SPECTRA OF UNITS

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1. The defining adjunction

If R is a ring spectrum, then we let $\mathrm{GL}_1(R)$ denote the preimage of $(\pi_0 R)^\times$ under the evident map $\Omega^\infty R \to \pi_0 R$. If R has an E_∞ structure then $\mathrm{GL}_1(R)$ is a grouplike E_∞ space and thus an infinite loop space, associated to a connective spectrum that we call $\mathrm{gl}_1(R)$, the *spectrum of units* for R. We also write $\mathrm{bgl}_1(R)$ for $\Sigma\,\mathrm{gl}_1(R)$.

Conversely, if X is an arbitrary spectrum then the additive structure makes $\Omega^\infty_- X$ into an E_∞ space and so makes $\Sigma^\infty_+\Omega^\infty_- X$ into an E_∞ ring spectrum. It is formally reasonable to expect an adjunction

$$\mathrm{Spectra}(X, \mathrm{gl}_1(R)) \simeq (E_\infty\mathrm{Rings})(\Sigma^\infty_+\Omega^\infty_- X, R).$$

Although this idea familiar to many people, we do not know of any rigorous theorem along these lines. One would expect that any such theorem would involve a number of technicalities about model structures, comparison of operads and so on. In this note we will explain a conjecture about how this should work out. We will use the framework described in [1].

Let $\mathcal{M}$ be the EKMM category of S-modules, and write $\wedge$ for the natural smash product in $\mathcal{M}$. Let $\mathcal{C}$ denote the category of commutative monoids in $\mathcal{M}$. Let S^k_c be the cofibrant k-sphere in $\mathcal{M}$.

Conjecture 1.1. There are functors $T\colon \mathcal{M} \to \mathcal{C}$ and $\mathrm{gl}_1\colon \mathcal{C} \to \mathcal{M}$ such that

(a) T is left adjoint to gl_1, and the adjunction is enriched and passes to homotopy.
(b) TS^0 is obtained from the free commutative ring PS^0 by inverting the tautological element $x \in \pi_0(PS^0)$.
(c) There is a natural map $\alpha_X\colon TX \to \Sigma^\infty_+\mathcal{M}(S^0_c, X)$ in $\mathcal{M}$, which is a weak equivalence for all X. The composite

$$x^{-1}PS^0 = \Sigma^\infty(\coprod_n B\Sigma_n)^+ \to TS^0 \xrightarrow{\alpha_{S^0}} \Sigma^\infty_+ QS^0$$

is the usual equivalence.

For the rest of this note, we will assume this conjecture.

Note that if $k<0$ then $\mathcal{M}(S^0_k, S^{-k}_c)$ is contractible, so TS^{-k}_c is weakly equivalent to S^0 in $\mathcal{C}$, so $\pi_k\,\mathrm{gl}_1(R) = \overline{h}\mathcal{C}(TS^{-k}_c, R) = 0$. Thus, the spectrum $\mathrm{gl}_1(R)$ is (-1)-connected for all R. Moreover, we have

$$\Omega^\infty_-\,\mathrm{gl}_1(R) = \mathcal{M}(S^0_c, \mathrm{gl}_1(R)) = \mathcal{C}(T(S^0_c), R) = \mathcal{C}(x^{-1}PS^0, R) \subset \mathcal{C}(PS^0, R) = \Omega^\infty_- R.$$

It is not hard to see that the relevant subspace of $\Omega^\infty_- R$ is just $GL_1(R)$ as defined earlier, and so $\pi_0\,\mathrm{gl}_1(R) = (\pi_0 R)^\times$.

Note that the functor $R \mapsto \overline{h}\mathcal{C}(TX, R) = [X, \mathrm{gl}_1(R)]$ takes values in the category of abelian groups, so TX is naturally a cogroup object in $\overline{h}\mathcal{C}$, with structure maps

$$S^0 \xleftarrow{\epsilon} TX \xrightarrow{\psi} TX \wedge TX$$

say. These should be compatible with the collapse and diagonal maps of the space $\Omega^\infty_- X$ under α_X.

The topological compatibility of our adjunction means that $\mathrm{gl}_1(F(K_+, R)) = F(K_+, \mathrm{gl}_1(R))$ for any space K. It follows that

$$[K_+, \mathrm{gl}_1(R)] = \pi_0\,\mathrm{gl}_1(F(K_+, R)) = (R^0 K)^\times.$$

2. Thom spectra

Suppose we have a spectrum X and a map $\zeta\colon \Omega X = \Sigma^{-1}X \to \mathrm{gl}_1(S)$, or equivalently a map $X \to \mathrm{bgl}_1(S)$, or a map $\zeta^{\#}\colon T\Omega X \to S$ in $\mathcal{C}$. We define the corresponding *Thom spectrum* $T\zeta$ to be the homotopy pushout in $\mathcal{C}$ of the maps

$$S^0 \xleftarrow{\epsilon} T\Omega X \xrightarrow{\zeta^{\#}} S^0.$$

We find that $\mathcal{C}(T\zeta, R)$ is the space of paths from 0 to $\mathrm{gl}_1(\eta)\circ\zeta$ in $\mathcal{M}(\Omega X, \mathrm{gl}_1(R))$. (I learned this point of view from Mike Hopkins; I do not know whether there is any earlier history.)

