THE STEINBERG MODULE AND THE HECKE ALGEBRA

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1. Introduction

This note aims to give a self-contained exposition of the Steinberg module and the Hecke algebra for $GL_n(\mathbb{F}_p)$, aiming towards the applications in algebraic topology. We have tried to use methods that are elementary, or failing that, familiar to topologists.

I have not tried to investigate the history of these ideas but it seems likely that it goes something like this.

(a) People worked out how to solve problems by matrix calculations, including row-reduction.
(b) Thinking more abstractly, other people worked out the ideas presented here.
(c) Generalising further, people developed a still more abstract theory of Coxeter groups, BN pairs and so on.

There is plenty of current literature written at levels (a) and (c), but not so much at level (b). Everything that we say is well-known to those who have digested the more general theory, so our purpose is purely expository.

To avoid trivialities, we assume $n > 1$ unless otherwise stated.

2. Length of permutations

For any permutation $\sigma \in \Sigma_n$, we put

$$\begin{aligned} L(\sigma) &= \{(i,j) \mid 0 < i < j \leq n \text{ and } \sigma(i) > \sigma(j)\} \\ L^+(\sigma) &= L(\sigma) \amalg \{(i,i) \mid 0 < i \leq n\} \\ &= \{(i,j) \mid 0 < i \leq j \leq n \text{ and } \sigma(i) \geq \sigma(j)\} \\ l(\sigma) &= |L(\sigma)|. \end{aligned}$$

The integer $l(\sigma)$ is called the *length* of σ.

Example 2.1. For $\sigma = (1\ 3\ 5)(2\ 4)$ we have

$$L(\sigma) = \{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5),(4,5)\},$$

so $l(\sigma) = 7$.

It is sometimes convenient to reformulate this slightly. We put

$$\overline{L}(\sigma) = \{\{i,j\} \mid (i,j) \in L(\sigma)\}.$$

Equivalently, $\overline{L}(\sigma)$ is the set of subsets $T \subseteq \{1,\dots,n\}$ such that $|T| = 2$ and $\sigma\colon T \to \sigma T$ is order-reversing. As $i < j$ for all $(i,j) \in L(\sigma)$ we see that $\overline{L}(\sigma)$ bijects with $L(\sigma)$, so $|\overline{L}(\sigma)| = l(\sigma)$.

Remark 2.2. Put $\Delta = \prod_{i<j}(x_j - x_i) \in \mathbb{Z}[x_1,\dots,x_n]$. One can then see that for $\sigma \in \Sigma_n$ we have $\sigma.\Delta = (-1)^{l(\sigma)}\Delta$. Using this, we find that the map $\sigma \mapsto (-1)^{l(\sigma)}$ is a nontrivial homomorphism $\Sigma_n \to \{\pm 1\}$, which must therefore be the same as the signature.

Lemma 2.3. $l(\sigma^{-1}) = l(\sigma)$.

Proof. σ induces a bijection $L(\sigma) \to L(\sigma^{-1})$. □

Definition 2.4. For $i = 1,\dots,n-1$ we let s_i denote the transposition that exchanges i and $i+1$.

Proposition 2.5. *$l(\sigma)$ is the least r such that σ can be written in the form $s_{i_1}s_{i_2}\dots s_{i_r}$.*

Lemma 2.6. *For permutations $\sigma, \tau \in \Sigma$ we have*

$$\overline{L}(\sigma\tau) = \overline{L}(\tau)\Delta\tau_*^{-1}\overline{L}(\sigma)$$

(where $A\Delta B$ is the symmetric difference, $(A \cup B) \setminus (A \cap B)$.)

Proof. Consider a pair of permutations σ, τ. The composite

$$\{i,j\} \xrightarrow{\tau} \tau_*\{i,j\} = \{\tau(i), \tau(j)\} \xrightarrow{\sigma} \sigma_*\tau_*\{i,j\}$$

is order-reversing iff precisely one of the two composed maps is order-reversing. □

Lemma 2.7. *If $\sigma(k) < \sigma(k+1)$ then $l(\sigma s_k) = l(\sigma) + 1$, otherwise $l(\sigma s_k) = l(\sigma) - 1$.*

Proof. Now take $\tau = s_k$ in the previous lemma, so $\overline{L}(\tau) = \{\{k, k+1\}\}$. It follows that $l(\sigma s_k)$ is $l(\sigma) - 1$ if $\{k, k+1\} \in s_{k*}^{-1}\overline{L}(\sigma)$, and $l(\sigma) + 1$ otherwise. As $s_{k*}\{k, k+1\} = \{k, k+1\}$ we have $\{k, k+1\} \in s_{k*}^{-1}\overline{L}(\sigma)$ iff $\{k, k+1\} \in \overline{L}(\sigma)$ iff $\sigma(k) > \sigma(k+1)$, as required. □

Proof of Proposition 2.5. If $\sigma = s_{i_1}s_{i_2}\dots s_{i_r}$, then it is immediate from the lemma that $l(\sigma) \leq r$. Conversely, suppose we have $l(\sigma) = r$. If $r = 0$ then σ is order-preserving and so must be the identity, which is compatible with the claim in the proposition. If $r > 0$ then σ is not order-preserving, so there must exist k with $\sigma(k+1) < \sigma(k)$. The lemma tells us that $l(\sigma s_k) = r - 1$. We may thus assume inductively that $\sigma s_k = s_{i_1}s_{i_2}\dots s_{i_{r-1}}$ for some $i_1, \dots, i_{r-1}$. It follows that $\sigma = s_{i_1}\dots s_{i_{r-1}}s_k$, which is an expression of the required form. □

Definition 2.8. Consider a word $w = s_{i_1}s_{i_2}\dots s_{i_r}$ in the variables $s_1, \dots, s_{n-1}$. The corresponding permutation then has length at most r. If the length is precisely r, we say that w is *reduced*.

Definition 2.9. We write ρ for the permutation $\rho(i) = n+1-i$, which satisfies $\rho^2 = 1$. Note that $\rho(i) > \rho(j)$ iff $i < j$, so $l(\rho) = n(n-1)/2$.

Definition 2.10. For $m \leq k$ we put $t_m^k = s_m s_{m+1}\dots s_{k-1}$ (to be interpreted as the identity when $m = k$). In cycle notation, this is

$$t_m^k = (m,\ m+1,\ \dots, k-1,\ k).$$

Proposition 2.11. *For any $\sigma \in \Sigma_n$ there is a unique sequence $m_1, \dots, m_n$ with $1 \leq m_k \leq k$ and*

$$\sigma = t_{m_n}^n t_{m_{n-1}}^{n-1} \dots t_{m_2}^2 t_{m_1}^1.$$

Moreover, we have $l(\sigma) = \sum_{k=1}^n (k - m_k)$.

Proof. Put $m_n = \sigma(n)$ and $\tau = (t_{m_n}^n)^{-1}\sigma$. Then $\tau(n) = n$, so τ can be regarded as an element of Σ_{n-1}, so by induction we have

$$\tau = t_{m_{n-1}}^{n-1} t_{m_{n-2}}^{n-2} \dots t_{m_2}^2 t_{m_1}^1$$

for some $m_{n-1}, \dots, m_1$, and $l(\tau) = \sum_{k=1}^{n-1}(k - m_k)$. It follows that

$$\sigma = t_{m_n}^n \tau = t_{m_n}^n t_{m_{n-1}}^{n-1} \dots t_{m_2}^2 t_{m_1}^1.$$

We also have

$$\overline{L}(t_{m_n}^n) = \{\{i, n\} \mid m_n \leq i < n\},$$

so

$$\tau_*^{-1}\overline{L}(t_{m_n}^n) = \{\{\tau^{-1}(i), n\} \mid m_n \leq i < n\},$$

whereas $\overline{L}(\tau)$ contains no pairs of the form $\{i, n\}$. Thus $\overline{L}(\tau)$ and $\tau_*^{-1}\overline{L}(t_{m_n}^n)$ are disjoint, and

$$l(\sigma) = |\overline{L}(\tau)| + |\tau_*^{-1}\overline{L}(t_{m_n}^n)| = l(\tau) + (n - m_n) = \sum_{k=1}^n (k - m_k).$$

□

Definition 2.12. Let $\widetilde{\Sigma}_n$ be the group freely generated by symbols $s_1, \dots, s_{n-1}$ subject to the relations $s_i^2 = 1$ and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and $s_i s_j = s_j s_i$ whenever $|i - j| > 1$. It is straightforward to check that the corresponding relations hold in Σ_n, so there is a canonical map $\epsilon\colon \widetilde{\Sigma}_n \to \Sigma_n$ sending s_i to s_i. We will use the notation t_m^k for the element $s_m s_{m+1}\dots s_{k-1}$ in $\widetilde{\Sigma}_n$ as well as the corresponding element in Σ_n.

Proposition 2.13. *The map $\epsilon\colon \widetilde{\Sigma}_n \to \Sigma_n$ is an isomorphism.*

Proof. Let $X_n \subseteq \widetilde{\Sigma}_n$ be the set of elements of the form $t^n_{m_n} t^{n-1}_{m_{n-1}} \dots t^1_{m_1}$ with $1 \leq m_k \leq k$, so $|X_n| \leq n!$. We see from Proposition 2.11 that $\epsilon\colon X_n \to \Sigma_n$ is surjective, and hence bijective by a counting argument. It will therefore suffice to show that $X_n = \widetilde{\Sigma}_n$. Note that $X_n = \bigcup_{m=1}^n t^n_m X_{n-1}$, and we may assume by induction that $X_{n-1} = \widetilde{\Sigma}_{n-1}$, so X_n is closed under right multiplication by $\widetilde{\Sigma}_{n-1}$. As $\widetilde{\Sigma}_n$ is generated by $\widetilde{\Sigma}_{n-1}$ and s_{n-1}, it will suffice to show that X_n is closed under right multiplication by s_{n-1}. This can be deduced from Lemma 2.14 below, after noting that s_{n-1} commutes with t^k_m for all $k < n-1$. □

Lemma 2.14. *If $k \leq n$ and $l < n$ then in $\widetilde{\Sigma}_n$ we have*

$$t^n_k t^{n-1}_l s_{n-1} = t^n_k t^n_l = \begin{cases} t^n_{l+1} t^{n-1}_k & \text{for } k \leq l \\ t^n_l t^{n-1}_{k-1} & \text{for } k > l. \end{cases}$$

Proof. (a) It is immediate from the definitions that $t^n_k t^{n-1}_l s_{n-1} = t^n_k t^n_l$.

