal maps are bijections. This is a symmetric bimonoidal category, under the definitions

$$\begin{aligned} (X \rightarrow U) \amalg (Y \rightarrow V) &= (X \amalg Y \rightarrow U \amalg V) \\ (X \rightarrow U) \otimes (Y \rightarrow V) &= ((X \times V \amalg U \times Y) \rightarrow (U \times V)). \end{aligned}$$

We have functors $j, e: CF \rightarrow \mathcal{F}_2$ given by $j(X) = (\emptyset \rightarrow X)$ and $e(X) = (X \rightarrow 1)$; these satisfy $j(X \amalg Y) = j(X) \amalg j(Y)$ and $j(X \times Y) = j(X) \otimes j(Y)$ and $e(X \amalg Y) = e(X) \otimes e(Y)$ (up to coherent natural isomorphism in all cases).

We write $G\mathcal{F}_2$ for the functor category $[bG, \mathcal{F}_2]$, which is isomorphic to the category of maps of finite G -sets.

Definition 3.3. Let I be a small category, and let $F: I \rightarrow \text{Cat}$ be a functor. We let ΔF be the category with

$$\begin{aligned} \text{obj}(\Delta F) &= \{(i, a) \mid i \in \text{obj}(I), a \in \text{obj}(F(i))\} \\ (\Delta F)((i, a), (j, b)) &= \{(u, f) \mid u \in I(i, j), f \in F(j)(u_* a, b)\}. \end{aligned}$$

The composition rule is $(v, g) \circ (u, f) = (vu, g \circ (v_* f))$.

Lemma 3.4. $B(\Delta F) = \text{holim}_I (B \circ F)$.

Proof. This must be in the literature somewhere. □

Corollary 3.5. Let \mathcal{C} be a category with an action of G . Let \mathcal{D} be the category with

$$\begin{aligned} \text{obj}(\mathcal{D}) &= \{(H, c) \mid H \leq G, c \in \text{obj}(\mathcal{C})^H\} \\ \mathcal{D}((H, c), (K, d)) &= \{(g, p) \mid g \in G/H, gHg^{-1} = K, p \in \mathcal{C}(gc, d)^K\} \\ (h, q) \circ (g, p) &= (hg, q \circ (h.p)). \end{aligned}$$

Then $\lambda^G \Sigma_+^\infty BC = \Sigma_+^\infty BD$.

Proof. Let \mathcal{O} be the category with

$$\begin{aligned} \text{obj}(\mathcal{O}) &= \{\text{subgroups of } G\} \\ \mathcal{O}(H, K) &= \{g \in G/H \mid gHg^{-1} = K\}. \end{aligned}$$

For any G -space X we have a functor $\mathcal{O} \rightarrow \text{Top}$ given by $H \mapsto X^H$, and one formulation of the tom Dieck splitting is that $\lambda^G \Sigma_+^\infty X$ is the homotopy colimit of this functor. Now take $X = BC$ and apply the lemma. □

Conjecture 3.6. $\lambda^G \Sigma_+^\infty BqG = K(G\mathcal{F}_2)$ as ring spectra. The obvious ring map $K(G\mathcal{F}) = \lambda^G S \rightarrow \lambda^G \Sigma_+^\infty BqG = K(G\mathcal{F}_2)$ comes from the functor $j: G\mathcal{F} \rightarrow G\mathcal{F}_2$. Moreover, the following diagram commutes:

$$\begin{array}{ccccc} \Sigma_+^\infty BG\mathcal{F} & \xrightarrow{\simeq} & \phi^G \Sigma_+^\infty BqG & \longrightarrow & \lambda^G \Sigma_+^\infty BqG \\ \text{Be} \downarrow & & & & \downarrow \simeq \\ \Sigma_+^\infty BG\mathcal{F}_2 & \longrightarrow & & \longrightarrow & K(G\mathcal{F}_2) \end{array}$$