In particular, for the case $\zeta = 0$ we have

$$\mathcal{C}(T(X \xrightarrow{0} \mathrm{bgl}_1(S)), R) = \Omega\mathcal{M}(\Omega X, \mathrm{gl}_1(R)) = \mathcal{M}(X, \mathrm{gl}_1(R)) = \mathcal{C}(TX, R),$$

so

$$T(X \xrightarrow{0} \mathrm{bgl}_1(R)) = TX \simeq \Sigma^\infty_+\Omega^\infty_- X.$$

Now suppose we have another map $\xi\colon Y \to \mathrm{bgl}_1(S)$. From the above description we see that

$$T\zeta \wedge T\xi = T(X \vee Y \xrightarrow{(\zeta,\xi)} \mathrm{bgl}_1(S)).$$

In particular, we can take $Y = X$ and $\zeta = \xi$, and then use a shearing isomorphism to get

$$T\zeta \wedge T\zeta \simeq T\zeta \wedge \Sigma^\infty_+\Omega^\infty_- X.$$

It follows that for every naive ring spectrum R with a map $T\zeta \to R$, we have a natural isomorphism $R_*T\zeta \simeq R_*\Sigma^\infty_+\Omega^\infty_- X$ of R_*-algebras.

Similarly, we can use the maps

$$X \xrightarrow{(1,1)} X \vee X \xrightarrow{(\zeta,0)} \mathrm{bgl}_1(S)$$

to get a map $T\zeta \to T\zeta \wedge TX$, giving a coaction of the cogroup TX on $T\zeta$.

3. Equivariant versions

We can enrich our original conjecture as follows. Let $\mathcal{L}$ be the category of infinite universes.

Conjecture 3.1. There are functors $T\colon \mathcal{L}\times\mathcal{M} \to \mathcal{C}$ and $\mathrm{gl}_1\colon \mathcal{L}\times\mathcal{C} \to \mathcal{M}$ such that

(a) For each universe $\mathbb{U}$, the functor $T(\mathbb{U}, -)$ is left adjoint to $\mathrm{gl}_1(\mathbb{U}, -)$, and the adjunction is enriched, natural in $\mathbb{U}$, and passes to homotopy.
(b) There is a natural map $T(\mathbb{U}, X) \to \Sigma^\infty_+\mathcal{M}(S(\mathbb{U}), X)$, which is a weak equivalence for all $\mathbb{U}$ and X.
(c) There is a natural equivalence $T(\mathbb{R}, X) \simeq TX$, where TX is as considered previously.

Now let G be a finite group, and let $G\mathcal{M}$ be the category of objects in $\mathcal{M}$ with an action of G, and so on. Let $\mathbb{U} = \mathbb{C}[G]^\infty$ be the standard complete G-universe. We give $G\mathcal{M}$ the model structure with $S(\mathbb{U}_G)$ as the basic cell, so the associated homotopy category is the usual category of genuine G-spectra. We will write $\lambda^G\colon G\mathcal{M} \to \mathcal{M}$ for the Lewis-May fixed point functor, which in our framework is given by $\lambda^G X = F_G(S(\mathbb{U}), X)$. We will also let ϕ^G denote the geometric fixed point functor, given on cofibrant objects by $\phi^G(X) = \lambda^G(\widetilde{EG}\wedge X)$.

For any $X \in G\mathcal{M}$ we combine the G-actions on X and $\mathbb{U}$ to get an action on $T(\mathbb{U}, X)$. Similarly, for $R \in G\mathcal{C}$ we get an action on $\mathrm{gl}_1(\mathbb{U}, R)$. The homeomorphism

$$\mathcal{M}(X, \mathrm{gl}_1(\mathbb{U}, R)) \simeq \mathcal{C}(T(\mathbb{U}, X), R)$$

is then G-equivariant, so we get an adjunction

$$G\mathcal{M}(X, \mathrm{gl}_1(\mathbb{U}, R)) \simeq G\mathcal{C}(T(\mathbb{U}, X), R).$$

Using this, we can make all our constructions equivariant.