(b) We claim that for $l < n$ we have $(t^n_{l+1})^{-1} t^n_l = t^{n-1}_l (t^n_l)^{-1}$. We first give a proof for the case $l = 5$ and $n = 10$, writing k instead of s_k for brevity, and e for the identity permutation.

$$\begin{aligned}
(t^{10}_6)^{-1} t^{10}_5 &= 987656789 \\
&= 987565789 && (656 = 565) \\
&= 598767895 && ([5,7] = [5,8] = [5,9] = e) \\
&= 598676895 && (767 = 676) \\
&= 569878965 && ([6,8] = [6,9] = e) \\
&= 569787965 && (878 = 787) \\
&= 567989765 && ([7,9] = e) \\
&= 567898765 && (989 = 898) \\
&= t^9_5 (t^{10}_5)^{-1}
\end{aligned}$$

This pattern can be converted to a formal proof as follows. The claim is clear when $l = n-1$, so we can work by downwards induction on l. Consider the relation $s_{l+1} s_l s_{l+1} = s_l s_{l+1} s_l$. We multiply on the right by t^n_{l+2} and on the left by $(t^n_{l+2})^{-1}$, noting that $(t^n_{l+2})^{-1} s_{l+1} = (t^n_{l+1})^{-1}$ and $s_l s_{l+1} t^n_{l+2} = t^n_l$ and that s_l commutes with t^n_{l+2}. This gives

$$(t^n_{l+1})^{-1} t^n_l = s_l (t^n_{l+2})^{-1} s_{l+1} t^n_{l+2} s_l.$$

We can use the relation $s_{l+1} t^n_{l+2} = t^n_{l+1}$ and the induction hypothesis to convert the right hand side to $s_l t^{n-1}_{l+1} (t^n_{l+1})^{-1} s_l$, and from the definitions, this is the same as $t^{n-1}_l (t^n_l)^{-1}$.

(c) Now suppose that $k \leq l$. Note that $t^m_k = t^l_k t^m_l$ and that t^l_k commutes with t^m_{l+1}. We can therefore multiply the identity in (b) on the left by t^l_k to get $(t^n_{l+1})^{-1} t^n_k = t^{n-1}_k (t^n_l)^{-1}$. This can be rearranged to give the case $k \leq l$ of the lemma.

(d) Now suppose we have $n \geq i > j$. Take $k = j$ and $l = i-1$ in (c) to get $(t^n_i)^{-1} t^n_j = t^{n-1}_j (t^n_{i-1})^{-1}$. From the definitions, the right hand side can be rewritten as $t^n_j (t^{n-1}_{i-1})^{-1}$, and the equation can then be rearranged to give $t^n_i t^n_j = t^n_j t^{n-1}_{i-1}$. Up to a change of notation, this is the same as the second case of the lemma.

□

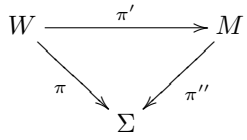
Definition 2.15. Let W_r be the set of words of length r in letters $s_1, \dots, s_n$, and put $W = \coprod_r W_r$, which is the free monoid generated by $s_1, \dots, s_n$. Let $\sim_r$ be the equivalence relation on W_r generated by the following rules:

- $u s_i s_j v \sim u s_j s_i v$ if $|i-j| > 1$
- $u s_i s_j s_i v = u s_j s_i s_j v$ if $|i-j| = 1$.

Put $M_r = W_r/\sim_r$ and $M = \coprod_r M_r$, so M is the quotient monoid of W by the relations $s_i s_j = s_j s_i$ (for $|i-j|>1$) and $s_i s_j s_i = s_j s_i s_j$ (for $|i-j|=1$). We then have

$$\Sigma = M/\langle s_i^2 = 1 \mid i = 1, \dots, n-1\rangle$$

Let



be the obvious projection maps. Recall that a word $w = s_{p_1} \cdots s_{p_r} \in W$ is *reduced* iff $l(\pi(w)) = r$. We write R_r for the set of reduced words of length r, and put $R = \coprod_r R_r$. Note that if $u \in R_r$ and $v \in W_r$ and $u \sim_r v$ then $v \in R_r$.

Theorem 2.16. *Let $u, v \in R_r$ be such that $\pi(u) = \pi(v)$; then $u \sim_r v$ (or equivalently $\pi'(u) = \pi'(v)$).*

The proof will be given after some preliminaries.

Definition 2.17. Given $\sigma, \tau \in \Sigma$, we say that σ *could end with* τ if the following equivalent conditions are satisfied:

(a) There exists $\rho \in \Sigma$ such that $\sigma = \rho\tau$ and $l(\sigma) = l(\rho) + l(\tau)$.
(b) $l(\sigma\tau^{-1}) = l(\sigma) - l(\tau)$.
(c) There are reduced words u, v such that $\pi(v) = \tau$ and $\pi(uv) = \sigma$.
(d) $\overline{L}(\tau) \subseteq \overline{L}(\sigma)$.

(It is straightforward to check that (a) to (c) are equivalent, and it follows from Lemma 2.6 that (a) is equivalent to (d).)

Lemma 2.18. *Suppose that $\sigma \in \Sigma$ and that σ could end with s_i, and also σ could end with s_j, where $j \neq i$.*

(a) *If $|i-j|>1$ then σ could end with $s_i s_j = s_j s_i$.*
(b) *If $|i-j|=1$ then σ could end with $s_i s_j s_i = s_j s_i s_j$.*

Proof. As σ could end with s_i we have $\overline{L}(s_i) = \{\{i, i{+}1\}\} \subseteq \overline{L}(\sigma)$, so $\sigma(i) > \sigma(i{+}1)$. Similarly $\sigma(j) > \sigma(j{+}1)$. If $|i-j|>1$ one checks that $\overline{L}(s_i s_j) = \{\{i, i+1\}, \{j, j+1\}\}$, so σ could end with $s_i s_j$. Suppose instead that $|i-j|=1$; we may assume without loss that $j = i+1$. We have $\sigma(i) > \sigma(i+1) = \sigma(j) > \sigma(j+1)$, so

$$\{\{i, i+1\}, \{i+1, i+2\}, \{i, i+2\}\} \subseteq \overline{L}(\sigma).$$

The permutation $\tau = s_i s_j s_i = s_j s_i s_j$ is then the 3-cycle $(i, i+1, i+2)$, so $\overline{L}(\tau) = \{\{i, i+1\}, \{i+1, i+2\}, \{i, i+2\}\}$, so σ could end with τ, as claimed. □

Proof. Proof of Theorem 2.16 We work by induction on r, noting that the cases $r = 0$ and $r = 1$ are easy. We may thus suppose that $r > 1$ and that $u = xs_i$, $v = ys_j$ for some $x, y \in R_{r-1}$ and $i, j \in \{1, \dots, n-1\}$. Put

$$\sigma = \pi(u) = \pi(v) = \pi(x)s_i = \pi(y)s_j.$$

If $i = j$ we see that $\pi(x) = \pi(y)$, so $x \sim y$ by the induction hypothesis, so $u = xs_i \sim ys_i = ys_j = v$ as required. If $|i-j|=1$ we see from Lemma 2.18 that there exists $z \in R_{r-3}$ with $\sigma = \pi(zs_i s_j s_i) = \pi(zs_j s_i s_j)$. The $i = j$ case now tells us that $u = xs_i \sim zs_i s_j s_i$ and $zs_j s_i s_j \sim ys_j = v$, and from the definitions we have $zs_i s_j s_i \sim zs_j s_i s_j$ so $u = v$. A very similar argument works when $|i-j|>1$. □

3. Standard notation

In some parts of our exposition, it will be convenient to work with an arbitrary finite-dimensional vector space W over $\mathbb{F}_p$. In others, it will be convenient to take $W = {\mathbb{F}_p}^n$. In that context, we let $e_1, \dots, e_n$ be the

standard basis, and put $E_i = \mathbb{F}_p\{e_1, \dots, e_i\}$. We let $G = GL_n(\mathbb{F}_p)$ be the automorphism group of ${\mathbb{F}_p}^n$, and put

$$\begin{aligned} T &= \{g \in G \mid ge_i \in \mathbb{F}_p e_i \text{for all } i\} \\ &= \{\text{invertible diagonal matrices }\} \\ B &= \{g \in G \mid gE_i = E_i \text{ for all } i\} \\ &= \{\text{invertible upper triangular matrices}\} \\ U &= \{g \in B \mid ge_i = e_i \pmod{E_{i-1}}\} \\ &= \{\text{upper unitriangular matrices}\}. \end{aligned}$$

Given $g \in G$ we let g_{ij} denote the element in the i'th row and j'th column of the corresponding matrix. If $n = 3$, a typical element of G then has the form

$$\begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}$$

With this convention, we have $(gh)_{ik} = \sum_j g_{ij}h_{jk}$ and $g.e_j = \sum_i g_{ij}e_i$ and $(g.x)_i = \sum_j g_{ij}x_j$. Moreover, we have

$$\begin{aligned} T &= \{g \in G \mid g_{ij} = 0 \text{ whenever } i \neq j\} \\ B &= \{g \in G \mid g_{ij} = 0 \text{ whenever } i > j\} \\ U &= \{g \in B \mid g_{ii} = 1 \text{ for all } i\}. \end{aligned}$$

We regard Σ_n as a subgroup of G in the usual way, so $\sigma.e_i = e_{\sigma(i)}$ and $(\sigma.x)_i = x_{\sigma^{-1}(i)}$. The corresponding matrix elements are $\sigma_{ij} = \delta_{i,\sigma(j)}$. Note that B fits in a split extension $U \to B \to T$, and we have

$$\begin{aligned} |T| &= (p-1)^n \\ |U| &= p^{n(n-1)/2} \\ |B| &= (p-1)^n p^{n(n-1)/2}. \end{aligned}$$

4. Jordan permutations

Let W be a vector space of dimension n over $\mathbb{F}_p$. We write $\mathrm{Flag}(W)$ for the set of complete flags $\underline{U} = (U_0 < U_1 < \ldots < U_n = W)$ (where necessarily $\dim(U_i) = i$ for all i). Now suppose we have two flags $\underline{U}, \underline{V} \in \mathrm{Flag}(W)$. For $0 < i, j \leq n$ we put

$$Q_{ij} = \frac{U_i \cap V_j}{(U_{i-1} \cap V_j) + (U_i \cap V_{j-1})}.$$

One checks that the natural map from this to

$$\frac{U_{i-1} + (U_i \cap V_j)}{U_{i-1} + (U_i \cap V_{j-1})}$$

is an isomorphism. Thus, the groups $Q_{i,1}, \dots, Q_{i,n}$ are the quotients in the filtration of the one-dimensional space U_i/U_{i-1} by the groups $(U_{i-1} + (U_i \cap V_j))/U_{i-1}$. It follows that for each i there is a unique index $j = \sigma(i)$ such that $Q_{i,\sigma(i)} \neq 0$. Similarly, for each j, there is a unique $i = \tau(j)$ such that $Q_{\tau(j),j} \neq 0$. It follows that σ and τ are inverse to each other, so both lie in Σ_n. We write $\delta(\underline{U}, \underline{V})$ for σ, and note that $\delta(\underline{V}, \underline{U}) = \delta(\underline{U}, \underline{V})^{-1}$.

