# OPERADS

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**Note:** these notes are unfinished and incomplete. In particular, very many references to the literature need to be added; little of the material is original. I have nonetheless put this document on the web so that I can refer to it in answer to some questions on mathoverflow.

## 1. Operads

Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category, with unit object 1. (We will also write 1 for the set $\{0\}$.) For us, an *operad* $E$ in $\mathcal{C}$ consists of objects $E(A) \in \mathcal{C}$ for each finite set $A$, functorial for bijections $A \to A'$, together with a map $\eta\colon 1 \to E(1)$ and maps $\gamma_p$ as follows. Given a map $p\colon A \to B$ of finite sets, we put $A_b = p^{-1}\{b\}$, and $E(p) = \bigotimes_{b\in B} E(A_b)$. We should then have a map

$$\gamma_p\colon E(B) \otimes E(p) \to E(A),$$

subject to two axioms. For the first, let $c_A$ be the unique map $A \to 1$, so $E(c_A) = E(A)$ and we have a map

$$E(A) = 1 \otimes E(A) \xrightarrow{\eta\otimes 1} E(1) \otimes E(A) \xrightarrow{\gamma_{c_A}} E(A).$$

This should be the identity. Next, suppose we have maps $A \xrightarrow{p} B \xrightarrow{q} C$ of finite sets. For $b \in B$ and $c \in C$ we put $A_b = p^{-1}\{b\}$ and $A_c = (qp)^{-1}\{c\}$ and $B_c = q^{-1}\{c\}$, so we have a map $p_c = p|_{A_c}\colon A_c \to B_c$ for all $c$, giving $\gamma_{p_c}\colon E(B_c) \otimes E(p_c) \to E(A_c)$. Note that $\bigotimes_c E(B_c) = E(q)$ and $\bigotimes_c E(p_c) = E(p)$ and $\bigotimes_c E(A_c) = E(qp)$, so the tensor product of the maps $\gamma_{p_c}$ gives a map

$$\gamma_{p,q}\colon E(q) \otimes E(p) \to E(qp).$$

The second axiom is that the following diagram should commute:

$$\begin{array}{ccc}
E(C)\otimes E(q)\otimes E(p) & \xrightarrow{1\otimes\gamma_{p,q}} & E(C)\otimes E(qp) \\
\downarrow{\scriptstyle \gamma_q\otimes 1} & & \downarrow{\scriptstyle \gamma_{qp}} \\
E(B)\otimes E(p) & \xrightarrow[\gamma_p]{} & E(A).
\end{array}$$

When defining operads, we will often silently leave to the reader the task of identifying the unit map $\eta\colon 1 \to E(1)$ and checking the unit axiom.

**Example 1.1.** We can define a rather trivial operad called 1 by taking $1(A) = 1$ (the unit for the symmetric monoidal structure) for all $A$. The algebras for this operad (in a sense that we have not yet recalled) are just the commutative monoids in $\mathcal{C}$, so this operad is often called Comm instead of 1.

**Example 1.2.** For any finite set $A$, let $\mathrm{Ord}(A)$ denote the set of total orderings of $A$. Given a map $p\colon A \to B$ and total orderings of $B$ and all the fibres $A_b$, we define an ordering of $A$ by $a < a'$ iff $p(a) < p(a')$ in $B$, or $p(a) = p(a') = b$ and $a < a'$ in the given order on $A_b$. This construction gives a map $\gamma_p\colon \mathrm{Ord}(B) \times \prod_B \mathrm{Ord}(A_b) \to \mathrm{Ord}(A)$, making Ord an operad. The algebras for this operad are just associative monoids, so the notation $\mathrm{Ass}(A)$ is often used instead of $\mathrm{Ord}(A)$.

**Remark 1.3.** Traditionally one formulates operads using only the sets $\underline{n} = \{1, \ldots, n\}$ rather than all finite sets. One can then identify $\mathrm{Ord}(\underline{n})$ with $\Sigma_n$. However, this leads one to think that the group structure of $\Sigma_n$ might be relevant, which is not really the case.

**Example 1.4.** Put $\Delta(A) = \{x\colon A \to I \mid \sum_a x(a) = 1\}$ (where $I = [0, 1]$). Given a map $p\colon A \to B$ we can identify $\Delta(p)$ with

$$\{x\colon A \to I \mid \forall b \sum_{a \in A_b} x(a) = 1\}$$

and then define $\gamma_p\colon \Delta(B) \times \Delta(p) \to \Delta(A)$ by $\gamma_p(y, x) = x.(y \circ p)$. This makes $\Delta$ an operad. We can define a suboperad $\mathring{\Delta}$ by $\mathring{\Delta}(A) = \{x \in \Delta(A) \mid x(a) > 0 \text{ for all } a\}$.

**Example 1.5.** We can define a similar operad $\Delta'$ by $\Delta'(A) = \{x\colon A \to I \mid \max\{x(a) \mid a \in A\} = 1\}$ and again $\gamma_p(y, x) = x.(y \circ p)$. There is a natural map $\pi\colon \Delta'(A) \to \Delta(A)$ given by $\pi(x) = x/\sum_a x(a)$, but this does not respect the operad structure. In fact, there is no operad map $f\colon \Delta' \to \Delta$. To see this, consider the elements $b_A \in \Delta'(A)$ given by $b_A(a) = 1$ for all $a$, and the elements $b'_A = b_A/|A| \in \Delta(A)$ (for $A \neq \emptyset$). As $b_A$ is invariant under permutations of $A$, and $b'_A$ is the unique element of $\Delta(A)$ with this invariance, we would have to have $f(b_A) = b'_A$. However, we have $\gamma_p(b_B; (b_{A_b})_{b \in B} = b_A$, but the elements $b'_A$ do not have the corresponding property, so $f$ cannot preserve the operad structure.