However, it is not easy to understand the equivariant homotopy type of the space

$$GL_1(R) = \mathcal{M}(S(\mathbb{U}), \mathrm{gl}_1(\mathbb{U}, R)) = \mathcal{C}(T(\mathbb{U}, S(\mathbb{U})), R).$$

In the case $G = 1$ we know that $T(\mathbb{U}, S(\mathbb{U})) = T(S^0_c)$ (which is equivalent to $\Sigma^\infty_+ QS^0$) is obtained from the free object PS^0_c by inverting a single homotopy element, which implies that $GL_1(R)$ is just the union of the invertible components in $\Omega^\infty R = \mathcal{M}(S^0_c, R) = \mathcal{C}(PS^0_c, R)$. The analogue for general G is more complicated. To explain it, we introduce several categories associated to G.

- bG is the category with one object, whose endomorphism monoid is G. This has $BbG = BG$.
- For any set T, we let eT be the category with object set T, and a single morphism between any pair of objects. Any bijection $T \to U$ gives an isomorphism $eT \to eU$ of categories. In particular, G acts on itself by left multiplication, and so acts on eG by automorphisms of categories. We find that $BeG = EG$ as G-spaces. We also find that $bG = (eG)/G$ as categories, and that $BbG = (BeG)/G$ as spaces.
- We let $\mathcal{F}$ be the category of finite sets and bijections, so $B\mathcal{F} \simeq \coprod_n B\Sigma_n$. We thus have $\Sigma^\infty_+ B\mathcal{F} = PS^0$. As $\mathcal{F}$ is a symmetric monoidal category, it has a K-theory spectrum $K(\mathcal{F})$, which in this case is just S^0.
- We write sG for the category whose objects are pairs (X, u) where X is a finite set, and $u\colon X \to G$. The morphisms from (X, u) to (Y, v) are the bijections $X \to Y$ (independent of u and v). The group G acts on sG by isomorphisms, by the rule $g.(X, u) = (X, gu)$, where $(gu)(x) = g.u(x)$. This is equivariantly equivalent to the free symmetric monoidal category generated by eG. We have $BsG = D(EG)$ (where D is the total extended power functor), and the group completion of this is $Q(EG_+)$. It is natural to think of $K(sG)$ as being closely related to the cofibrant G-sphere $S(\mathbb{U}_G)$, although we do not know of a way to make this precise. (It would not be strong enough to say that $K(sG)$ is weakly equivalent to $S(\mathbb{U}_G)$ in the usual model structure on $G\mathcal{M}$, essentially because the non-cofibrant unit sphere S^0 (with trivial action) is already weakly equivalent to $S(\mathbb{U}_G)$.) As part of the evidence, recall that
$$\Omega^\infty S(\mathbb{U}_G) = \mathcal{M}(S(\mathbb{R}^\infty), S(\mathbb{U}_G)) = Q\mathcal{L}(\mathbb{U}_G, \mathbb{R}^\infty)_+,$$
and $\mathcal{L}(\mathbb{U}_G, \mathbb{R}^\infty)$ is a model for EG, so $\Omega^\infty S(\mathbb{U}_G) = Q(EG_+) = \Omega^\infty K(sG)$ as G-spaces.
- We write $qG = [eG, \mathcal{F}] = \mathrm{SymMon}(sG, \mathcal{F})$. If we had a sufficently good K-theory functor, we would expect a natural map $B\,\mathrm{SymMon}(\mathcal{A}, \mathcal{B}) \to \mathcal{M}(K(\mathcal{A}), K(\mathcal{B}))$. In particular, we would get an equivariant map
$$BqG \to \mathcal{M}(K(sG), K(\mathcal{F})) \simeq \mathcal{M}(S(\mathbb{U}_G), S(\mathbb{R}^\infty)) = Q_G S^0.$$
I think it should be possible to construct this by more pedestrian methods (eg configuration spaces).
- We write $G\mathcal{F} = [bG, \mathcal{F}]$ for the category of finite G-sets and isomorphisms, and $G\mathcal{T}$ for the subcategory of transitive G-sets. One can check that $G\mathcal{F}$ is the free symmetric monoidal category generated by $G\mathcal{T}$, so $BG\mathcal{F} = D(BG\mathcal{T})$ and $K(G\mathcal{F}) = \Sigma^\infty_+ BG\mathcal{T}$. The tom Dieck splitting says that $\lambda^G S^0$ can also be identified with $\Sigma^\infty_+ BG\mathcal{T}$, so $K(G\mathcal{F}) = \lambda^G S^0$. We also have $(qG)^G = [(eG)/G, \mathcal{F}] = [bG, \mathcal{F}] = G\mathcal{F}$, so $(BqG)^G = BG\mathcal{F}$. More generally, for any $H \leq G$ one checks that eG is H-equivariantly equivalent to eH, and so $(qG)^H = H\mathcal{F}$ and $(BqG)^H = BH\mathcal{F}$.
- We also write fG for the subcategory of free G-sets in $G\mathcal{F}$. We have a functor from bG to fG, sending the unique object in bG to $G \in \mathrm{obj}(fG)$, and sending the morphism $g \in bG$ to the morphism $(x \mapsto xg^{-1}) \in fG(G, G)$. This extends to an equivalence from the free symmetric monoidal category on bG to fG. It follows that $BfG = D(BG)$ and $K(fG) = \Sigma^\infty_+ BG$.