Example 4.1. We claim that $\delta(\sigma\underline{E}, \underline{E}) = \sigma$. This is more or less clear except perhaps for the fact that it is σ and not σ^{-1}. To check this we put

$$U_i = \sigma E_i = \mathrm{span}\{e_{\sigma(1)}, \dots, e_{\sigma(i)}\} = \mathrm{span}\{e_k \mid \sigma^{-1}(k) \leq i\}$$

and $V_j = E_j = \mathrm{span}\{e_1, \dots, e_j\}$, and we consider the quotients

$$Q_{ij} = \frac{U_i \cap V_j}{(U_{i-1} \cap V_j) + (U_i \cap V_{j-1})}$$

Put $A_i = \{e_k \mid \sigma^{-1}(k) \leq i\}$ and $B_j = \{e_k \mid k \leq j\}$. Then $A_i \cap B_j$ is a basis for $U_i \cap V_j$, so the set

$$C_{ij} = (A_i \cap B_j) \setminus ((A_{i-1} \cap B_j) \cup (A_i \cap B_{j-1})$$

is a basis for Q_{ij}. However, we have

$$C_{ij} = (A_i \setminus A_{i-1}) \cap (B_j \setminus B_{j-1}) = \{e_{\sigma(i)}\} \cap \{e_j\},$$

so $Q_{ij} = 0$ unless $j = \sigma(i)$.

Example 4.2. Take $W = {\mathbb{F}_p}^5$. Given elements $a, \dots, g \in \mathbb{F}_p$, put

$$u_1 = \begin{bmatrix} a \\ b \\ c \\ d \\ 1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} e \\ f \\ g \\ 1 \\ 0 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad u_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad u_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then Q_{15} is spanned by u_1, Q_{24} is spanned by u_2, Q_{31} is spanned by u_3, Q_{42} is spanned by u_4, Q_{53} is spanned by u_5, and all other Q_{ij}'s are zero. It follows that $\delta(\underline{U}, \underline{E}) = (1\ 5\ 3)(2\ 4)$.

Remark 4.3. As the definition of $\delta(\underline{U}, \underline{V})$ is completely natural, we have $\delta(g\underline{U}, g\underline{V}) = \delta(\underline{U}, \underline{V})$ for all $\underline{U}, \underline{V} \in \mathrm{Flag}(W)$ and $g \in \mathrm{Aut}(W)$.

Remark 4.4. It is not true in general that

$$\delta(\underline{T}, \underline{V}) = \delta(\underline{T}, \underline{U})\delta(\underline{U}, \underline{V}).$$

Indeed, if $n = 2$ then $\mathrm{Flag}(V)$ is natually identified with the set PV of one-dimensional subspaces of V. If we let ρ be the nontrivial element of Σ_2, then $\delta(L, L) = 1$, and $\delta(L, M) = \rho$ whenever $L \neq M$. It follows that if L, M and N are all distinct, then

$$\delta(L, N) \neq \delta(L, M)\delta(M, N).$$

We now consider the cases where $\delta(\underline{U}, \underline{V})$ is the identity or one of the adjacent transpositions s_i. These could also be extracted from the more general theory in the next section but it is instructive to treat them directly.

Lemma 4.5. *Suppose we have flags $\underline{U}, \underline{V}$ with $\delta(\underline{U}, \underline{V}) = \sigma$. If $U_{i-1} = V_{i-1}$ then $\sigma(i) = i$ iff $U_i = V_i$.*

Proof. Put $A = U_{i-1} = V_{i-1}$, so $\dim(A) = i - 1$. We have $U_{i-1} \cap V_i = V_{i-1} \cap V_i = V_{i-1} = A$ and similarly $U_i \cap V_{i-1} = A$ so $Q_{ii} = (U_i \cap V_i)/A$. Thus $\sigma(i) = i$ iff $Q_{ii} \neq 0$ iff $U_i \cap V_i > A$. As U_i and V_i have dimension i and A has dimension $i - 1$, we have $U_i \cap V_i > A$ iff $U_i = V_i$. □

Corollary 4.6. *If $\delta(\underline{U}, \underline{V}) = 1$ then $\underline{U} = \underline{V}$.* □

Proposition 4.7. *We have $\delta(\underline{U}, \underline{V}) = s_i$ iff $U_j = V_j$ for all $j \neq i$ but $U_i \neq V_i$.*

Proof. Suppose that $\delta(\underline{U}, \underline{V}) = s_i$. Lemma 4.5 tells us that $U_j = V_j$ for $j < i$, but $U_i \neq V_i$. Put $A = U_{i-1} = V_{i-1}$. As $U_i \neq V_i$ we see that $U_i \cap V_i$ must be strictly smaller than U_i, but it contains A, which has codimension one in U_i, so $U_i \cap V_i = A$. Using this we find that $Q_{i,i+1} = (U_i \cap V_{i+1})/A$. As $\delta(\underline{U}, \underline{V}) = s_i$ we have $Q_{i,i+1} \neq 0$, so $\dim(U_i \cap V_{i+1}) \geq \dim(A) + 1 = i$, but $\dim(U_i) = i$ so $U_i \cap V_{i+1} = U_i$, so $U_i < V_{i+1}$. Of course we also $V_i < V_{i+1}$ and

$$\dim(U_i + V_i) = \dim(U_i) + \dim(V_i) - \dim(U_i \cap V_i) = i + i - (i - 1) = i + 1 = \dim(V_{i+1}),$$

so $V_{i+1} = U_i + V_i$. Symmetrically, we have $U_{i+1} = U_i + V_i$. For $j > i + 1$ we have $s_i(j) = j$, so we can use Lemma 4.5 to show that $U_j = V_j$ for all such j.

We leave the converse to the reader. □

5. Schubert cells

Given a flag $\underline{V} \in \mathrm{Flag}(W)$ and a permutation $\sigma \in \Sigma_n$ we put

$$Y = Y(\sigma, \underline{V}) = \{\underline{U} \mid \delta(\underline{U}, \underline{V}) = \sigma\} \subset \mathrm{Flag}(W).$$

We write $Y(\sigma)$ for $Y(\sigma, \underline{E}) \subset \mathrm{Flag}({\mathbb{F}_p}^n)$. We also put

$$X = X(\sigma) = U \cap U^{(\sigma\rho)^{-1}},$$

where $\rho(i) = n + 1 - i$ as in Definition 2.9.

Lemma 5.1. *A matrix $g = (g_{ij})_{i,j=1}^n \in M_n(\mathbb{F}_p)$ lies in $X(\sigma)$ iff we have*

$$g_{ij} = \begin{cases} 1 & \text{if } i = j \\ \text{arbitrary} & \text{if } (i,j) \in L(\sigma^{-1}) \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we have $|X(\sigma)| = p^{l(\sigma)}$.

Proof. We have $g \in U$ iff $g_{ii} = 1$, and $g_{ij} = 0$ whenever $i > j$. Next, we have $g \in U^\tau$ iff $\tau g \tau^{-1} \in U$ iff $a_{ij} = 0$ whenever $\tau(i) > \tau(j)$. In particular, we have $g \in U^{(\sigma\rho)^{-1}} = U^{\rho\sigma^{-1}}$ iff $g_{ij} = 0$ whenever $\rho\sigma^{-1}(i) > \rho\sigma^{-1}(j)$, or equivalently $\sigma^{-1}(i) < \sigma^{-1}(j)$. The claim follows. □

Proposition 5.2. *The map $\phi\colon g \mapsto g\sigma\underline{E}$ gives a bijection $X(\sigma) \to Y(\sigma)$. In particular, we have $|Y(\sigma)| = p^{l(\sigma)}$, and thus $|Y(\sigma, \underline{V})| = p^{l(\sigma)}$ for any $\underline{V}$.*

Proof. First note that $X(\sigma) \subseteq U \subseteq B$, so for $g \in X(\sigma)$ we have $g^{-1}\underline{E} = \underline{E}$, so

$$\delta(g\sigma\underline{E}, \underline{E}) = \delta(\sigma\underline{E}, g^{-1}\underline{E}) = \delta(\sigma\underline{E}, \underline{E}) = \sigma,$$

so the flag $\phi(g) = g\sigma\underline{E}$ lies in $Y(\sigma)$ as claimed.

Next, observe that the stabiliser of $\sigma\underline{E}$ in G is $B^{\sigma^{-1}}$, so the stabiliser in $X(\sigma) = U \cap U^{\rho\sigma^{-1}}$ is contained in the group

$$H = B^{\sigma^{-1}} \cap U^{\rho\sigma^{-1}} = (B \cap U^\rho)^{\sigma^{-1}}.$$

Now B consists of upper triangular matrices, and U^ρ consists of lower unitriangular matrices, so $B \cap U^\rho = 1$, so $H = 1$. It follows that $X(\sigma)$ acts freely on $\sigma\underline{E}$, so $\phi\colon X(\sigma) \to Y(\sigma)$ is injective.

Now put

$$T_i = \mathbb{F}_p\{e_m \mid m \leq \sigma(i),\ \sigma^{-1}(m) \geq i\} \leq E_{\sigma(i)}.$$

Let $\epsilon_i\colon {\mathbb{F}_p}^n \to \mathbb{F}_p$ be the i'th coordinate projection.