Another way to say this is to introduce the operad $\mathrm{Comm}_0 \subset \mathrm{Comm}$ given by $\mathrm{Comm}_0(\emptyset) = \emptyset$ and $\mathrm{Comm}(A) = 1$ for $A \neq \emptyset$. The elements $b_A$ give an operad map $\mathrm{Comm}_0 \to \Delta'$, but there is no operad map $\mathrm{Comm}_0 \to \Delta$.

## 2. Little cubes

We now recall the little cubes operad. Put $J = [0, 1)$, and consider $J^k$ as a partially ordered set in the obvious way. Given $x, y \in [0, 1]^k$ with $x < y$, define $f_{xy}\colon J^k \to J^k$ by $f_{xy}(t)_i = t_i y_i + (1 - t_i)x_i$, so $f_{xy}(J^k) = [x, y)$. Let $C_k(1)$ be the set of all maps of the form $f_{xy}$ for some $x < y$, which is a monoid under composition. Given any map $f\colon A \times J^k \to J^k$ we define $f_a\colon J^k \to J^k$ by $f_a(x) = f(a, x)$. We let $C_k(A)$ be the set of all injective maps $f\colon A \times J^k \to J^k$ for which all the maps $f_a$ lie in $C_k(1)$. These spaces give an operad $C_k$ in a well-known way: given a map $p\colon A \to B$, the space $C_k(p) = \prod_b C_k(A_b)$ can be identified naturally with a subspace of the space of injective maps $A \times J^k \to B \times J^k$, and the map

$$\gamma_p\colon C_k(B) \times C_k(p) \to C_k(A)$$

is then just $\gamma_p(f, g) = f \circ g$.

If we let $m$ be the centre of $J^k$ we get an embedding $i\colon A \to A \times J^k$ given by $i(a) = (a, m)$, and thus a map $i^*\colon C_k(A) \to \mathrm{Inj}(A, J^k) \simeq \mathrm{Inj}(A, \mathbb{R}^k)$. It is not hard to see that this is a homotopy equivalence. In the case $A = 2 = \{0, 1\}$ we also have a map $\mathrm{Inj}(2, \mathbb{R}^k) \to S^{k-1}$ given by $x \mapsto (x(1) - x(0))/\|x(1) - x(0)\|$, which is again a homotopy equivalence, so $C_k(2) \simeq S^{k-1}$.

It follows that the homology groups $H_* C_k(A)$ give an operad in the category of graded abelian groups (under tensor product). It is known that this is the operad whose algebras are Poisson algebras. In more detail, a *$k$-Poisson algebra* is a graded abelian group $A_*$ together with an associative product $A_i \otimes A_j \to A_{i+j}$ that is commutative in the graded sense, together with a bracket operation $[,]\colon A_i \otimes A_j \to A_{i+j+k-1}$ satisfying

$$[a, b] + (-1)^{\overline{a}\overline{b}}[b, a] = 0$$

$$(-1)^{\overline{a}\overline{b}}[a, [b, c]] + (-1)^{\overline{b}\overline{c}}[b, [c, a]] + (-1)^{\overline{c}\overline{a}}[c, [a, b]] = 0$$

$$[a, bc] = [a, b]c + (-1)^{\overline{a}|b|}[a, c]b$$

where $\overline{a} = |a| + k - 1$.

The obvious generator of $H_0 C_k(2)$ gives the commutative product. The Lie bracket comes from the generator of $H_{k-1} C_k(2) = H_{k-1} S^{k-1}$.

Let Lie denote the operad for Lie algebras. This is an operad in ungraded abelian groups, but we can consider it as a graded operad in degree zero. The signs for Poisson algebras are supposed to work out so that if $R_*$ is a Poisson algebra, then $\Sigma^{1-k} R_*$ is a Lie algebra. This works out to give a map $\mathrm{Lie}(A) \to H_{(|A|-1)(k-1)} C_k(n)$.

It seems quite hard for a small operad to map to the little cubes operad. As evidence for this, let $\mathcal{F}$ be the free operad generated by a single binary operation, let $\delta \in \mathcal{C}_1(2)$ correspond to the obvious map $J \amalg J \simeq (0, 1) \setminus \{\frac{1}{2}\} \to J$, and let $\phi\colon \mathcal{F} \to C_1$ be the map sending the generator to $\delta$. Say that an subset $U \subseteq [0, 1]$ is a *standard interval* if it has the form $[i/2^k, (i + 1)/2^k]$ for some $i$. Given a point $f \in C_1(A)$, let

$\mathcal{T}$ be the set of subsets $U \subseteq A$ such that $\overline{f(U \times J)}$ is a standard interval. I think that if $f = \phi(t)$ then $T$ is a binary $A$-tree which determines $t$. This implies that $\phi\colon \mathcal{F} \to C_1$ is injective. I think that the same holds if we replace $\delta$ by any other element of $C_1(2)$.

## 3. Spheres

We next want to make the assignment $A \mapsto BWA$ into an operad in unbased spaces, and $A \mapsto S^{WA}$ into an operad in based spaces. We will do this in two different ways, one of which starts from the operation $(s,t) \mapsto st$ on $I = [0,1]$, and the other from $(s,t) \mapsto \min(s,t)$. In order to interpolate between these, we give the construction in slightly greater generality: we assume given a commutative and associative binary operation $*$ on $I$ such that $1 * t = t$ for all $t$, and $s * t = 0$ iff ($s = 0$ or $t = 0$).