Now let $A_+G = \pi_0(G\mathcal{F}) = \pi_0 BG\mathcal{F} = \pi_0^G BqG$ be the Burnside semiring for G. The group completion of this is the Burnside ring AG, which can also be described as $\pi_0^G(Q_G S^0)$. Let U be the set of subsets of G, with its evident action of G, and note that this contains a copy of G/H for each subgroup H. For any $x \in AG$ we have $x + n[U] \in A_+G$ for $n \gg 0$, so AG is the colimit of the sequence
$$A_+G \xrightarrow{+[U]} A_+G \xrightarrow{+[U]} A_+G \xrightarrow{+[U]} \cdots$$
This comes from a sequence
$$BqG \to BqG \to BqG \to \cdots$$
of G-spaces, and I think one can show that the homotopy colimit is $Q_G S^0$. Now let u be the element of $\mathbb{Z}[A_+G]$ corresponding to $[U]$, or its image under the natural stabilisation map
$$\mathbb{Z}[A_+G] = \mathbb{Z}\{\pi_0^G BqG\} \to \pi_0^G \Sigma^\infty_+ BqG.$$

We deduce that $\Sigma^\infty_+ Q_G S^0 = (\Sigma^\infty_+ BqG)[u^{-1}]$ as ring spectra. It is not hard to deduce that this can be regarded as an equivalence of strict modules over T, where T is a strictly commutative ring equivalent to $\Sigma^\infty_+ BqG$. We can then form the Bousfield localisation of T in the category of commutative T-algebras, with respect to the module $T[u^{-1}]$; this gives us a commutative T-algebra that is equivalent to $T[u^{-1}]$ as an T-module, and is equivalent to $\Sigma^\infty_+ Q_G S^0$ as an T-algebra. It follows that $G\mathcal{C}(\Sigma^\infty_+ Q_G S^0, R)$ is the union of certain components in the space $G\mathcal{C}(\Sigma^\infty_+ BqG, R)$, for any $R \in G\mathcal{C}$. If we replace R by $F(G_+, R)$, we deduce that the G-space $GL_1(R) = \mathcal{C}(\Sigma^\infty_+ Q_G S^0, R)$ is the union of certain components in the space $\mathcal{C}(\Sigma^\infty_+ BqG, R)$.

It is illuminating to consider the image of the equivalence $\Sigma^\infty_+ Q_G S^0 = u^{-1}\Sigma^\infty_+ BqG$ under various functors. Firstly, we have the geometric fixed point functors ϕ^H, which satisfy $\phi^H \Sigma^\infty_+ X = \Sigma^\infty_+ X^H$. In particular, we have

$$\phi^H \Sigma^\infty_+ BqG = \Sigma^\infty_+ (BqG)^H = \Sigma^\infty_+ BH\mathcal{F},$$

so

$$\phi^H u^{-1} \Sigma^\infty_+ BqG = u^{-1}\Sigma^\infty_+ BH\mathcal{F} = \Sigma^\infty_+ (Q_H S^0)^H.$$

We also know that $Q_G S^0$ is H-equivariantly equivalent to $Q_H S^0$, so

$$\Sigma^\infty_+ (Q_H S^0)^H = \phi^H \Sigma^\infty_+ Q_G S^0,$$

as expected.

It is also interesting to consider $\pi_0^G \Sigma^\infty_+ BqG$ and $\pi_0^G \Sigma^\infty_+ Q_G S^0$. For this, we need still more categories.

Definition 3.2. $\mathcal{F}_2$ is the category whose objects are maps of finite sets, and whose morphisms are commutative squares in which the horizontal maps are bijections. This is a symmetric bimonoidal category, under the definitions

$$(X \to U) \amalg (Y \to V) = (X \amalg Y \to U \amalg V)$$
$$(X \to U) \otimes (Y \to V) = ((X \times V \amalg U \times Y) \to (U \times V)).$$

We have functors $j, e \colon CF \to \mathcal{F}_2$ given by $j(X) = (\emptyset \to X)$ and $e(X) = (X \to 1)$; these satisfy $j(X \amalg Y) = j(X) \amalg j(Y)$ and $j(X \times Y) = j(X) \otimes j(Y)$ and $e(X \amalg Y) = e(X) \otimes e(Y)$ (up to coherent natural isomorphism in all cases).

We write $G\mathcal{F}_2$ for the functor category $[bG, \mathcal{F}_2]$, which is isomorphic to the category of maps of finite G-sets.

Definition 3.3. Let I be a small category, and let $F \colon I \to \text{Cat}$ be a functor. We let ΔF be the category with

$$\text{obj}(\Delta F) = \{(i, a) \mid i \in \text{obj}(I)\ ,\ a \in \text{obj}(F(i))\}$$
$$(\Delta F)((i, a), (j, b)) = \{(u, f) \mid u \in I(i, j)\ ,\ f \in F(j)(u_* a, b)\}.$$

The composition rule is $(v, g) \circ (u, f) = (vu, g \circ (v_* f))$.