Consider a flag $\underline{V} \in Y(\sigma)$. We claim that there is a unique element $v_i \in V_i \cap T_i$ such that $\epsilon_{\sigma(i)}(v_i) = 1$, and moreover that $v_1, \dots, v_i$ is a basis for V_i over $\mathbb{F}_p$. We will prove this by induction on i, so we assume that the corresponding fact holds for all $j < i$. Put

$$S_i = \mathbb{F}_p\{v_j \mid j < i \text{ and } \sigma(j) < \sigma(i)\} \leq V_i \cap E_{\sigma(i)}$$

The leading terms of the vectors v_j in S_i are precisely the vectors e_m with $m < \sigma(i)$ but $i > \sigma^{-1}(m)$. Using this, we see that $E_{\sigma(i)} = S_i \oplus T_i$, and thus that $V_i \cap E_{\sigma(i)} = S_i \oplus L_i$ for some (unique) subspace $L_i \leq T_i$. We next claim that $S_i = V_{i-1} \cap E_{\sigma(i)}$. This is straightforward, given that $\{v_1, \dots, v_{i-1}\}$ is a basis for V_{i-1} and the leading term in v_j is $e_{\sigma(j)}$. It follows that the space

$$L_i \simeq (V_i \cap E_{\sigma(i)})/(V_{i-1} \cap E_{\sigma(i)})$$

has dimension at most one. On the other hand, because $\delta(\underline{V}, \underline{E}) = \sigma$ we see that $\epsilon_{\sigma(i)}\colon L_i \to \mathbb{F}_p$ must be surjective, so there is a unique element v_i as described.

Now define $g\colon {\mathbb{F}_p}^n \to {\mathbb{F}_p}^n$ by $g(e_i) = v_{\sigma^{-1}(i)}$, so $g\sigma(e_i) = v_i$, so $g\sigma(\underline{E}) = \underline{V}$. As the leading term of $v_{\sigma^{-1}(i)}$ is e_i, we have $g \in U$. We also have

$$g\sigma(e_k) = v_k \in T_k \leq \mathbb{F}_p\{e_m \mid \sigma^{-1}(m) \geq k\} = \mathbb{F}_p\{e_{\sigma(k)}, e_{\sigma(k+1)}, \dots, e_{\sigma(m)}\},$$

so $\sigma^{-1}g\sigma$ is a lower-triangular matrix, so $\rho\sigma^{-1}g\sigma\rho$ is upper-triangular, so $g \in B^{\rho\sigma^{-1}}$. We also know that the diagonal entries in g are all equal to 1, so the same is true of the diagonal entries in $\rho\sigma^{-1}g\sigma\rho$, so $g \in U^{\rho\sigma^{-1}}$. This means that $g \in X(\sigma)$, and $\phi(g) = \underline{V}$. We conclude that ϕ is surjective as well as injective. □

Example 5.3. Consider the permutation $\sigma = (1\ 5\ 3)(2\ 4) \in \Sigma_5$, corresponding to the matrix

$$\sigma = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then

$$L(\sigma^{-1}) = \{(1,4), (1,5), (2,4), (2,5), (3,4), (3,5), (4,5)\},$$

so $X(\sigma)$ is the set of matrices of the form

$$g = \begin{bmatrix} 1 & 0 & 0 & b & a \\ 0 & 1 & 0 & d & c \\ 0 & 0 & 1 & f & e \\ 0 & 0 & 0 & 1 & g \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For such g, we have

$$g\sigma = \begin{bmatrix} a & e & 1 & 0 & 0 \\ b & f & 0 & 1 & 0 \\ c & g & 0 & 0 & 1 \\ d & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The columns of this are the vectors u_i as in Example 4.2, so $\phi(g)$ is the flag $\underline{U}$ considered in Example 4.2.

Corollary 5.4. *Consider triples $(V, \underline{U}, \underline{W})$ where V is an n-dimensional vector space over $\mathbb{F}_p$ and $\underline{U}$ and $\underline{W}$ are complete flags in V. Then such triples are classified up to isomorphism by the invariant $\delta(\underline{U}, \underline{W}) \in \Sigma$. In particular, if $\underline{U}, \underline{W}, \underline{W}' \in \mathrm{Flag}(V)$ then there is an automorphism $g \in \mathrm{Aut}(V)$ with $g\underline{U} = \underline{U}$ and $g\underline{W} = \underline{W}'$ if and only iff $\delta(\underline{U}, \underline{W}) = \delta(\underline{U}, \underline{W}')$.*

Proof. Let $(V, \underline{U}, \underline{W})$ be a pair as above, and put $\sigma = \delta(\underline{U}, \underline{W})$. It will suffice to show that $(V, \underline{U}, \underline{W}) \simeq (\mathbb{F}_p{}^n, \sigma\underline{E}, \underline{E})$. Choose elements $w_i \in W_i \setminus W_{i-1}$ and define $f\colon \mathbb{F}_p{}^n \to V$ by $f(a) = \sum_i a_i w_i$. This gives an isomorphism $\mathbb{F}_p{}^n \to V$ sending $\underline{E}$ to $\underline{W}$. This must send some other flag $\underline{F}$ to $\underline{U}$. By naturality we have $\delta(\underline{F}, \underline{E}) = \delta(\underline{U}, \underline{W}) = \sigma$, so $\underline{F} \in Y(\sigma)$, so $\underline{F} = x\sigma\underline{E}$ for some $x \in X(\sigma) \leq B$. The map x^{-1} now gives an isomorphism $(\mathbb{F}_p{}^n, \underline{F}, \underline{E}) \to (\mathbb{F}_p{}^n, \sigma\underline{E}, \underline{E})$. □

Corollary 5.5. *For a triple $(V, \underline{U}, \underline{W})$ as above, we have $\delta(\underline{U}, \underline{W}) = \sigma$ iff there exists a basis $v_1, \dots, v_n$ such that for all i we have*

$$U_i = \mathrm{span}\{v_{\sigma(1)}, \dots, v_{\sigma(i)}\}$$
$$W_i = \mathrm{span}\{v_1, \dots, v_i\}.$$

If so, then the number of such bases is $(p-1)^n p^{l(\sigma^{-1}\rho)}$.

Proof. A basis is the same thing as an isomorphism $\mathbb{F}_p{}^n \to V$; a basis with properties as above is the same as an isomorphism $(\mathbb{F}_p{}^n, \sigma\underline{E}, \underline{E}) \to (V, \underline{U}, \underline{W})$. Thus, such bases exist iff $\delta(\underline{U}, \underline{W}) = \sigma$, and if so, the number of such bases is $|\mathrm{Aut}(\mathbb{F}_p{}^n, \sigma\underline{E}, \underline{E})|$. This automorphism group is just $B \cap B^{\sigma^{-1}}$, which is conjugate to $B^\sigma \cap B$. This fits into a short exact sequence

$$X(\sigma^{-1}\rho) = U^\sigma \cap U \to B^\sigma \cap B \to T,$$

so $|B^\sigma \cap B| = |T||X(\sigma^{-1}\rho)| = (p-1)^n p^{l(\sigma^{-1}\rho)}$ as claimed. □

6. The Bruhat decomposition

Proposition 6.1. *We have $G = \coprod_{\sigma \in \Sigma_n} B\sigma B$. Moreover, for each $\sigma \in \Sigma_n$ we have a bijection*

$$X(\sigma) \times B = (U \cap U^{\rho\sigma^{-1}}) \times B \to B\sigma B$$

given by $(g, b) \mapsto g\sigma b$. We thus have

$$|B\sigma B| = p^{l(\sigma)}|B| = p^{l(\sigma)+n(n-1)/2}(p-1)^n.$$

Proof. Given $g \in G$, put $\pi(g) = \delta(g\underline{E}, \underline{E}) \in \Sigma_n$. If $b, b' \in B$ then $b\underline{E} = \underline{E} = b'\underline{E}$, so

$$\pi(b^{-1}gb') = \delta(b^{-1}gb'\underline{E}, \underline{E}) = \delta(gb'\underline{E}, b\underline{E}) = \delta(g\underline{E}, \underline{E}) = \pi(g).$$

Moreover, if $\sigma \in \Sigma_n \leq G$ then $\sigma E_i = \mathbb{F}_p\{e_{\sigma(j)} \mid j \leq i\}$, from which it follows directly that $\pi(\sigma) = \sigma$. This means that $\pi(B\sigma B) = \{\sigma\}$.

Conversely, suppose that $\pi(h) = \sigma$. Propsition 5.2 tells us that there is a unique element $g \in X(\sigma)$ such that $g\sigma\underline{E} = h\underline{E}$. If we put $b = (g\sigma)^{-1}h$ we find that $b \in B$ and $h = g\sigma b$. In particular, this shows that $h \in B\sigma B$, so $\pi^{-1}\{\sigma\} = B\sigma B$, so $G = \coprod_{\sigma \in \Sigma_n} B\sigma B$. We also see that our map $X(\sigma) \times B \to B\sigma B$ is surjective. To see that it is also injective, suppose we have $h = g'\sigma b'$ for some other $g' \in X(\sigma)$ and $b' \in B$. As $b\underline{E} = \underline{E} = b'\underline{E}$ we see that $g\sigma\underline{E} = g'\sigma\underline{E}$. Proposition 5.2 now tells us that $g = g'$, and as $g\sigma b = g'\sigma b'$ we also have $b = b'$. □

Corollary 6.2. $|B \cap B^{\rho\sigma^{-1}}| = |B\sigma B|/|U|$.

Proof. We have $B = T \times U$ and $T^\tau = T$ for all $\tau \in \Sigma_n$, so

$$|B \cap B^{\rho\sigma^{-1}}| = |T|.|U \cap U^{\rho\sigma^{-1}}| = |T|.p^{l(\sigma)}.$$

On the other hand,

$$|B\sigma B| = p^{l(\sigma)}|B| = |U||T|p^{l(\sigma)},$$

and the claim follows easily. □

Corollary 6.3. *For each $\sigma \in \Sigma_n$ we have a bijection*

$$B \times X(\sigma^{-1}) = B \times (U \cap U^{\rho\sigma}) \to B\sigma B$$

given by $(b, g) \mapsto b\sigma g$.