The evidence is as follows. Let $\mathcal{J} \subset G\mathcal{F}_2$ be the subcategory of objects $(X \rightarrow U)$ for which U is a G -orbit, or equivalently, the category of \sqcup -indecomposable objects in $G\mathcal{F}_2$. It is not hard to see that $G\mathcal{F}_2$ is the free symmetric monoidal category generated by \mathcal{J} , so $K(G\mathcal{F}_2) = \Sigma_+^\infty B\mathcal{J}$. It will therefore suffice to show that \mathcal{J} is equivalent to the category \mathcal{D} in Corollary 3.5, based on $\mathcal{C} = qG$. In that case $\mathcal{C}^H \simeq H\mathcal{F}$, so an object of \mathcal{D} is essentially a pair (H, X) , where $H \leq G$ and X is an H -set, giving an object $G \times_H X \rightarrow G/H$ in \mathcal{J} . One checks that this construction gives the required equivalence $\mathcal{D} \rightarrow \mathcal{J}$. I have not checked all the additional claims but they seem very plausible.

We deduce that $\pi_0^G \Sigma_+^\infty BqG = K_0(G\mathcal{F}_2)$. This is the free abelian group generated by the isomorphism classes of indecomposables in $G\mathcal{F}_2$. The indecomposables are all of the form $(G \times_H X \rightarrow G/H)$ for some $H \leq G$ and some H -set X . For fixed H the construction $[X] \mapsto [G \times_H X \rightarrow G/H]$ gives us a map $\mathbb{Z}[A_+H] \rightarrow K_0(G\mathcal{F}_2)$, and by putting these together we get an isomorphism

$$\bigoplus_{(H)} \mathbb{Z}[(A_+H)/(W_G H)] \rightarrow K_0(G\mathcal{F}_2).$$

Addition of $[U]$ gives a well-defined endomorphism of $(A_+H)/(W_G H)$ for all H , corresponding to multiplication by $u \in \pi_0^G \Sigma_+^\infty BqG$. This leads to an isomorphism

$$\pi_0^G \Sigma_+^\infty Q_G S^0 = u^{-1} K_0(G\mathcal{F}_2) = \bigoplus_{(H)} \mathbb{Z}[(AH)/(W_G H)].$$

This can also be obtained more directly from the tom Dieck splitting:

$$\begin{aligned} \pi_0^G \Sigma_+^\infty Q_G S^0 &= \pi_0 \Sigma_+^\infty \operatorname{holim}_{\mathcal{O}} (Q_G S^0)^H = \mathbb{Z} \{ \lim_{\mathcal{O}} \pi_0^H (Q_G S^0) \} \\ &= \mathbb{Z} \{ \coprod_{(H)} A(H)/W_G H \} = \bigoplus_{(H)} \mathbb{Z}[(AH)/(W_G H)]. \end{aligned}$$

Example 3.7. Let p be prime, and let G be a group of order p . Let x, y and z be the elements of $K(G\mathcal{F}_2)$ corresponding to the maps $1 \rightarrow 1$, $G \rightarrow 1$ and $\emptyset \rightarrow G$. Then $x^i y^j = [(i.1 \amalg j.G) \rightarrow 1]$ and $x^i z = [i.G \rightarrow G]$. Any indecomposable object of $G\mathcal{F}_2$ has one of these two forms, so x, y and z generate $K_0(\mathcal{F}_2)$. One checks that in fact

$$K_0(G\mathcal{F}_2) = \mathbb{Z}[x, y, z] / ((x^p - y)z, (z - p)z).$$

4. SMALLER MODELS

We now set up smaller model for qG .

Let \mathcal{F}' be the usual skeleton of \mathcal{F} , with object set \mathbb{N} and morphism set $\coprod_n \Sigma_n$. We have a functor $\sqcup: \mathcal{F}' \times \mathcal{F}' \rightarrow \mathcal{F}'$, given by $n \sqcup m = n + m$

$$(\sigma \sqcup \tau)(i) = \begin{cases} \sigma(i) & \text{if } 0 \leq i < n \\ \tau(i - n) + n & \text{if } n \leq i < n + m \end{cases}$$

This is associative and unital on the nose. We have a symmetry isomorphism $\gamma_{n,m}: n \sqcup m \rightarrow m \sqcup n$ given by

$$\gamma_{n,m}(i) = \begin{cases} i + m & \text{if } 0 \leq i < n \\ i - n & \text{if } n \leq i < m. \end{cases}$$

This makes \mathcal{F}' a symmetric monoidal category.