Lemma 3.4. *$B(\Delta F) = \underset{\longrightarrow\ I}{\text{holim}}(B \circ F)$.*

Proof. This must be in the literature somewhere. □

Corollary 3.5. *Let $\mathcal{C}$ be a category with an action of G. Let $\mathcal{D}$ be the category with*

$$\text{obj}(\mathcal{D}) = \{(H, c) \mid H \leq G\ ,\ c \in \text{obj}(\mathcal{C})^H\}$$
$$\mathcal{D}((H, c), (K, d)) = \{(g, p) \mid g \in G/H\ ,\ gHg^{-1} = K\ ,\ p \in \mathcal{C}(gc, d)^K\}$$
$$(h, q) \circ (g, p) = (hg, q \circ (h.p)).$$

Then $\lambda^G \Sigma^\infty_+ B\mathcal{C} = \Sigma^\infty_+ B\mathcal{D}$.

Proof. Let $\mathcal{O}$ be the category with

$$\text{obj}(\mathcal{O}) = \{\text{ subgroups of } G\}$$
$$\mathcal{O}(H, K) = \{g \in G/H \mid gHg^{-1} = K\}.$$

For any G-space X we have a functor $\mathcal{O} \to \text{Top}$ given by $H \mapsto X^H$, and one formulation of the tom Dieck splitting is that $\lambda^G \Sigma^\infty_+ X$ is the homotopy colimit of this functor. Now take $X = B\mathcal{C}$ and apply the lemma. □

Conjecture 3.6. $\lambda^G \Sigma^\infty_+ BqG = K(G\mathcal{F}_2)$ as ring spectra. The obvious ring map $K(G\mathcal{F}) = \lambda^G S \to \lambda^G \Sigma^\infty_+ BqG = K(G\mathcal{F}_2)$ comes from the functor $j\colon G\mathcal{F} \to G\mathcal{F}_2$. Moreover, the following diagram commutes:

$$\begin{array}{ccccc} \Sigma^\infty_+ BG\mathcal{F} & \xrightarrow{\simeq} & \phi^G \Sigma^\infty_+ BqG & \longrightarrow & \lambda^G \Sigma^\infty_+ BqG \\ {\scriptstyle Be}\downarrow & & & & \downarrow{\scriptstyle \simeq} \\ \Sigma^\infty_+ BG\mathcal{F}_2 & & \longrightarrow & & K(G\mathcal{F}_2) \end{array}$$

The evidence is as follows. Let $\mathcal{J} \subset G\mathcal{F}_2$ be the subcategory of objects $(X \to U)$ for which U is a G-orbit, or equivalently, the category of $\sqcup$-indecomposable objects in $G\mathcal{F}_2$. It is not hard to see that $G\mathcal{F}_2$ is the free symmetric monoidal category generated by $\mathcal{J}$, so $K(G\mathcal{F}_2) = \Sigma^\infty_+ B\mathcal{J}$. It will therefore suffice to show that $\mathcal{J}$ is equivalent to the category $\mathcal{D}$ in Corollary 3.5, based on $\mathcal{C} = qG$. In that case $\mathcal{C}^H \simeq H\mathcal{F}$, so an object of $\mathcal{D}$ is essentially a pair (H, X), where $H \leq G$ and X is an H-set, giving an object $G \times_H X \to G/H$ in $\mathcal{J}$. One checks that this construction gives the required equivalence $\mathcal{D} \to \mathcal{J}$. I have not checked all the additional claims but they seem very plausible.

We deduce that $\pi_0^G \Sigma^\infty_+ BqG = K_0(G\mathcal{F}_2)$. This is the free abelian group generated by the isomorphism classes of indecomposables in $G\mathcal{F}_2$. The indecomposables are all of the form $(G \times_H X \to G/H)$ for some $H \leq G$ and some H-set X. For fixed H the construction $[X] \mapsto [G \times_H X \to G/H]$ gives us a map $\mathbb{Z}[A_+H] \to K_0(G\mathcal{F}_2)$, and by putting these together we get an isomorphism

$$\bigoplus_{(H)} \mathbb{Z}[(A_+H)/(W_GH)] \to K_0(G\mathcal{F}_2).$$

Addition of $[U]$ gives a well-defined endomorphism of $(A_+H)/(W_GH)$ for all H, corresponding to multiplication by $u \in \pi_0^G \Sigma^\infty_+ BqG$. This leads to an isomorphism

$$\pi_0^G \Sigma^\infty_+ Q_G S^0 = u^{-1} K_0(G\mathcal{F}_2) = \bigoplus_{(H)} \mathbb{Z}[(AH)/(W_GH)].$$