Proof. We have a bijection $\phi\colon X(\sigma^{-1}) \times B \to B\sigma^{-1}B$ given by $\phi(g, b) = g\sigma b$. Define $\chi\colon G \to G$ by $\chi(g) = g^{-1}$. As B and $X(\sigma^{-1})$ are groups, they are preserved by χ. We also have $\chi(B\sigma^{-1}B) = B\sigma B$. We therefore have a bijection $B \times X(\sigma^{-1}) \to B\sigma B$ given by

$$(b, g) \mapsto \chi(\phi(\chi(g), \chi(b))) = (g^{-1}\sigma^{-1}b^{-1})^{-1} = b\sigma g.$$

□

Proposition 6.4. *If $l(\sigma\tau) = l(\sigma) + l(\tau)$ then there is a bijection $\phi\colon X(\sigma) \times X(\tau) \to X(\sigma\tau)$ given by $\phi(g, h) = g\, h^{\sigma^{-1}} = g\sigma h\sigma^{-1}$.*

Proof. Recall first that the condition $l(\sigma\tau) = l(\sigma)+l(\tau)$ is equivalent to the following: for any $T \subseteq \{1, \dots, n\}$ with $|T| = 2$, at most one of the two maps

$$T \xrightarrow{\tau} \tau(T) \xrightarrow{\sigma} \sigma\tau(T)$$

is order-reversing.

We now show that $\phi(g, h) \in X(\sigma\tau)$, or equivalently that $g\, h^{\sigma^{-1}} \in U \cap U^{\rho\tau^{-1}\sigma^{-1}}$. We are given that $g \in U$ and $h \in U^{\rho\tau^{-1}}$, so it will suffice to show that $h^{\sigma^{-1}} \in U$ and $g \in U^{\rho\tau^{-1}\sigma^{-1}}$. For the first of these, recall that the (i, j)'th matrix element in $h^{\sigma^{-1}}$ is $h_{\sigma^{-1}(i),\sigma^{-1}(j)}$. Thus, if $h^{\sigma^{-1}} \notin U$, we have some $i > j$ with $h_{\sigma^{-1}(i),\sigma^{-1}(j)} \neq 0$. As $h \in X(\tau)$ we must have $\sigma^{-1}(i) < \sigma^{-1}(j)$ and $\tau^{-1}\sigma^{-1}(i) > \tau^{-1}\sigma^{-1}(j)$. This means that both the maps

$$\{\tau^{-1}\sigma^{-1}(i), \tau^{-1}\sigma^{-1}(j)\} \xrightarrow{\tau} \{\sigma^{-1}(i), \sigma^{-1}(j)\} \xrightarrow{\sigma} \{i, j\}$$

are order-reversing, contrary to our hypothesis; so $h^{\sigma^{-1}} \in U$ after all. For the second claim, suppose that $g \notin U^{\rho\tau^{-1}\sigma^{-1}}$, so $g^{\sigma\tau} \notin U^\rho$. Then there must exist $i < j$ with $g_{\sigma\tau(i),\sigma\tau(j)} \neq 0$. As $g \in X(\sigma)$, this means that $\sigma\tau(i) < \sigma\tau(j)$ and $\sigma^{-1}\sigma\tau(i) > \sigma^{-1}\sigma\tau(j)$, or equivalently $\tau(i) > \tau(j)$. This means that both the maps

$$\{i, j\} \xrightarrow{\tau} \{\tau(i), \tau(j)\} \xrightarrow{\sigma} \{\sigma\tau(i), \sigma\tau(j)\}$$

are order-reversing, which again gives a contradiction. This completes the proof that $\phi(g, h) \in X(\sigma\tau)$.

We now show that ϕ is surjective. Consider an arbitrary element $k \in X(\sigma\tau)$. Then $k \in B$, so $k\sigma \in B\sigma B$. Corollary 6.1 tells us that there is a unique pair $(g, b) \in X(\sigma) \times B$ with $g\sigma b = k\sigma$. Similarly, we have $b\tau \in B\tau B$, so there is a unique $(h, c) \in X(\tau) \times B$ with $b\tau = h\tau c$. It follows that

$$k\sigma\tau = g\sigma b\tau = g\sigma h\tau c = \phi(g, h)\sigma\tau c$$

We can now apply Corollary 6.1 to the permutation $\sigma\tau$ to deduce that $k = \phi(g, h)$ and $c = 1$. This shows that ϕ is surjective, but

$$|X(\sigma\tau)| = p^{l(\sigma\tau)} = p^{l(\sigma)+l(\tau)} = |X(\sigma) \times X(\tau)|,$$

so ϕ is actually a bijection. □

Proposition 6.5. *If $l(\sigma\tau) = l(\sigma) + l(\tau)$ then $B\sigma B\tau B = B\sigma\tau B$.*

Proof. It is clear that $B\sigma\tau B \subseteq B\sigma B\tau B$, so it will suffice to show that

$$|B\sigma B\tau B| \leq |B\sigma\tau B| = |B|p^{l(\sigma\tau)} = |B|p^{l(\sigma)}p^{l(\tau)}.$$

For this we note that $B\sigma B = X(\sigma)\sigma B$ and $B\tau B = B\tau X(\tau^{-1})$ so

$$B\sigma B\tau B = X(\sigma)\sigma B\tau X(\tau^{-1}).$$

As $|X(\sigma)| = p^{l(\sigma)}$ and $|X(\tau^{-1})| = p^{l(\tau^{-1})} = p^{l(\tau)}$, this gives the required inequality. □

7. Parabolic subgroups

Definition 7.1. For any set $I \subseteq \{1, \dots, n-1\}$, we put $\Sigma_I = \langle s_i \mid i \in I\rangle \leq \Sigma_n$. Suppose that $\{0, \dots, n\} \setminus I = \{i_0, i_1, \dots, i_r\}$ with $0 = i_0 < i_1 < \cdots < i_r = n$. Then Σ_I preserves the sets,

$$(0, i_1], (i_1, i_2], \dots, (i_{r-1}, i_r].$$

In fact, it is the largest subgroup of Σ_n that preserves these sets, so

$$\Sigma_I \simeq \Sigma_{i_1} \times \Sigma_{i_2-i_1} \times \cdots \times \Sigma_{i_r-i_{r-1}}.$$

Definition 7.2. Next, for any n-dimensional vector space V over $\mathbb{F}_p$ we let $\mathrm{Flag}_I(V)$ be the set of flags

$$\underline{W} = (0 = W_{i_0} < W_{i_1} < W_{i_2} < \cdots < W_{i_r} = V)$$

with $\dim(W_j) = j$ for all $j \in I^c$. This gives a functor from n-dimensional vector spaces to sets.

In the case $V = {\mathbb{F}_p}^n$ we have an obvious flag $\underline{E} \in \mathrm{Flag}_I({\mathbb{F}_p}^n)$ given by $E_j = \mathrm{span}\{e_1, \dots, e_j\}$ for all $j \in I^c$. We let P_I denote the stabiliser of this flag. We also write P_i for $P_{\{i\}}$ and P_{ij} for $P_{\{i,j\}}$, and so on. The subgroups P_I are called *standard parabolic subgroups* of G. More generally a subgroup $P \leq G$ is *parabolic* if it contains a conjugate of B.

For example, we have $P_\emptyset = B$ and $P_{\{1,\dots,n-1\}} = G$. In the case $n = 7$ and $I = \{1, 3, 4, 6\}$, the group P_I consists of invertible matrices of the following shape:

$$\left[\begin{array}{cc|ccc|cc} * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ \hline 0 & 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * \\ \hline 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * \end{array}\right]$$

Here $I^c = \{0, 2, 5, 7\}$, and the sizes of the diagonal blocks are $2 - 0$, $5 - 2$ and $7 - 5$.

Proposition 7.3. *$P_I = B\Sigma_I B$.*

Proof. Firstly, it is clear from the definitions that $\Sigma_I \leq P_I$ and $B \leq P_I$ so $B\Sigma_I B \subseteq P_I$. Conversely, suppose that $g \in P_I$ and put $U_j = gE_j$, so for $a \in I^c$ we have $U_a = E_a$. Let σ be the permutation such that $g \in B\sigma B$. Recall from Section 6 that this is characterised by characterised by $Q_{i,\sigma(i)} \neq 0$, where

$$Q_{ij} = (U_i \cap E_j)/((U_{i-1} \cap E_j) + (U_i \cap E_{j-1})).$$

Suppose we have $a < i \leq b$, with $a, b \in I^c$. We claim that $a < \sigma(i) \leq b$, or in other words that $Q_{ij} = 0$ for $j \leq a$ or $j > b$. Indeed, if $j \leq a$ then $E_j \leq E_a = U_a \leq U_{i-1}$, so $(U_i \cap E_j) \leq (U_{i-1} \cap E_j)$, so $Q_{ij} = 0$. Similarly, if $j > b$ then $E_{j-1} \geq E_b = U_b \geq U_i$, so $(U_i \cap E_j) \leq (U_i \cap E_{j-1})$, so $Q_{ij} = 0$. This shows that σ preserves the interval $(a, b]$ as claimed, for all $a, b \in I^c$. It follows that $\sigma \in \Sigma_I$. □

Lemma 7.4. *If $\sigma(k) > \sigma(k+1)$ then $\sigma \in B\sigma Bs_kB$.*

Proof. Define $g \in G$ by $g(e_k) = e_k + e_{k+1}$ and $g(e_i) = e_i$ for $i \neq k$. We claim that $g \in Bs_kB$. In the case $n = 2$ and $k = 1$ this follows from the equation

$$\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right].$$

The general case follows by taking block sums with suitable identity matrices. Next, we claim that the element $g = \sigma g \sigma^{-1}$ lies in B. Indeed, we have $b(e_{\sigma(k)}) = e_{\sigma(k+1)}$ and $b(e_i) = e_i$ for all other i, so the claim follows from the fact that $\sigma(k+1) < \sigma(k)$. We now have $\sigma = b^{-1}\sigma g \in B\sigma g$ and $g \in Bs_kB$ so $\sigma \in B\sigma Bs_kB$ as claimed. □

Proposition 7.5. *Let P be a subgroup of G such that $P \geq B$. Then $P = P_I$ for some I (so P is a standard parabolic subgroup).*

Proof. Put $I = \{i \mid s_i \in P\}$, so $P_I \leq P$. Suppose that $g \in P$, and put $\sigma = \pi(g)$. As $BgB = B\sigma B$ and $B \leq P_I \leq P$ we have $\sigma \in P$, and $g \in P_I$ iff $\sigma \in P_I$. If $l(\sigma) = 0$ then $\sigma = 1 \in P$. If $l(\sigma) = 1$ then $\sigma = s_i$ for some i and $\sigma \in P$ so $i \in I$ so $\sigma \in P_I$. Now suppose that $l(\sigma) > 1$, so we have $\sigma(k) > \sigma(k+1)$ for some k. Lemma 7.4 tells us that $\sigma = b\sigma c s_k d$ for some $b, c, d \in B$. As $\sigma, b, c, d \in P$ we conclude that $s_k \in P$, so $k \in I$, so $s_k \in P_I$. It also follows that the element $\tau = \sigma s_k$ lies in P. As $\sigma(k+1) < \sigma(k)$ we have $l(\tau) = l(\sigma) - 1$, so we can assume by induction that $\tau \in P_I$. It follows that $\sigma = \tau s_k \in P_I$ and thus that $g \in P_I$. □