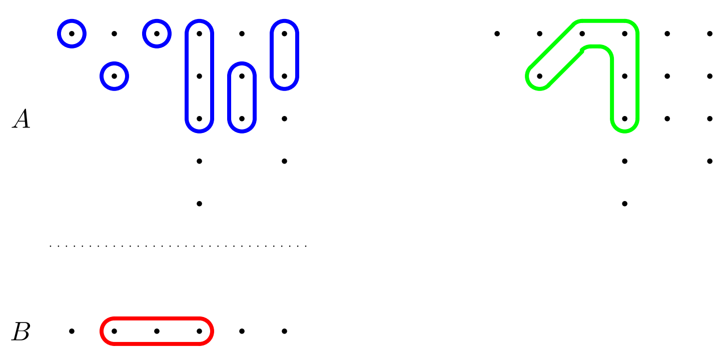
Firstly, we identify $BWA$ with $\{x\colon A \to I \mid \max_{a\in A} x(a) = 1\}$ (so $BW\emptyset = \emptyset$). This identifies $SWA = \partial BWA$ with $\{x \in BWA \mid \min_{a\in A} x(a) = 0\}$. We put $S^{WA} = BWA/\partial BWA$, so $S^{W\emptyset} = 0$ and $S^{W1} = S^0$.

Now suppose we have a map $p\colon A \to B$, and put $A_b = p^{-1}\{b\}$. We identify $\prod_b BWA_b$ with the set of maps $x\colon A \to I$ such that $\max_{a\in A_b} x(a) = 1$ for all $b$. We then define $\gamma_p\colon BWB \times \prod_{b\in B} BWA_b \to BWA$ by $\gamma_p(y,x)(a) = y(p(a)) * x(a)$. This is easily seen to give an operad structure. Moreover, we have $\gamma_p(y,x) \in \partial BWA$ iff $(y,x)$ lies in the boundary of the domain, so we get an induced map $\overline{\gamma}_p\colon S^{WB} \wedge \bigwedge_b S^{WA_b} \to S^{WA}$, making $A \mapsto S^{WA}$ into an operad in the category of pointed spaces.

If we start with the operation $s * t = st$, it is not hard to see that the maps $\overline{\gamma}_p$ are homeomorphisms. Indeed, if $\gamma_p(y,x) = z \in BWA \setminus \partial BWA$ then $y(b) = \max\{z(a) \mid a \in A_b\}$ and $x(a) = z(a)/y(p(a))$.

On the other hand, for the operation $s * t = \min(s,t)$ we can obtain the resulting operad structure on $BWA$ as the geometric realisation of an operad in the category of posets. To see this, let $\mathcal{C}A$ be the poset of nonempty subsets of $A$. We can define $f\colon \mathcal{C}A \to BWA$ by $f(A) = \chi_A$, and this extends linearly to give a map $f\colon |\mathcal{C}A| \to \mathrm{Map}(A, I)$. We claim that this actually gives a homeomorphism $|\mathcal{C}A| \to BWA$. Indeed, a point in $|\mathcal{C}A|$ is a map $x\colon \mathcal{C}A \to I$ such that $x^{-1}(0,1]$ is a chain in $\mathcal{C}A$, and thus has the form $\{U_0 \subset U_1 \subset \cdots \subset U_r\}$, with $x(U_i) = t_i > 0$ say, and $\sum_i t_i = 1$. We then have $f(x) = \sum_i t_i \chi_{U_i}$, so $f(x)(a) = 1$ for $a \in U_0$. As $U_0 \in \mathcal{C}A$ we have $U_0 \neq \emptyset$ so $\max\{f(x)(a) \mid a \in A\} = 1$, so $f(x) \in BWA$. It is not hard to prove the rest of the claim but we will not do so here.

Next, given $p\colon A \to B$, we identify $\mathcal{C}(p) = \prod_b \mathcal{C}(A_b)$ with the set of subsets $U \subseteq A$ such that $p(U) = B$. We then define $\gamma_p\colon \mathcal{C}(B) \times \mathcal{C}(p) \to \mathcal{C}(A)$ by $\gamma_p(U,V) = U \cap p^{-1}V$. An example is shown below:




On the left we have the sets $A$ and $B$, and $p\colon A \to B$ is given by vertical projection. The subset $V \subseteq B$ is shown in red, and the subsets $U_b \subseteq A_b$ are shown in blue. The resulting subset $W = \gamma_p(V; (U_b)_{b\in B}) \subseteq A$ is shown on the right.