This can also be obtained more directly from the tom Dieck splitting:

$$\begin{aligned} \pi_0^G \Sigma^\infty_+ Q_G S^0 &= \pi_0 \Sigma^\infty_+ \operatorname{holim}_{\overrightarrow{\mathcal{O}}} (Q_G S^0)^H = \mathbb{Z}\{\varinjlim_{\mathcal{O}} \pi_0^H (Q_G S^0)\} \\ &= \mathbb{Z}\{\coprod_{(H)} A(H)/W_GH\} = \bigoplus_{(H)} \mathbb{Z}[(AH)/(W_GH)]. \end{aligned}$$

Example 3.7. Let p be prime, and let G be a group of order p. Let x, y and z be the elements of $K(G\mathcal{F}_2)$ corresponding to the maps $1 \to 1$, $G \to 1$ and $\emptyset \to G$. Then $x^i y^j = [(i.1 \amalg j.G) \to 1]$ and $x^i z = [i.G \to G]$. Any indecomposable object of $G\mathcal{F}_2$ has one of these two forms, so x, y and z generate $K_0(\mathcal{F}_2)$. One checks that in fact

$$K_0(G\mathcal{F}_2) = \mathbb{Z}[x, y, z]/((x^p - y)z, (z - p)z).$$

4. Smaller models

We now set up smaller model for qG.

Let $\mathcal{F}'$ be the usual skeleton of $\mathcal{F}$, with object set $\mathbb{N}$ and morphism set $\coprod_n \Sigma_n$. We have a functor $\sqcup\colon \mathcal{F}' \times \mathcal{F}' \to \mathcal{F}'$, given by $n \sqcup m = n + m$

$$(\sigma \sqcup \tau)(i) = \begin{cases} \sigma(i) & \text{if } 0 \leq i < n \\ \tau(i-n) + n & \text{if } n \leq i < n+m \end{cases}$$

This is associative and unital on the nose. We have a symmetry isomorphism $\gamma_{n,m}\colon n \sqcup m \to m \sqcup n$ given by

$$\gamma_{n,m}(i) = \begin{cases} i + m & \text{if } 0 \leq i < n \\ i - n & \text{if } n \leq i < m. \end{cases}$$

This makes $\mathcal{F}'$ a symmetric monoidal category.

As the inclusion $\mathcal{F}' \to \mathcal{F}$ is an equivalence, we see that qG is equivalent to $q'G := [eG, \mathcal{F}']$. Let $X\colon eG \to \mathcal{F}'$ be a functor. As the objects of eG are all isomorphic, and $\mathcal{F}'$ is skeletal, we see that there is a number

$n \in \mathbb{N}$ such that $X(g) = n$ for all g. Next, for each $x \in G$ we have a unique morphism $1 \to x^{-1}$ in eG and thus an element $\phi_X(x) = X(1 \to x^{-1}) \in \Sigma_n$. We then have $\phi_X(1) = 1$, and the image under X of the unique morphism $x \to y$ is $\phi_X(y^{-1})\phi_X(x^{-1})^{-1}$. If we have another object $Y \in q'G$, then a morphism $f\colon X \to Y$ is just a system of permutations $f_x \in \Sigma_n$ for each $x \in G$, making the following squares commute:

$$\begin{array}{ccc} n & \xrightarrow{\phi_X(x)} & n \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_{x^{-1}}} \\ n & \xrightarrow[\phi_Y(x)]{} & n \end{array}$$

This means that the maps f_y are all determined by f_1, which can be arbitrary. It follows that $q'G$ is isomorphic to the category of pairs (n, ϕ), where $n \in \mathbb{N}$ and $\phi\colon G \to \Sigma_n$ is a pointed map (not necessarily a homomorphism). There are no morphisms from (n, ϕ) to (m, ψ) unless $n = m$, in which case the morphism set is Σ_n (with the usual composition).

The symmetric monoidal structure on $q'G$ is

$$(n, \phi) \sqcup (m, \psi) = (n + m, \phi \sqcup \psi),$$

where $\phi \sqcup \psi$ means the map

$$G \xrightarrow{(\phi,\psi)} \Sigma_n \times \Sigma_m \xrightarrow{\sqcup} \Sigma_{n+m}.$$

The map

$$\sqcup\colon q'G((n, \phi), (n, \phi')) \times q'G((m, \psi), (m, \psi')) \to q'G((n + m, \phi \sqcup \psi), (n + m, \phi' \sqcup \psi'))$$

is just the map

$$\sqcup\colon \Sigma_n \times \Sigma_m \to \Sigma_{n+m}$$

as described previously.