Corollary 7.6. *Every parabolic subgroup is conjugate to a standard parabolic subgroup.*

Proposition 7.7. *If $I \subseteq J$ then $\mathrm{Map}_G(\mathrm{Flag}_I(V), \mathrm{Flag}_J(V))$ is a singleton; otherwise, it is empty.*

Proof. We first observe that the orbits of B in V are precisely the sets $E_k \setminus E_{k-1}$, and thus that the spaces E_k are the only B-invariant subspace of V. This means that the point $\underline{E}_I \in \mathrm{Flag}_I(V)$ (which we call the *basepoint*) is the unique B-fixed point in $\mathrm{Flag}_I(V)$. Thus, any map from $\mathrm{Flag}_I(V)$ to $\mathrm{Flag}_J(V)$ must send the basepoint to the basepoint, and this determines it uniquely because G acts transitively on $\mathrm{Flag}_I(V)$. As the stabilisers of the basepoints are P_I and P_J, there is a unique map when $P_I \leq P_J$, and no maps otherwise. It is also clear that $P_I \leq P_J$ iff $I \subseteq J$. □

Definition 7.8. We say that a finite G-set X is *parabolic* if every point has parabolic isotropy group. We write $\mathcal{P}$ for the category of parabolic finite G-sets and equivariant maps. We also write $\mathcal{P}'$ for the category of finite sets Y equipped with a list $(Y_1, \dots, Y_{n-1})$ of subsets. We define a functor $F\colon \mathcal{P} \to \mathcal{P}'$ by $FX = X^B$, equipped with the subsets $F_i X = X^{P_i}$ for $i = 1, \dots, n-1$.

$$FX = (X^B; X^{P_1}, \dots, X^{P_{n-1}}).$$

Proposition 7.9. *$F\colon \mathcal{P} \to \mathcal{P}'$ is an equivalence of categories.*

Proof. Consider an object $Y \in \mathcal{P}'$. For $y \in Y$ we put $I_y = \{i \mid y \in Y_i\}$. We then put $F'Y = \coprod_{y \in Y} G/P_{I_y} \in \mathcal{P}$. Now consider a morphism $f\colon Y \to Z$ in $\mathcal{P}'$. As $f(Y_i) \subseteq Z_i$ we have $I_y \subseteq I_{f(y)}$, so there is a unique G-map $G/P_{I_y} \to G/P_{I_{f(y)}} \subseteq F'Z$. Putting these together, we get a map $F'f\colon F'Y \to F'Z$, and this gives us a functor $\mathcal{P}' \to \mathcal{P}$. Note that

$$(G/P_I)^{P_i} = \mathrm{Map}_G(\mathrm{Flag}_i(V), \mathrm{Flag}_I(V)) = \begin{cases} 1 & \text{if } i \in I \\ \emptyset & \text{otherwise.} \end{cases}$$

Using this we see that $FF' = 1_{\mathcal{P}'}$. Next, consider an object $X \in \mathcal{P}$. This can be written as a disjoint union of orbits, each of which is isomorphic to G/P for some parabolic P. We then note that P is conjugate to P_I for some I, so G/P is isomorphic to G/P_I. The only B-fixed point in G/P_I is the basepoint, and the basepoint is fixed by P_i iff $i \in I$. It now follows that $X = F'FX$, and thus that F is an equivalence as claimed. □

8. The Hecke algebra

Let $\mathcal{V}$ be the category of n-dimensional vector spaces over $\mathbb{F}_p$, with linear isomorphisms as the morphisms. Let $\mathcal{F}$ be the category of finite sets, and let $\mathcal{A}$ be the category of finitely generated free $\mathbb{Z}_{(p)}$-modules. We write $\mathcal{V}\mathcal{F}$ and $\mathcal{V}\mathcal{A}$ for the functor categories $[\mathcal{V}, \mathcal{F}]$ and $[\mathcal{V}, \mathcal{A}]$.

We will be interested in various functors $X \in \mathcal{V}\mathcal{F}$ such as

$$\mathrm{Flag}(V) = \{ \text{ complete flags in } V\}$$
$$\mathrm{Base}(V) = \{ \text{ bases for } V\}.$$

Given $X \in \mathcal{V}\mathcal{F}$ we define $\mathbb{Z}_{(p)}[X] \in \mathcal{V}\mathcal{A}$ by $\mathbb{Z}_{(p)}[X](V) = \mathbb{Z}_{(p)}[X(V)]$, the free $\mathbb{Z}_{(p)}$-module generated by the finite set $X(V)$.

Definition 8.1. The *Hecke algebra* is the ring $\mathcal{H} = \mathrm{End}(\mathbb{Z}[\mathrm{Flag}])$.

This formulation is convenient for conceptual purposes, but for calculations a more explicit version is useful. If X is a functor from $\mathcal{V}$ to sets then $X({\mathbb{F}_p}^n)$ is a G-set (where $G = GL_n(\mathbb{F}_p)$), and this construction gives an equivalence $[\mathcal{V}, \{\text{sets}\}] = \{G - \text{sets}\}$. This identifies $\mathbb{Z}_{(p)}[\text{Flag}]$ with the $\mathbb{Z}_{(p)}[G]$-module $\mathbb{Z}_{(p)}[G/B]$, and so identifies $\mathcal{H}$ with $\text{End}_{\mathbb{Z}_{(p)}[G]}(\mathbb{Z}_{(p)}[G/B])$. We will analyse this in more detail later.

First, however, we discuss some general constructions giving maps between functors. Note that $\mathbb{Z}_{(p)}[X]$ has an obvious inner product, given by $\langle [x], [x'] \rangle = \delta_{xx'}$. Given $f\colon \mathbb{Z}_{(p)}[X] \to \mathbb{Z}_{(p)}[Y]$ we let $f^t\colon \mathbb{Z}_{(p)}[Y] \to \mathbb{Z}_{(p)}[X]$ be the adjoint with respect to this inner product. In particular, if f comes from a map $f\colon X \to Y$ then $f^t[y] = \sum_{f(x)=y}[x]$.

Definition 8.2. Define $Z(\sigma) \in \mathcal{V}\mathcal{F}$ by

$$Z(\sigma)(V) = \{(\underline{U}, \underline{W}) \in \text{Flag}(V)^2 \mid \delta(\underline{U}, \underline{W}) = \sigma\}$$

(where $\delta(\underline{U}, \underline{V})$ is the Jordan permutation, as in Section 4). This has projection maps

$$\text{Flag} \xleftarrow{\pi_0} Z(\sigma) \xrightarrow{\pi_1} \text{Flag}$$

and we put $T_\sigma = \pi_1 \pi_0^t\colon \mathbb{Z}_{(p)}[\text{Flag}] \to \mathbb{Z}_{(p)}[\text{Flag}]$, so $T_\sigma \in \mathcal{H}$ and

$$T_\sigma[\underline{U}] = \sum_{\delta(\underline{U},\underline{W})=\sigma} [\underline{W}].$$

It is clear from this that $T_\sigma^t = T_{\sigma^{-1}}$. We will write T_i for T_{s_i}.

Proposition 8.3. *The maps T_σ give a basis for $\mathcal{H}$ over $\mathbb{Z}_{(p)}$.*

Proof. Consider an element $f\colon \mathbb{Z}_{(p)}[\text{Flag}] \to \mathbb{Z}_{(p)}[\text{Flag}]$. For any $V \in \mathcal{V}$ and any $\underline{U}, \underline{W} \in \text{Flag}(V)$ we let $n(V, \underline{U}, \underline{W})$ be the coefficient of $[\underline{W}]$ in $f([\underline{U}])$. As f is natural, this coefficient depends only on the isomorphism class of the triple $(V, \underline{U}, \underline{W})$. Thus, by Corollary 5.4, there are well-defined numbers n_σ for $\sigma \in \Sigma$ such that $n(V, \underline{U}, \underline{W}) = n_{\delta(\underline{U},\underline{W})}$. It follows that $f = \sum_\sigma n_\sigma T_\sigma$. □

Proposition 8.4. *Fix $i \in \{1, \dots, n-1\}$. Let $\underline{U}$ be a flag, let F be the set of flags $\underline{W}$ such that $W_j = U_j$ for all $j \neq i$, and put $a = \sum_{\underline{W} \in F} [\underline{W}]$. Then $T_i[\underline{U}] = a - [\underline{U}]$ and*

$$(T_i - p)(T_i + 1)[\underline{U}] = (T_i^2 - (p-1)T_i - p)[\underline{U}] = 0.$$

Proof. The formula $T_i[\underline{U}] = a - [\underline{U}]$ is immediate from Proposition 4.7. Note also that the definition of a depends only on the spaces U_j for $j \neq i$, so for $\underline{W} \in F$ we have $T_i[\underline{W}] = a - [\underline{W}]$. It follows that

$$T_i(a) = \sum_{\underline{W} \in F} T_i[\underline{W}] = \sum_{\underline{W} \in F} (a - [\underline{W}]) = |F|a - \sum_{\underline{W}} [\underline{W}] = (|F| - 1)a.$$

Moreover, F bijects with the set of one-dimensional subspaces of $U_{i+1}/U_{i-1} \simeq {\mathbb{F}_p}^2$, so $|F| = p + 1$, so $T_i(a) = pa$. It follows that $(T_i - p)(T_i + 1)[\underline{W}] = (T_i - p)(a) = 0$ as claimed. □

Definition 8.5. We define $\hat{e}\colon \mathbb{Z}_{(p)}[\text{Flag}(V)] \to \mathbb{Z}_{(p)}[\text{Flag}(V)]$ by

$$\hat{e}[\underline{U}] = |\,\text{Flag}(V)|^{-1} \sum_{\underline{W} \in \text{Flag}(V)} [\underline{W}]$$

(so $\hat{e}[\underline{U}]$ is independent of $\underline{U}$). This is natural, and so defines an element of $\mathcal{H}$. We also define a map $\widehat{\xi}\colon \mathcal{H} \to \mathbb{Z}_{(p)}$ by $\widehat{\xi}(\sum_\sigma n_\sigma T_\sigma) = \sum_\sigma p^{l(\sigma)}$.