This makes $\mathcal{C}$ into an operad in posets, and the map $f\colon \mathcal{C}A \to BWA$ is an operad morphism with respect to the structure on $BWA$ derived from any binary operation with the properties discussed above. We claim that if we use the operation $s * t = \min(s,t)$, then $f$ gives an operad isomorphism $|\mathcal{C}A| \to BWA$. A point in $|\mathcal{C}(A)| \times \prod_B |\mathcal{C}(A_b)| = |\mathcal{C}(A) \times \mathcal{C}(p)|$ is a map $x\colon \mathcal{C}(A) \times \mathcal{C}(p) \to I$ whose support is a chain $(V_0, U_0) < \ldots < (V_r, U_r)$ say. Put $t_i = x(U_i, V_i) > 0$, so $\sum_i t_i = 1$. This corresponds to the point $(z,y) \in BWB \times \prod_B BWA_b$, where $z = \sum_i t_i \chi_{V_i}$ and $y = \sum_i t_i \chi_{U_i}$. It follows that

$$\gamma(z,y)(a) = \min(\sum_i t_i \chi_{U_i}(p(a)), \sum_i t_i \chi_{V_i}(a)).$$

Let $n$ be minimal such that $a \in p^{-1}U_n$ (or $n = r+1$ if $a \notin p^{-1}U_r$). Let $m$ be minimal such that $a \in V_m$ (or $m = r+1$ if $a \notin V_r$). Put $k = \max(n, m)$, which is minimal such that $a \in p^{-1}U_k \cap V_k = \gamma(V_k, U_k)$. We then have

$$\gamma(z, y)(a) = \min(\sum_{i \geq n} t_i, \sum_{i \geq m} t_i) = \sum_{i \geq \max(n,m)} t_i = \sum_i t_i \chi_{\gamma(V_k, U_k)}(a).$$

It follows from this that $f$ is an operad map as claimed.

We would next like to interpolate between the operations $(s, t) \mapsto st$ and $(s, t) \mapsto \min(s, t)$. Given $d \in [1, \infty)$ we define $\phi_d \colon I \to I$ by $\phi_d(t) = 1 - (1-t)^d$. This is clearly a strictly increasing homeomorphism with $\phi_d(0) = 0$ and $\phi_d(1) = 1$, and $\phi_d \circ \phi_e = \phi_{de}$. We put $s *_d t = \phi_d^{-1}(\phi_d(s)\phi_d(t))$. This is a binary operation with all the properties discussed previously, and $s *_1 t = st$. We also put $s *_\infty t = \min(s, t)$, and define $\psi \colon [1, \infty] \times I^2 \to I$ by $\psi(d, s, t) = s *_d t$. This is continuous, by the following lemma:

**Lemma 3.1.** *For all $d, s, t \in I$ we have $|s *_d t - s *_\infty t| \leq 2^{1/d} - 1$.*

*Proof.* By symmetry, we may assume that $s \leq t$, so $s *_\infty t = s$. Put $u = 1 - s$ and $v = 1 - t$, so $0 \leq v \leq u \leq 1$ and

$$1 - s *_d t = (1 - (1 - u^d)(1 - v^d))^{1/d} = (u^d + v^d(1 - u^d))^{1/d}.$$

We have $0 \leq v^d \leq u^d$ and $0 \leq 1 - u^d \leq 1$ so $u^d \leq u^d + v^d(1 - u^d) \leq 2u^d$, so $u \leq 1 - s *_d t \leq 2^{1/d}u$. Subtracting $u$ from both sides gives

$$0 \leq s - s *_d t = s *_\infty t - s *_d t \leq (2^{1/d} - 1)u \leq 2^{1/d} - 1.$$

□

**Remark 3.2.** It is natural to ask about algebras for the operad $\mathcal{C}$, but these do not seem to have a natural description. An idempotent commutative monoid certainly gives a $\mathcal{C}$-algebra, but idempotence cannot be forced by operadic conditions. A $\mathcal{C}$-algebra has three binary operations. In an idempotent monoid, one of these is the product, and the other two are projections $M \leftarrow M^2 \to M$; but projections are not operadically mentionable. I think that $\mathcal{C}$-algebras are sets with two associative binary operations (written $a * b$ and $a : b$) such that $*$ is commutative and $a : b : c = a : c : b = a : (b * c) = a : (c * b)$ and $(a : b) * (c : d) = (a * c) : (b * d)$.

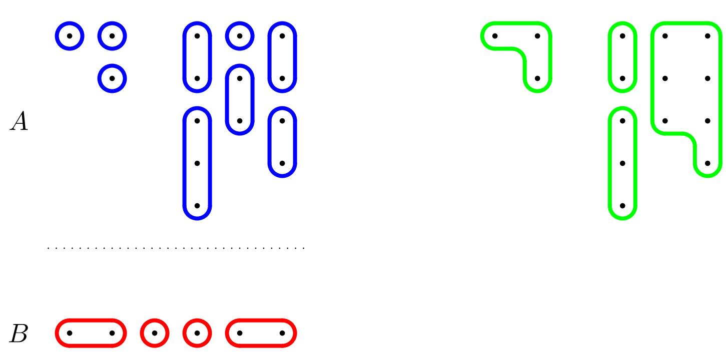
## 4. Partitions

For any finite set $A$, a *partition* of $A$ is a set $\pi$ of disjoint nonempty subsets (called *blocks*) whose union is $A$. We write $\mathcal{P}(A)$ for the set of all partitions, ordered by $\pi \leq \pi'$ iff every block of $\pi$ is a union of some collection of blocks of $\pi'$ (which implies $|\pi| \leq |\pi'|$).

Let $p \colon A \to B$ be a map of finite sets, and put $A_b = p^{-1}\{b\}$ as usual. Let $\pi$ be a partition of $B$, and let $\omega_b$ be a partition of $A_b$. There may be some points $b \in B$ such that $\{b\}$ is a block of $\pi$. For each such $b$, we note that $\omega_b$ is a family of disjoint subsets of $A$. All other blocks of $\pi$ will be sets $U \subseteq B$ with $|U| > 1$, and for each of these we have a subset $p^{-1}U \subseteq A$ (which might be empty). We put

$$\sigma = \gamma_p(\pi; (\omega_b)_{b \in B}) = \{p^{-1}U \mid U \in \pi,\ |U| > 1,\ p^{-1}U \neq \emptyset\} \amalg \coprod_{\{b\} \in \pi} \omega_b.$$

An example is shown below:



On the left we have the sets $A$ and $B$, and $p\colon A \to B$ is given by vertical projection. The partition $\pi$ of $B$ is shown in red, and the partitions $\omega_b$ of $A_b$ are shown in blue. The resulting partition $\sigma = \gamma_p(\pi; (\omega_b)_{b\in B})$ of $A$ is shown on the right.