As the inclusion $\mathcal{F}' \rightarrow \mathcal{F}$ is an equivalence, we see that qG is equivalent to $q'G := [eG, \mathcal{F}']$. Let $X: eG \rightarrow \mathcal{F}'$ be a functor. As the objects of eG are all isomorphic, and \mathcal{F}' is skeletal, we see that there is a number

$n \in \mathbb{N}$ such that $X(g) = n$ for all g . Next, for each $x \in G$ we have a unique morphism $1 \rightarrow x^{-1}$ in eG and thus an element $\phi_X(x) = X(1 \rightarrow x^{-1}) \in \Sigma_n$. We then have $\phi_X(1) = 1$, and the image under X of the unique morphism $x \rightarrow y$ is $\phi_X(y^{-1})\phi_X(x^{-1})^{-1}$. If we have another object $Y \in q'G$, then a morphism $f: X \rightarrow Y$ is just a system of permutations $f_x \in \Sigma_n$ for each $x \in G$, making the following squares commute:

$$\begin{array}{ccc} n & \xrightarrow{\phi_X(x)} & n \\ f_1 \downarrow & & \downarrow f_{x^{-1}} \\ n & \xrightarrow{\phi_Y(x)} & n \end{array}$$

This means that the maps f_y are all determined by f_1 , which can be arbitrary. It follows that $q'G$ is isomorphic to the category of pairs (n, ϕ) , where $n \in \mathbb{N}$ and $\phi: G \rightarrow \Sigma_n$ is a pointed map (not necessarily a homomorphism). There are no morphisms from (n, ϕ) to (m, ψ) unless $n = m$, in which case the morphism set is Σ_n (with the usual composition).

The symmetric monoidal structure on $q'G$ is

$$(n, \phi) \sqcup (m, \psi) = (n + m, \phi \sqcup \psi),$$

where $\phi \sqcup \psi$ means the map

$$G \xrightarrow{(\phi, \psi)} \Sigma_n \times \Sigma_m \xrightarrow{\sqcup} \Sigma_{n+m}.$$

The map

$$\sqcup: q'G((n, \phi), (n, \phi')) \times q'G((m, \psi), (m, \psi')) \rightarrow q'G((n + m, \phi \sqcup \psi), (n + m, \phi' \sqcup \psi'))$$

is just the map

$$\sqcup: \Sigma_n \times \Sigma_m \rightarrow \Sigma_{n+m}$$

as described previously.

One checks that the action of G on $\text{obj}(q'G)$ is given by $g.(n, \phi) = (n, g\phi)$, where $(g\phi)(x) = \phi(xg)\phi(g)^{-1}$. This is an action because

$$\begin{aligned} (g(h\phi))(x) &= (h\phi)(xg)(h\phi)(g)^{-1} = \phi(xgh)\phi(h)^{-1}\phi(h)\phi(gh)^{-1} \\ &= \phi(xgh)\phi(gh)^{-1} = ((gh)\phi)(x). \end{aligned}$$

We also have an action

$$g: (q'G)((n, \phi), (n, \psi)) \rightarrow (q'G)((n, g\phi), (n, g\psi))$$

given by $g.\sigma = \psi(g)\sigma\phi(g)^{-1}$. This is an action because

$$\begin{aligned} g.(h.\sigma) &= g.(\psi(h)\sigma\phi(h)^{-1}) = (h\psi)(g)\psi(h)\sigma\phi(h)^{-1}(h\phi)(g)^{-1} \\ &= \psi(gh)\psi(h)^{-1}\psi(h)\sigma\phi(h)^{-1}\phi(h)\phi(gh)^{-1} \\ &= \psi(gh)\sigma\phi(gh)^{-1} = (gh).\sigma \end{aligned}$$

If we have another morphism $\tau: (n, \psi) \rightarrow (n, \chi)$, then

$$(g.\tau)(g.\sigma) = \chi(g)\tau\psi(g)^{-1}\psi(g)\sigma\phi(g)^{-1} = \chi(g)\tau\sigma\phi(g)^{-1} = g.(\tau\sigma),$$

so we have defined a functor $g: q'G \rightarrow q'G$. One checks that this is the same as the action coming from the description $q'G = [eG, \mathcal{F}']$.