One checks that the action of G on $\mathrm{obj}(q'G)$ is given by $g.(n, \phi) = (n, g\phi)$, where $(g\phi)(x) = \phi(xg)\phi(g)^{-1}$. This is an action because

$$\begin{aligned} (g(h\phi))(x) &= (h\phi)(xg)(h\phi)(g)^{-1} = \phi(xgh)\phi(h)^{-1}\phi(h)\phi(gh)^{-1} \\ &= \phi(xgh)\phi(gh)^{-1} = ((gh)\phi)(x). \end{aligned}$$

We also have an action

$$g\colon (q'G)((n, \phi), (n, \psi)) \to (q'G)((n, g\phi), (n, g\psi))$$

given by $g.\sigma = \psi(g)\sigma\phi(g)^{-1}$. This is an action because

$$\begin{aligned} g.(h.\sigma) &= g.(\psi(h)\sigma\phi(h)^{-1}) = (h\psi)(g)\,\psi(h)\sigma\phi(h)^{-1}\,(h\phi)(g)^{-1} \\ &= \psi(gh)\psi(h)^{-1}\psi(h)\sigma\phi(h)^{-1}\phi(h)\phi(gh)^{-1} \\ &= \psi(gh)\sigma\phi(gh)^{-1} = (gh).\sigma \end{aligned}$$

If we have another morphism $\tau\colon (n, \psi) \to (n, \chi)$, then

$$(g.\tau)(g.\sigma) = \chi(g)\tau\psi(g)^{-1}\psi(g)\sigma\phi(g)^{-1} = \chi(g)\tau\sigma\phi(g)^{-1} = g.(\tau\sigma),$$

so we have defined a functor $g\colon q'G \to q'G$. One checks that this is the same as the action coming from the description $q'G = [eG, \mathcal{F}']$.

5. Symmetric monoidal functors

Let $(\mathcal{A}, \oplus, \otimes)$ be a bipermutative category, so $K(\mathcal{A})$ can be constructed as an object in $\mathcal{C}$. It is reasonable to suppose that the space

$$\mathcal{C}(\Sigma^\infty_+ BqG, K(\mathcal{A})) = \mathcal{C}(K(\mathrm{Free}(qG)), K(\mathcal{A}))$$

should be well-related to the classifying space of the category of bipermutative functors $\mathrm{Free}(qG) \to \mathcal{A}$, or the category of permutative functors $qG \to (\mathcal{A}, \otimes)$. Even in the case $G = 1$ the relationship is not too close because of group completion issues, but nonetheless this suggests that it would be interesting to understand the categories $\mathrm{SymMon}(qG, \mathcal{Q})$ for various symmetric monoidal categories $\mathcal{Q}$.

As a warm-up, we consider the category $\mathrm{SymMon}(\mathcal{F}',\mathcal{F}')$. An object is a pair (F,ζ), where $F\colon \mathcal{F}'\to\mathcal{F}'$ and ζ is a natural map $\zeta_{a,b}\colon F(a\sqcup b)\to F(a)\sqcup F(b)$ such that the following diagrams commute:

$$
\begin{array}{ccc}
F(a\sqcup b\sqcup c) & \xrightarrow{\zeta_{a,b\sqcup c}} & F(a)\sqcup F(b\sqcup c) \\
\downarrow{\scriptstyle \zeta_{a\sqcup b,c}} & & \downarrow{\scriptstyle 1\sqcup\zeta_{b,c}} \\
F(a\sqcup b)\sqcup F(c) & \xrightarrow[\zeta_{a,b}\sqcup 1]{} & F(a)\sqcup F(b)\sqcup F(c)
\end{array}
\qquad
\begin{array}{ccc}
F(a\sqcup b) & \xrightarrow{\zeta_{a,b}} & F(a)\sqcup F(b) \\
\downarrow{\scriptstyle F(\gamma_{a,b})} & & \downarrow{\scriptstyle \gamma_{F(a),F(b)}} \\
F(b\sqcup a) & \xrightarrow[\zeta_{b,a}]{} & F(b)\sqcup F(a)
\end{array}
$$

We have not given any axioms about the unit, because they are not needed. The existence of $\zeta_{0,0}$ implies that $F(0)=0$, and thus that $\zeta_{0,0}=1_0$. If we put $b=c=0$ in the left hand diagram above, we find that $\zeta_{0,a}^2=\zeta_{0,a}$ and so $\zeta_{0,a}=1$. Similarly, we have $\zeta_{a,0}=1$.

The morphisms from (F,ζ) to (G,ξ) are the natural maps $\alpha\colon F\to G$ such that the following diagram commutes:

$$
\begin{array}{ccc}
F(a\sqcup b) & \xrightarrow{\zeta_{a,b}} & F(a)\sqcup F(b) \\
\downarrow{\scriptstyle \alpha_{a+b}} & & \downarrow{\scriptstyle \alpha_a\sqcup\alpha_b} \\
G(a\sqcup b) & \xrightarrow[\xi_{a,b}]{} & G(a)\sqcup G(b)
\end{array}
$$

We define another category $\mathcal{D}$ as follows. An object is a pair (d,α), where $d\in\mathbb{N}$ and α is a sequence of elements $\alpha_m\in\Sigma_{md}$ (for $m\geq 0$) with $\alpha_0=1$ and $\alpha_1=1$. There are no morphisms from (d,α) to (d',α') unless $d'=d$, in which case the set of morphisms is Σ_d. Composition is just multiplication of permutations. We will show that $\mathrm{SymMon}(\mathcal{F}',\mathcal{F}')$ is isomorphic to $\mathcal{D}$.