**Proposition 4.1.** *The above maps give an operad structure on $\mathcal{P}$.*

*Proof.* Consider the following data:

- Finite sets $A$, $B$ and $C$.
- Maps $A \xrightarrow{p} B \xrightarrow{q} C$ (using which we define $A_b = p^{-1}\{b\}$ and $A_c = (qp)^{-1}\{c\}$ and $B_c = q^{-1}\{c\}$).
- A partition $\tau \in \mathcal{P}(C)$.
- Partitions $\sigma_c \in \mathcal{P}(B_c)$ for all $c \in C$.
- Partitions $\rho_b \in \mathcal{P}(A_b)$ for all $b \in B$.

These data can be combined in two different ways to give partitions $\theta, \phi \in \mathcal{P}(A)$. The main axiom that we need to check is that $\theta = \phi$. In more detail, we put

$$\begin{aligned}
\lambda_c &= \gamma_p(\sigma_c; (\rho_b)_{b\in B_c}) \in \mathcal{P}(A_c)\\
\mu &= \gamma_q(\tau; (\sigma_c)_{c\in C}) \in \mathcal{P}(B)\\
\theta &= \gamma_p(\mu; (\rho_b)_{b\in B}) \in \mathcal{P}(A)\\
\phi &= \gamma_{qp}(\tau; (\lambda_c)_{c\in C}) \in \mathcal{P}(A).
\end{aligned}$$

From the definitions, we have

$$\begin{aligned}
\theta = \phi =& \{(qp)^{-1}(W) \mid W \in \gamma,\ |W| > 1, (qp)^{-1}(W) \neq \emptyset\} \amalg \\
& \coprod_{\{c\}\in\tau} \{p^{-1}(V) \mid V \in \sigma_c,\ |V| > 1,\ p^{-1}(V) \neq \emptyset\} \amalg \\
& \coprod_{\{c\}\in\tau} \coprod_{\{b\}\in\sigma_c} \alpha_b.
\end{aligned}$$

□

**Corollary 4.2.** *By geometric realisation, the (contractible) spaces $PA = |\mathcal{P}(A)|$ form an operad.* □

## 5. Trees

**Definition 5.1.** Let $A$ be a finite set. A *tree* on $A$ is a set $\mathcal{T} \subseteq \mathcal{C}(A)$ such that

- If $U, V \in \mathcal{T}$ then $U \subseteq V$ or $V \subseteq U$ or $U \cap V = \emptyset$
- The minimal sets in $\mathcal{T}$ cover $A$.

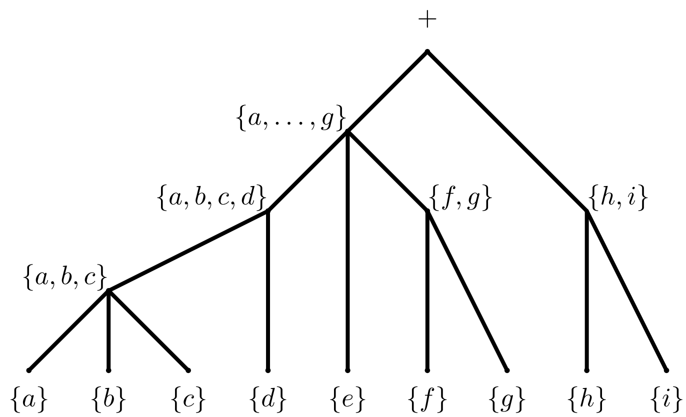
We write Trees$(A)$ for the set of such trees (which we regard as a poset with $\mathcal{T} \leq \mathcal{T}'$ iff $\mathcal{T} \subseteq \mathcal{T}'$). Note that if $|A| \leq 1$ then $A$ has precisely one tree.

We say that a tree $\mathcal{T}$ is is *separated* if every singleton lies in $\mathcal{T}$. For a general tree $\mathcal{T}$, the minimal sets in $\mathcal{T}$ form a partition $\pi$ of $A$, and every set in $\mathcal{T}$ is a union of some of the blocks of $\pi$, so $\mathcal{T}$ comes from a separated tree $\overline{\mathcal{T}}$ on $\pi$.

**Remark 5.2.** Given a tree $\mathcal{T}$ as defined above, we can construct a graph $\Gamma$ as follows. We regard $\mathcal{T}$ as a poset as before, and adjoin an element $+$ with $T < +$ for all $T \in \mathcal{T}$. We take $\mathcal{T} \amalg \{+\}$ as the vertex set of $\Gamma$, and we connect $a$ to $b$ by an edge whenever $a < b$ but there is no $x$ with $a < x < b$. The vertex $+$ is called the *root*, and the vertices corresponding to minimal sets are called *leaves* (so our trees are upside down). We find that $\Gamma$ is a tree in a more traditional sense, that the root and the leaves are the only univalent vertices, and there are no bivalent vertices. If $A = \{a, \dots, i\}$ and

$$\mathcal{T} = \{\{x\} \mid x \in A\} \cup \{\{a, b, c\}, \{f, g\}, \{h, i\}, \{a, b, c, d\}, \{a, b, c, d, e, f, g\}\}$$

then the tree $\Gamma$ is as shown below.