5. SYMMETRIC MONOIDAL FUNCTORS

Let $(\mathcal{A}, \oplus, \otimes)$ be a bipermutative category, so $K(\mathcal{A})$ can be constructed as an object in \mathcal{C} . It is reasonable to suppose that the space

$$\mathcal{C}(\Sigma_+^\infty BqG, K(\mathcal{A})) = \mathcal{C}(K(\text{Free}(qG)), K(\mathcal{A}))$$

should be well-related to the classifying space of the category of bipermutative functors $\text{Free}(qG) \rightarrow \mathcal{A}$, or the category of permutative functors $qG \rightarrow (\mathcal{A}, \otimes)$. Even in the case $G = 1$ the relationship is not too close because of group completion issues, but nonetheless this suggests that it would be interesting to understand the categories $\text{SymMon}(qG, \mathcal{Q})$ for various symmetric monoidal categories \mathcal{Q} .

As a warm-up, we consider the category $\text{SymMon}(\mathcal{F}', \mathcal{F}')$. An object is a pair (F, ζ) , where $F: \mathcal{F}' \rightarrow \mathcal{F}'$ and ζ is a natural map $\zeta_{a,b}: F(a \sqcup b) \rightarrow F(a) \sqcup F(b)$ such that the following diagrams commute:

$$\begin{array}{ccc} F(a \sqcup b \sqcup c) & \xrightarrow{\zeta_{a,b \sqcup c}} & F(a) \sqcup F(b \sqcup c) \\ \zeta_{a \sqcup b, c} \downarrow & & \downarrow 1 \sqcup \zeta_{b,c} \\ F(a \sqcup b) \sqcup F(c) & \xrightarrow{\zeta_{a,b \sqcup 1}} & F(a) \sqcup F(b) \sqcup F(c) \end{array} \quad \begin{array}{ccc} F(a \sqcup b) & \xrightarrow{\zeta_{a,b}} & F(a) \sqcup F(b) \\ F(\gamma_{a,b}) \downarrow & & \downarrow \gamma_{F(a), F(b)} \\ F(b \sqcup a) & \xrightarrow{\zeta_{b,a}} & F(b) \sqcup F(a) \end{array}$$

We have not given any axioms about the unit, because they are not needed. The existence of $\zeta_{0,0}$ implies that $F(0) = 0$, and thus that $\zeta_{0,0} = 1_0$. If we put $b = c = 0$ in the left hand diagram above, we find that $\zeta_{0,a}^2 = \zeta_{0,a}$ and so $\zeta_{0,a} = 1$. Similarly, we have $\zeta_{a,0} = 1$.

The morphisms from (F, ζ) to (G, ξ) are the natural maps $\alpha: F \rightarrow G$ such that the following diagram commutes:

$$\begin{array}{ccc} F(a \sqcup b) & \xrightarrow{\zeta_{a,b}} & F(a) \sqcup F(b) \\ \alpha_{a+b} \downarrow & & \downarrow \alpha_a \sqcup \alpha_b \\ G(a \sqcup b) & \xrightarrow{\xi_{a,b}} & G(a) \sqcup G(b) \end{array}$$

We define another category \mathcal{D} as follows. An object is a pair (d, α) , where $d \in \mathbb{N}$ and α is a sequence of elements $\alpha_m \in \Sigma_{md}$ (for $m \geq 0$) with $\alpha_0 = 1$ and $\alpha_1 = 1$. There are no morphisms from (d, α) to (d', α') unless $d' = d$, in which case the set of morphisms is Σ_d . Composition is just multiplication of permutations. We will show that $\text{SymMon}(\mathcal{F}', \mathcal{F}')$ is isomorphic to \mathcal{D} .