Proposition 8.6. *$\hat{e}$ is a central idempotent in $\mathcal{H}$, with $\hat{e}^t = \hat{e}$. Moreover, $\widehat{\xi}$ is a ring map, with $\widehat{\xi}(a^t) = \widehat{\xi}(a)$ and $a\hat{e} = \hat{e}a = \widehat{\xi}(a)\hat{e}$ for all $a \in \mathcal{H}$.*

Proof. Consider an object $V \in \mathcal{V}$, and put $x = |\,\text{Flag}(V)|^{-1} \sum_{\underline{W} \in \text{Flag}(V)} [\underline{W}]$, so $\hat{e}[\underline{U}] = x$ for all $\underline{W}$. It is easy to see that $\hat{e}(x) = x$ also, and thus that $\hat{e}^2 = \hat{e}$. We have $\langle \hat{e}[\underline{U}], [\underline{W}] \rangle = |\,\text{Flag}\,|^{-1} = \langle [\underline{U}], \hat{e}[\underline{W}] \rangle$ for all $\underline{U}$ and $\underline{W}$, so $\hat{e}$ is self-adjoint, so $\hat{e}^t = \hat{e}$.

Next, we have

$$\hat{e}T_\sigma[\underline{U}] = \sum_{\delta(\underline{U},\underline{W})=\sigma} \hat{e}[\underline{W}] = |\{\underline{W} \mid \delta(\underline{W}, \underline{U}) = \sigma^{-1}\}|x = p^{l(\sigma^{-1})}x = p^{l(\sigma)}\hat{e}[\underline{W}].$$

(using Proposition 5.2 and the fact that $l(\sigma^{-1}) = l(\sigma)$). It follows that $\hat{e}T_\sigma = p^{l(\sigma)}\hat{e}$, and thus that $\hat{e}a = \widehat{\xi}(a)\hat{e}$ for all $a \in \mathcal{H}$. By expanding $\hat{e}ab$ in two ways we deduce that $\widehat{\xi}$ is a ring map.

Now take the transpose of $\hat{e}a = \widehat{\xi}(a)\hat{e}$ to get $a^t\hat{e} = \widehat{\xi}(a)\hat{e}$. Replacing a by a^t gives $a\hat{e} = \xi(a^t)\hat{e}$. We also have $T_\sigma^t = T_{\sigma^{-1}}$ and $l(\sigma^{-1}) = l(\sigma)$ so $\widehat{\xi}(a^t) = \widehat{\xi}(a)$. Our equation now reads $a\hat{e} = \widehat{\xi}(a)\hat{e}$, so $a\hat{e} = \hat{e}a$, so $\hat{e}$ is central. □

Lemma 8.7. *Under the identification $\mathcal{H} = \mathrm{End}_{\mathbb{Z}_{(p)}[G]}(\mathbb{Z}_{(p)}[G/B])$, we have*

$$T_{\sigma^{-1}}[gB] = T_\sigma^t[gB] = \sum_{x \in X(\sigma)} [gx\sigma B].$$

Proof. As $T_{\sigma^{-1}}$ is a G-map it will suffice to prove this for $g = 1$. Using the identification $G/B = \mathrm{Flag}({\mathbb{F}_p}^n)$ (given by $hB \mapsto h\underline{E}$) it will suffice to check that $T_{\sigma^{-1}}[\underline{E}] = \sum_{x \in X(\sigma)}[x\sigma\underline{E}]$. By definition we have $T_{\sigma^{-1}}[\underline{E}] = \sum_{\delta(\underline{E},\underline{U})=\sigma^{-1}}[\underline{U}]$. Note that $\delta(\underline{E},\underline{U}) = \sigma^{-1}$ if and only if $\delta(\underline{U},\underline{E}) = \sigma$, and the Bruhat decomposition tells us that any flag $\underline{U}$ with $\delta(\underline{U},\underline{E}) = \sigma$ can be expressed uniquely as $x\sigma\underline{E}$ with $x \in X(\sigma)$, as required. □

To give another reformulation of this, note that $\mathrm{Base}(V)$ is canonically the same as $\mathrm{Iso}({\mathbb{F}_p}^n, V)$, so an element $g \in G = \mathrm{Aut}({\mathbb{F}_p}^n)$ gives a map $g^*\colon \mathrm{Base} \to \mathrm{Base}$, with $(gh)^* = h^*g^*$.

Corollary 8.8. *The following diagram commutes:*

$$\begin{array}{ccc} \mathbb{Z}_{(p)}[\mathrm{Base}] & \xrightarrow{\pi} & \mathbb{Z}_{(p)}[\mathrm{Flag}] \\ {\scriptstyle \sum_{x\in X(\sigma)}(x\sigma)^*}\downarrow & & \downarrow{\scriptstyle T_\sigma^t} \\ \mathbb{Z}_{(p)}[\mathrm{Base}] & \xrightarrow[\pi]{} & \mathbb{Z}_{(p)}[\mathrm{Flag}]. \end{array}$$

Proof. The main point to note is that all the maps make sense so that the claim is meaningful. It then reduces to the lemma using the equivalence $\mathcal{VA} = \{\mathbb{Z}_{(p)}[G] - \mathrm{modules}\}$. □

Corollary 8.9. *If $l(\sigma\tau) = l(\sigma) + l(\tau)$ then $T_\sigma T_\tau = T_{\sigma\tau}$.*

Proof. It will suffice to prove that $T_\tau^t T_\sigma^t[gB] = T_{\sigma\tau}^t[gB]$. The right hand side is $\sum_{z\in X(\sigma\tau)}[gz\sigma\tau B]$. Proposition 6.4 converts this to

$$\sum_{x\in X(\sigma)}\sum_{y\in X(\tau)}[gx\sigma y\sigma^{-1}\sigma\tau B] = \sum_{x,y}[gx\sigma y\tau B] = \sum_x T_\tau^t[gx\sigma B] = T_\tau^t T_\sigma^t[gB],$$

as required. □

Corollary 8.10. *If $s_{i_1}\dots s_{i_r}$ is a reduced word for σ then $T_{i_1}\dots T_{i_r} = T_\sigma$* □

Proposition 8.11. *The ring $\mathcal{H}$ is generated over $\mathbb{Z}_{(p)}$ by the elements T_i, subject only to the relations*

$$\begin{aligned} T_i^2 &= p + (p-1)T_i \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} \\ T_iT_j &= T_jT_i \qquad \text{if } |i-j| > 1. \end{aligned}$$

(The first relation can also be written as $(T_i + 1)(T_i - p) = 0$.)

Proof. For any $\sigma \in \Sigma$ of length r we can choose a reduced word $s_{i_1}\dots s_{i_r}$ representing σ, and we find from Corollary 8.10 that $T_\sigma = T_{i_1}\dots T_{i_r}$. This shows that the elements T_i generate $\mathcal{H}$. Moreover, if $s_{j_1}\dots s_{j_r}$ is another reduced word representing σ then $T_{j_1}\dots T_{j_r} = T_\sigma = T_{i_1}\dots T_{i_r}$. The second and third relations above are instances of this. Now consider T_i^2. Fix a flag $\underline{U} \in \mathrm{Flag}(V)$ and put $A = \{W \mid U_{i-1} < W < U_{i+1}\}$, so $|A| = p + 1$. Put

$$\phi(W) = (0 = U_0 < \dots < U_{i-1} < W < U_{i+1} < \dots < U_n = V),$$

so $\phi(U_i) = \underline{U}$. One checks that the flags with $\delta(\underline{W}, \underline{U}) = s_i$ are precisely those of the form $\phi(W)$ for some $W \in A \setminus \{U_i\}$. It follows that $T_i[\underline{U}] = \sum_{W \neq U_i} [\phi(W)]$ and thus that

$$\begin{aligned} T_i^2[\underline{U}] &= \sum_{W \neq U_i} T_i[\phi(W)] = \sum_{W \neq U_i} \sum_{W' \neq W} [\phi(W')] \\ &= \sum_{W'} [\phi(W')].|\{W \mid U_i \neq W \neq W'\}| = p[\underline{U}] + (p-1) \sum_{W' \neq U_i} [\phi(W')] \\ &= (p + (p-1)T_i)[\underline{U}] \end{aligned}$$

as claimed.

Now let $\mathcal{H}'$ be generated by symbols $T'_1, \dots, T'_{n-1}$ subject only to the relations in the statement of the proposition. As these relations are satisfied in $\mathcal{H}$ and the T_i generate, there is a well-defined and surjective ring map $\theta \colon \mathcal{H}' \to \mathcal{H}$ given by $\theta(T'_i) = T_i$.

Now consider an arbitrary element $\sigma \in \Sigma$. Choose any reduced word $u = s_{i_1} \cdots s_{i_r}$ such that $\pi(u) = \sigma$, and put $T'_\sigma = T'_{i_1} \cdots T'_{i_r} \in \mathcal{H}'$. This is well-defined by Theorem 2.16, and we have $\theta(T'_\sigma) = T_\sigma$ by Corollary 8.10. Define a map $\phi \colon \mathcal{H} \to \mathcal{H}'$ by $\phi(T_\sigma) = T'_\sigma$, so $\theta\phi = 1_{\mathcal{H}}$, so ϕ is injective.