Conversely, suppose we are given a connected graph $\Gamma$ with no cycles or bivalent vertices, a distinguished univalent vertex called $+$, and a surjection $\lambda$ from $A$ to the set of remaining univalent vertices. Given a vertex $v$ we let $T_v$ be the set of those $a \in A$ such that the shortest path from $\lambda(a)$ to $+$ passes through $v$. Then the set $\mathcal{T} = \{T_v \mid v \in \mathrm{vert}(\Gamma)\}$ is an tree according to our definition.

**Definition 5.3.** Let $\mathcal{T}$ be a separated tree, and put $\mathcal{T}' = \{T \in \mathcal{T} \mid |T| > 1\}$. For $T \in \mathcal{T}$, a *child* of $T$ is a maximal element in the set $\{U \in \mathcal{T} \mid U \subset T\}$. We write $\delta T$ for the set of children of $T$, and observe that these sets partition of $T$. A *grown child* of $T$ is a child $U \in T$ such that $|U| > 1$ (so $U$ has children of its own). We write $\delta' T = \mathcal{T}' \cap \delta T$ for the set of grown children of $T$.

**Definition 5.4.** Let $p\colon A \to B$ be a map of finite sets. We say that a tree $\mathcal{T} \in \mathrm{Trees}(A)$ is *p-fibred* if for each $T \in \mathcal{T}$ we have $|p(T)| = 1$. The set of $p$-fibred trees is easily identified with $\mathrm{Trees}(p) = \prod_{b \in B} \mathrm{Trees}(A_b)$. Given $\mathcal{U} \in \mathrm{Trees}(B)$ and $\mathcal{T} \in \mathrm{Trees}(p)$ we define

$$\gamma_p(\mathcal{U}, \mathcal{T}) = \mathcal{T} \cup \{p^{-1}U \mid U \in \mathcal{U},\ p^{-1}U \neq \emptyset\}.$$

**Proposition 5.5.** *The above construction gives a map*

$$\gamma_p\colon \mathrm{Trees}(B) \times \mathrm{Trees}(p) \to \mathrm{Trees}(A),$$

*which makes* Trees *into an operad.*

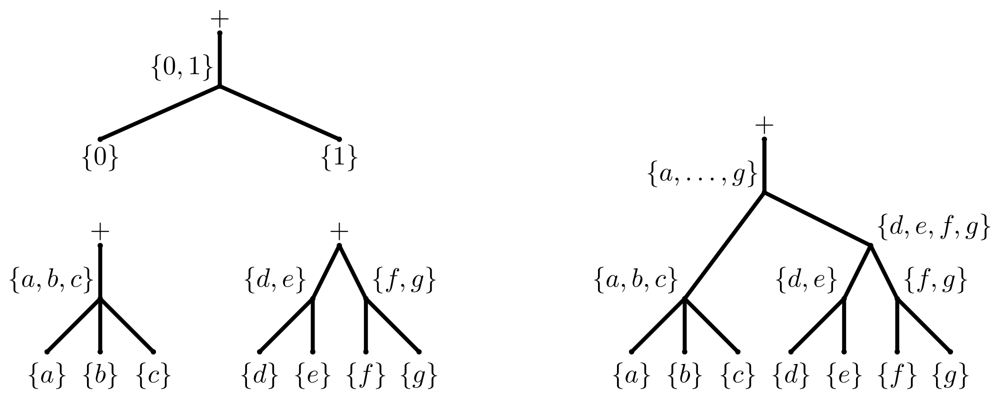
**Remark 5.6.** This is essentially the same as the well-known operation of grafting trees, but our formulation makes it easier to deal with degenerate cases. For example, consider the case where $A = \{a, \dots, g\}$ and $B = \{0, 1\}$ with $p$ sending $a, b, c$ to 0 and $d, e, f, g$ to 1. Consider the trees

$$\begin{aligned}
\mathcal{U} &= \{\{0\}, \{1\}, \{0, 1\}\} \in \mathrm{Trees}(B) \\
\mathcal{T}_0 &= \{\{a\}, \{b\}, \{c\}, \{a, b, c\}\} \in \mathrm{Trees}(A_0) \\
\mathcal{T}_1 &= \{\{d\}, \{e\}, \{f\}, \{g\}, \{d, e\}, \{f, g\}\} \in \mathrm{Trees}(A_1)
\end{aligned}$$

Then the tree $\mathcal{T}' = \gamma_p(\mathcal{U}; \mathcal{T}_0, \mathcal{T}_1)$ is given by

$$\mathcal{T}' = \{\{a\} \mid a \in A\} \cup \{\{a, b, c\}, \{d, e\}, \{e, f\}, \{a, b, c\}, \{d, e, f, g\}, A\}$$

Pictures are as follows:




*Proof.* First suppose we have $\mathcal{U} \in \mathrm{Trees}(B)$ and $\mathcal{T} \in \mathrm{Trees}(p)$ and put $\mathcal{X} = \gamma_p(\mathcal{U}, \mathcal{T})$. The minimal sets in $\mathcal{T}$ remain minimal in $\mathcal{X}$ and they cover $A$. Given $X, X' \in \mathcal{X}$ with $X \cap X' \neq \emptyset$, we must show that either $X \subseteq X'$ or $X' \subseteq X$. This is clear if both $X$ and $X'$ lie in $\mathcal{T}$. Next, consider the case where $X = p^{-1}U \neq \emptyset$ and $X' = p^{-1}U' \neq \emptyset$, with $U, U' \in \mathcal{U}$. As $X \cap X' \neq \emptyset$ we must have $U \cap U' \neq \emptyset$, so $U$ and $U'$ are nested, so $X$ and $X'$ are nested. Finally, consider the case where $X \in \mathcal{T}$ (so $p(X) = \{b\}$ for some $b \in B$) and $X' = p^{-1}U'$ for some $U' \in \mathcal{U}$. As $X \cap X' \neq \emptyset$ we must have $b \in U'$ and so $X \subseteq X'$. This shows that $\mathcal{X}$ is a tree, so we have a map

$$\gamma_p \colon \mathrm{Trees}(B) \times \mathrm{Trees}(p) \to \mathrm{Trees}(A).$$

Now suppose we have maps $A \xrightarrow{p} B \xrightarrow{q} C$ and elements $\mathcal{T} \in \mathrm{Trees}(p)$ and $\mathcal{U} \in \mathrm{Trees}(q)$ and $\mathcal{V} \in \mathrm{Trees}(C)$. This gives $\mathcal{X} = \gamma_{p,q}(\mathcal{U}, \mathcal{T}) \in \mathrm{Trees}(qp)$ and $\mathcal{Y} = \gamma_q(\mathcal{V}, \mathcal{U}) \in \mathrm{Trees}(B)$ and $\mathcal{Z} = \gamma_{qp}(\mathcal{V}, \mathcal{X}) \in \mathrm{Trees}(A)$ and $\mathcal{Z}' = \gamma_p(\mathcal{Y}, \mathcal{T}) \in \mathrm{Trees}(A)$. We must show that $\mathcal{Z} = \mathcal{Z}'$. In fact, we have

$$\begin{aligned}
\mathcal{X} &= (\mathcal{T} \cup \{p^{-1}(U) \mid U \in \mathcal{U}\}) \setminus \{\emptyset\} \\
\mathcal{Y} &= (\mathcal{U} \cup \{q^{-1}(V) \mid V \in \mathcal{V}\}) \setminus \{\emptyset\} \\
\mathcal{Z} &= (\mathcal{T} \cup \{p^{-1}(U) \mid U \in \mathcal{U}\} \cup \{(qp)^{-1}(V) \mid V \in \mathcal{V}\}) \setminus \{\emptyset\} \\
&= \mathcal{Z}'.
\end{aligned}$$

□

TODO:

- How close is $\gamma$ (in Definition 7.2) to being a homeomorphism?
- Use an inverse of some version of $\gamma$ to give a cooperad structure on $\widehat{P}(A)$.
- Why is the homology of this the Lie operad.
- Understand Ching's remarks about comparing this to little cubes/Poisson.
- Given $h \in H(A; \mathcal{T})$, we can extend over all of $\mathcal{C}A$ by $h(U) = h(\overline{U})$, where $\overline{U}$ is the smallest set in $\mathcal{T}$ containing $U$. This embeds $H(A; \mathcal{T})$ in $\mathrm{Map}(\mathcal{C}A, I)$. Does this extend to an embedding of $\widehat{P}(A)$?
- How does this interact with the constructions $P'A$, $\overline{P}A$, $\widehat{P}A$, $\widehat{P}_*A$ and so on?

## 6. Height functions

**Definition 6.1.** A *height function* on $A$ is a function $h \colon \mathcal{C}A \to I$ such that

- if $|U| = 1$ then $h(U) = 0$
- if $U \subseteq V$ then $h(U) \leq h(V)$
- if $U \cap V \neq \emptyset$ then $h(U \cup V) = \max(h(U), h(V))$.

We write $H(A)$ for the space of all height functions.

**Remark 6.2.** We will show later that this is homeomorphic to the geometric realisation of the poset $\mathcal{P}A$ of partitions of $A$, and also to a certain space of trees in which node of the tree has a specified height. The latter description gives rise to the name "height function".

**Example 6.3.** Suppose we have $A = \{a, b, c\}$, and we put

$$X = \{(\alpha, \beta, \gamma) \in I^3 \mid \max(\alpha, \beta) = \max(\beta, \gamma) = \max(\gamma, \alpha)\}.$$

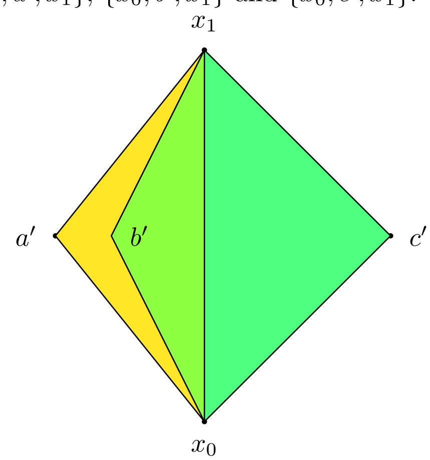
One checks that $H(A) \simeq X$, by the map

$$h \mapsto (h(\{b, c\}), h(\{a, c\}), h(\{a, b\})).$$

Moreover, we can triangulate $X$ (and thus $H(A)$) as follows. The vertices are

$$x_0 = (0, 0, 0)$$
$$x_1 = (1, 1, 1)$$
$$a' = (0, 1, 1)$$
$$b' = (1, 0, 1)$$
$$c' = (1, 1, 0),$$

and the maximal simplices are $\{x_0, a', x_1\}$, $\{x_0, b', x_1\}$ and $\{x_0, c', x_1\}$.