To prove this, we need a bipermutative structure on \mathcal{F}' . We can define a functor $\otimes: \mathcal{F}' \times \mathcal{F}' \rightarrow \mathcal{F}'$ by $n \otimes m = nm$ and $(\sigma \otimes \tau)(i + nj) = \sigma(i) + n\tau(j)$ for $0 \leq i < n$ and $0 \leq j < m$. This is associative and unital on the nose. We find that

$$\alpha \otimes (\beta \sqcup \gamma) = (\alpha \otimes \beta) \sqcup (\alpha \otimes \gamma).$$

We can thus define an object $(M_d, 1) \in \text{SymMon}(\mathcal{F}', \mathcal{F}')$ by $M_d(n) = dn$ and $M_d(\sigma) = 1_d \otimes \sigma$. Given $\phi \in \Sigma_d$, we have an automorphism ϕ_* of M_d given by

$$\phi_n = \phi \otimes 1_n \in \Sigma_{nd} = \mathcal{F}'(M_d(n), M_d(n)).$$

If $\alpha: (M_d, 1) \rightarrow (M_d, 1)$ is a morphism in SymMon then we must have $\alpha_{n+m} = \alpha_n \sqcup \alpha_m$. If we put $\phi = \alpha_1$ then we find inductively that $\alpha_j = \phi_j$ for all j , so $\alpha = \phi_*$. It follows that $\text{End}(M_d, 1) = \Sigma_d$.

Now suppose we have $(d, \alpha) \in \text{obj}(\mathcal{D})$. We define $M_d^\alpha: \mathcal{F}' \rightarrow \mathcal{F}'$ by $M_d^\alpha(n) = nd$ and

$$M_d^\alpha(\sigma) = (nd \xrightarrow{\alpha_n^{-1}} nd \xrightarrow{1_d \otimes \sigma} nd \xrightarrow{\alpha_n} nd).$$

This is defined so that $\alpha: M_d \rightarrow M_d^\alpha$ is a natural isomorphism. We also define

$$\zeta_{n,m}^\alpha: M_d^\alpha(n \sqcup m) \rightarrow M_d^\alpha(n) \sqcup M_d^\alpha(m)$$

to be the composite

$$(n+m)d \xrightarrow{\alpha_{n+m}^{-1}} (n+m)d \xrightarrow{\alpha_n \sqcup \alpha_m} (n+m)d.$$

This makes $(M_d^\alpha, \zeta^\alpha)$ into an object of $\text{SymMon}(\mathcal{F}', \mathcal{F}')$, isomorphic via α to M_d . Next, given a morphism

$$(d, \alpha) \xrightarrow{\phi} (d, \beta)$$

in \mathcal{D} , we have a map $(M_d^\alpha, \zeta^\alpha) \rightarrow (M_d^\beta, \zeta^\beta)$ in $\text{SymMon}(\mathcal{F}', \mathcal{F}')$ given by the composite

$$(M_d^\alpha, \zeta^\alpha) \xrightarrow{\alpha^{-1}} (M_d, 1) \xrightarrow{\beta} (M_d^\beta, \zeta^\beta).$$

This construction gives us a functor $M: \mathcal{D} \rightarrow \text{SymMon}(\mathcal{F}', \mathcal{F}')$, which is easily seen to be full and faithful. We must show that it is bijective on objects.

Consider an object $(G, \xi) \in \text{SymMon}(\mathcal{F}', \mathcal{F}')$. Put $d = G(1) \in \text{obj}(\mathcal{F}') = \mathbb{N}$. It is easy to see that $G(n) = nd$ for all n . Define $\omega_0 = 1 \in \Sigma_0$ and $\omega_1 = 1 \in \Sigma_d$, and

$$\omega_n = (nd = G((n-1) \sqcup 1) \xrightarrow{\xi_{n-1,1}} G(n-1) \sqcup G(1) \xrightarrow{\omega_{n-1} \sqcup 1} nd)$$

for $n > 1$. We claim that ω is a natural isomorphism $(G, \xi) \rightarrow (M_d, 1)$ and thus that $(G, \xi) = (M_d^\alpha, \zeta^\alpha)$, where $\alpha_n = \omega_n^{-1}$. The details still need to be written out.

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