Now put $A = \phi(\mathcal{H}) \leq \mathcal{H}'$. We claim that A is closed under right multiplication by T'_i for all i. To see this, consider an element $T'_\sigma \in A$. If $\sigma(i) > \sigma(i+1)$ then $l(\sigma s_i) = l(\sigma) + 1$. We choose any reduced word w for σ and note that ws_i is a reduced word for σs_i, so $T'_\sigma T'_i = T'_{\sigma s_i} \in A$. Otherwise we choose a reduced word w for the permutation $\tau = \sigma s_i$ and note that ws_i is a reduced word for σ. It follows that

$$T'_\sigma T'_i = T'_\tau (T'_i)^2 = T'_\tau (p + (p-1)T'_i) = pT'_\tau + (p-1)T'_{\tau s_i} \in A.$$

As $1 \in A$ and A is closed under multiplication by the generators of $\mathcal{H}'$ we see that $A = \mathcal{H}'$, so ϕ is surjective as well as injectve, so ϕ is an isomorphism. As $\theta\phi = 1$ is an isomorphism it follows that θ is also an isomorphism. □

9. The Steinberg module

Definition 9.1. We define $\pi \colon \text{Base} \to \text{Flag}$ by

$$\pi(v_1, \dots, v_n) = (0 < \text{span}\{v_1\} < \text{span}\{v_1, v_2\} < \cdots < \text{span}\{v_1, \dots, v_n\}).$$

This induces $\pi \colon \mathbb{Z}_{(p)}[\text{Base}] \to \mathbb{Z}_{(p)}[\text{Flag}]$ and $\pi^t \colon \mathbb{Z}_{(p)}[\text{Flag}] \to \mathbb{Z}_{(p)}[\text{Base}]$. Next, we define

$$\omega = |G/U|^{-1} \sum_{\sigma \in \Sigma} \text{sgn}(\sigma)\sigma^* \colon \mathbb{Z}_{(p)}[\text{Base}] \to \mathbb{Z}_{(p)}[\text{Base}].$$

We note that $(\sigma^*)^t = (\sigma^{-1})^*$ and so $\omega^t = \omega$. We put $\mu = \pi\omega \colon \mathbb{Z}_{(p)}[\text{Base}] \to \mathbb{Z}_{(p)}[\text{Flag}]$, and let M_{St} denote the image of μ. Note that $M_{\text{St}}(\mathbb{F}_p{}^n)$ is a module over $\mathbb{Z}_{(p)}[G]$; we call this the *Steinberg module*. We also put

$$\begin{aligned} e &= \mu\pi^t = \pi\omega\pi^t = e^t \colon \mathbb{Z}_{(p)}[\text{Flag}] \to \mathbb{Z}_{(p)}[\text{Flag}] \\ e_{\text{St}} &= \pi^t\mu = \pi^t\pi\omega \colon \mathbb{Z}_{(p)}[\text{Base}] \to \mathbb{Z}_{(p)}[\text{Base}] \end{aligned}$$

so $e \in \mathcal{H}$ and $e_{\text{St}} \in \text{End}(\mathbb{Z}_{(p)}[\text{Base}]) = \mathbb{Z}_{(p)}[G]^{\text{op}}$. We call e_{St} the *Steinberg idempotent*. (We will show later that both e_{St} and e are indeed idempotent.)

Proposition 9.2. *The map $\xi \colon \mathcal{H} \to \mathbb{Z}_{(p)}$ given by $\xi(T_\sigma) = \text{sgn}(\sigma)$ is a ring map, with $\xi(a^t) = \xi(a)$ for all a. Moreover, for all $a \in \mathcal{H}$ we have $a\mu = \xi(a)\mu$ and $ae = ea = \xi(a)e$ (so e is central in $\mathcal{H}$).*

Proof. We see from Proposition 8.4 that $(1 + T_i)[\underline{U}]$ depends only on the spaces U_j for $j \neq i$. On the other hand, we see from the definitions that $\pi s_i^*(\underline{v})$ only differs from $\pi(\underline{v})$ in the i'th space, so $(1+T_i)\pi = (1+T_i)\pi s_i^*$, so $(1 + T_i)\mu = (1 + T_i)\pi s_i^*\omega$. However, we have $s_i^*\omega = -\omega$, so $(1 + T_i)\pi s_i^*\omega = -(1 + T_i)\mu$. It follows that $(1 + T_i)\mu = 0$, or in other words $T_i\mu = -\mu$. Given any $\sigma \in \Sigma$ we can find a reduced word $s_{i_1} \dots s_{i_r}$ for Σ. We then have

$$T_\sigma\mu = T_{i_1} \dots T_{i_r}\mu = (-1)^r\mu = \widehat{\xi}(T_\sigma)\mu.$$

As the T_σ span $\mathcal{H}$ this gives $a\mu = \widehat{\xi}(a)\mu$ for all a. It follows that $\xi(ab)\mu = \xi(a)\xi(b)\mu$ and one sees directly that $\mu \neq 0$ so ξ is a ring map. (This could also have been deduced from Proposition 8.11.) We also have $T_\sigma^t = T_{\sigma^{-1}}$ and $\text{sgn}(\sigma^{-1}) = \text{sgn}(\sigma)$ so $\xi(a^t) = \xi(a)$.

Next, as $e = \mu\pi^t$ and $a\mu = \widehat{\xi}(a)\mu$ we have $ae = \xi(a)e$. We observed in Definition 9.1 that $e^t = e$, so we can transpose the above equation to get $ea^t = \xi(a)e$, and replace a by a^t to get $ea = \xi(a^t)e$. As $\xi(a^t) = \xi(a)$ this gives $ea = \xi(a)e$, showing that e is central. □

Proposition 9.3. $e = |G/B|^{-1} \sum_\sigma \operatorname{sgn}(\sigma) p^{l(\sigma^{-1}\rho)} T_\sigma$.

Proof. Corollary 5.5 tells us that

$$\langle \pi\sigma^*\pi^t[\underline{U}], [\underline{W}]\rangle = \langle \sigma^*\pi^t[\underline{U}], [\underline{W}]\rangle = \begin{cases} |B/U| p^{l(\sigma^{-1}\rho)} & \text{if } \delta(\underline{U}, \underline{W}) = \sigma \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\pi\sigma^*\pi^t = |B/U| p^{l(\sigma^{-1}\rho)} T_\sigma$, and thus that

$$e = |G/U|^{-1} \sum_\sigma \operatorname{sgn}(\sigma) \pi\sigma^*\pi^t = |G/B|^{-1} \sum_\sigma \operatorname{sgn}(\sigma) p^{l(\sigma^{-1}\rho)} T_\sigma$$

as claimed. □

Proposition 9.4. *We have $\mu\pi^t\mu = \mu$. The maps e and e_{St} are idempotent, and* $\operatorname{image}(e) = M_{St}$. *The map $\mu\colon \mathbb{Z}_{(p)}[\mathrm{Base}] \to M_{St}$ restricts to give an isomorphism* $\operatorname{image}(e_{\mathrm{St}}) \to M_{St}$, *with inverse given by the restriction of π^t.*

Proof. First, we have

$$\xi(e) = |G/B|^{-1} \textstyle\sum_\sigma p^{l(\sigma^{-1}\rho)} = |G/B|^{-1} \left|\coprod_\sigma X(\sigma^{-1}\rho)\right| = |G/B|^{-1} \left|\coprod_\tau X(\tau)\right| = 1.$$

It follows that $\mu\pi^t\mu = e\mu = \widehat{\xi}(e)\mu = \mu$, and thus that

$$e^2 = \mu\pi^t\mu\pi^t = \mu\pi^t = e$$
$$e_{\mathrm{St}}^2 = \pi^t\mu\pi^t\mu = \pi^t\mu = e_{\mathrm{St}},$$

so e and e_{St} are idempotent. As $e = \mu\pi^t$ we have $\operatorname{image}(e) \leq \operatorname{image}(\mu)$, and as $\mu = e\mu$ we have $\operatorname{image}(\mu) \leq \operatorname{image}(e)$, so $\operatorname{image}(e) = \operatorname{image}(\mu) = M_{\mathrm{St}}$. Now put $M'_{\mathrm{St}} = \operatorname{image}(e_{\mathrm{St}})$, and let $\alpha\colon M'_{\mathrm{St}} \to M_{\mathrm{St}}$ be the restriction of μ. As $e_{\mathrm{St}} = \pi^t\mu$ we see that $\pi^t(M_{\mathrm{St}}) = \operatorname{image}(\pi^t\mu) = M'_{\mathrm{St}}$. Thus, if we let β be the restriction of π^t to M_{St}, we see that β gives an epimorphism $M_{\mathrm{St}} \to M'_{\mathrm{St}}$. As the map $\mu\pi^t = e$ restricts to 1 on M_{St} we see that $\alpha\beta = 1$, and as β is an epimorphism this tells us that α and β are mutually inverse isomorphisms. □

Proposition 9.5. *M_{St} is projective and self dual in $\mathcal{VA}$, and has rank $p^{n(n-1)/2}$. The natural map $\mathbb{Z}_{(p)} \to \operatorname{End}_{\mathcal{VA}}(M_{St})$ is an isomorphism. If we put $N = \operatorname{image}(1-e)$ then $\operatorname{Hom}_{\mathcal{VA}}(M_{St}, N) = 0$ and $\operatorname{Hom}_{\mathcal{VA}}(N, M_{St}) = 0$.*

Proof. There is a Yoneda isomorphism $\operatorname{Hom}_{\mathcal{VA}}(\mathbb{Z}_{(p)}[\mathrm{Base}], A) = A(\mathbb{F}_{p^n})$, so $\mathbb{Z}_{(p)}[\mathrm{Base}]$ is projective. We have seen that M_{St} is isomorphic to the image of e_{St}, which is a summand in $\mathbb{Z}_{(p)}[\mathrm{Base}]$, so M_{St} is projective. On the other hand, M_{St} is also the image of a self-adjoint idempotent on $\mathbb{Z}_{(p)}[\mathrm{Flag}]$, so the standard perfect pairing on $\mathbb{Z}_{(p)}[\mathrm{Flag}]$ restricts to give one on M_{St}, so M_{St} is self dual. We also see that the rank of M_{St} is the trace of e. If $\sigma \neq 1$ then $\delta(\underline{U}, \underline{U}) \neq \sigma$ so one sees from the definitions that T_σ has trace zero. For $\sigma = 1$ the map T_σ is the identity, with trace $|\operatorname{Flag}| = |G/B|$. We therefore see from Proposition 9.3 that $\operatorname{trace}(e) = p^{l(\rho)} = p^{n(n-1)/2}$.

Next, note that $\mathbb{Z}_{(p)}[\mathrm{Flag}] = M_{\mathrm{St}} \oplus N$, so any map from M_{St} to itself (or from M_{St} to N, or from N to M_{St}) can be extended to an endomorphism of $\mathbb{Z}_{(p)}[\mathrm{Flag}]$, or in other words an element $a \in \mathcal{H}$. Any such element satisfies $ae = \xi(a)e = ea$, which means that a preserves $M_{\mathrm{St}} = \operatorname{image}(e)$ and $N = \ker(e)$, and acts as a scalar on M_{St}. Thus $\operatorname{End}_{\mathcal{VA}}(M_{\mathrm{St}}) = \mathbb{Z}_{(p)}$ and $\operatorname{Hom}_{\mathcal{VA}}(M_{\mathrm{St}}, N) = 0$ and $\operatorname{Hom}_{\mathcal{VA}}(N, M_{\mathrm{St}}) = 0$. □