To prove this, we need a bipermutative structure on $\mathcal{F}'$. We can define a functor $\otimes\colon \mathcal{F}'\times\mathcal{F}'\to\mathcal{F}'$ by $n\otimes m=nm$ and $(\sigma\otimes\tau)(i+nj)=\sigma(i)+n\tau(j)$ for $0\leq i<n$ and $0\leq j<m$. This is associative and unital on the nose. We find that

$$\alpha\otimes(\beta\sqcup\gamma)=(\alpha\otimes\beta)\sqcup(\alpha\otimes\gamma).$$

We can thus define an object $(M_d,1)\in\mathrm{SymMon}(\mathcal{F}',\mathcal{F}')$ by $M_d(n)=dn$ and $M_d(\sigma)=1_d\otimes\sigma$. Given $\phi\in\Sigma_d$, we have an automorphism ϕ_* of M_d given by

$$\phi_n=\phi\otimes 1_n\in\Sigma_{nd}=\mathcal{F}'(M_d(n),M_d(n)).$$

If $\alpha\colon (M_d,1)\to(M_d,1)$ is a morphism in SymMon then we must have $\alpha_{n+m}=\alpha_n\sqcup\alpha_m$. If we put $\phi=\alpha_1$ then we find inductively that $\alpha_j=\phi_j$ for all j, so $\alpha=\phi_*$. It follows that $\mathrm{End}(M_d,1)=\Sigma_d$.

Now suppose we have $(d,\alpha)\in\mathrm{obj}(\mathcal{D})$. We define $M_d^\alpha\colon\mathcal{F}'\to\mathcal{F}'$ by $M_d^\alpha(n)=nd$ and

$$M_d^\alpha(\sigma)=(nd\xrightarrow{\alpha_n^{-1}}nd\xrightarrow{1_d\otimes\sigma}nd\xrightarrow{\alpha_n}nd).$$

This is defined so that $\alpha\colon M_d\to M_d^\alpha$ is a natural isomorphism. We also define

$$\zeta^\alpha_{n,m}\colon M^d_\alpha(n\sqcup m)\to M_d^\alpha(n)\sqcup M_d^\alpha(m)$$

to be the composite

$$(n+m)d\xrightarrow{\alpha_{n+m}^{-1}}(n+m)d\xrightarrow{\alpha_n\sqcup\alpha_m}(n+m)d.$$

This makes $(M_d^\alpha,\zeta^\alpha)$ into an object of $\mathrm{SymMon}(\mathcal{F}',\mathcal{F}')$, isomorphic via α to M_d. Next, given a morphism

$$(d,\alpha)\xrightarrow{\phi}(d,\beta)$$

in $\mathcal{D}$, we have a map $(M_d^\alpha,\zeta^\alpha)\to(M_d^\beta,\zeta^\beta)$ in $\mathrm{SymMon}(\mathcal{F}',\mathcal{F}')$ given by the composite

$$(M_d^\alpha,\zeta^\alpha)\xrightarrow{\alpha^{-1}}(M_d,1)\xrightarrow{\beta}(M_d^\beta,\zeta^\beta).$$

This construction gives us a functor $M\colon\mathcal{D}\to\mathrm{SymMon}(\mathcal{F}',\mathcal{F}')$, which is easily seen to be full and faithful. We must show that it is bijective on objects.

Consider an object $(G,\xi)\in\mathrm{SymMon}(\mathcal{F}',\mathcal{F}')$. Put $d=G(1)\in\mathrm{obj}(\mathcal{F}')=\mathbb{N}$. It is easy to see that $G(n)=nd$ for all n. Define $\omega_0=1\in\Sigma_0$ and $\omega_1=1\in\Sigma_d$, and

$$\omega_n=(nd=G((n-1)\sqcup 1)\xrightarrow{\xi_{n-1,1}}G(n-1)\sqcup G(1)\xrightarrow{\omega_{n-1}\sqcup 1}nd)$$

for $n > 1$. We claim that ω is a natural isomorphism $(G, \xi) \to (M_d, 1)$ and thus that $(G, \xi) = (M_d^\alpha, \zeta^\alpha)$, where $\alpha_n = \omega_n^{-1}$. The details still need to be written out.

References

[1] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules and algebras in stable homotopy theory*, Amer. Math. Soc. Surveys and Monographs, vol. 47, American Mathematical Society, 1996.