We next exhibit a cell structure on $H(A)$, with cells indexed by the set $\mathrm{Trees}(A)$.

**Definition 6.4.** Let $h$ be a height function on $A$. We say that a set $U \in \mathcal{C}(A)$ is *$h$-critical* if $h(U) < \min\{h(V) \mid U \subset V \subseteq A\}$. Here $\min(\emptyset)$ is interpreted as 1, so that $A$ is $h$-critical iff $h(A) < 1$, whereas a proper subset $U \subset A$ is $h$-critical iff every strict superset $V$ has $h(V) > h(U)$. We write $\tau(h)$ for the collection of $h$-critical sets.

**Lemma 6.5.** $\tau(h) \in \mathrm{Trees}(A)$.

*Proof.* Suppose that $U, V \in \tau(h)$ and $U \cap V \neq \emptyset$; we must show that either $U \subseteq V$ or $V \subseteq U$. We may assume without loss that $h(U) \leq h(V)$, so $h(U \cup V) = \max(h(U), h(V)) = h(V)$. As $V$ is $h$-critical and $V \subseteq U \cup V$ and $h(V) = h(U \cup V)$, we must have $V = U \cup V$, so $U \subseteq V$ as required.

Next, we must show that any $a \in A$ is contained in a minimal $h$-critical set. Put $\mathcal{X} = \{U \subseteq A \mid a \in U \text{ and } h(U) = 0\}$. As $h$ is a height function, this contains $\{a\}$ and is closed under unions. Put $X = \bigcup_{U \in \mathcal{X}} U$, which is the largest element of $\mathcal{X}$. This is easily seen to be $h$-critical, and to be minimal in $\tau(h)$, and it contains $a$ as required. □

**Definition 6.6.** Let $\mathcal{T}$ be an $A$-tree. A *$\mathcal{T}$-height function* on $A$ is a map $h \colon \mathcal{T} \to I$ such that

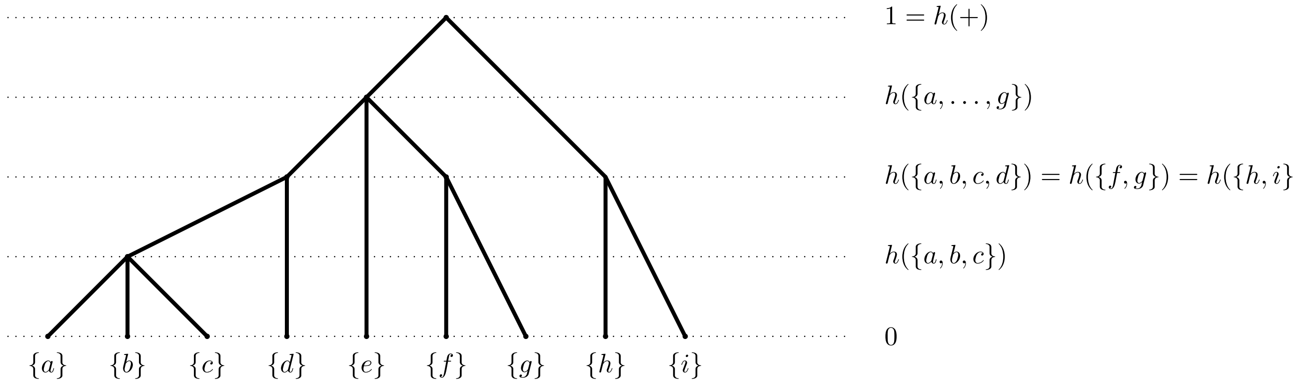
- if $U$ is minimal then $h(U) = 0$
- if $U \subseteq V$ then $h(U) \leq h(V)$.

We write $H(A; \mathcal{T})$ for the space of such functions. We also let $\mathcal{T}'$ be the collecton of non-minimal sets in $\mathcal{T}$, and put $d(\mathcal{T}) = |\mathcal{T}'|$. We say that a function $h \in H(A; \mathcal{T})$ is *singular* if for some $U \in \mathcal{T}$ we have

$$\min\{h(V) \mid U \subset V \in \mathcal{T}\} = h(U).$$

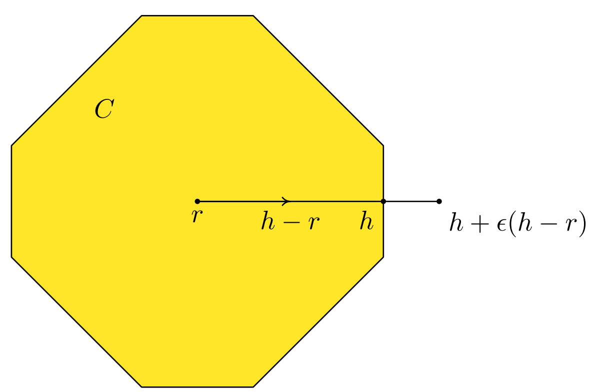
(In particular, if $h(U) = 1$ for some $U$ then $h$ is singular.) We write $\dot{H}(A; \mathcal{T})$ for the space of singular $\mathcal{T}$-height functions.

**Remark 6.7.** A picture of $\mathcal{T}$ as in Remark 5.2 implicitly gives a $\mathcal{T}$-height function, as illustrated below:




**Lemma 6.8.** *$H(A;\mathcal{T})$ is homeomorphic to a closed ball of dimension $d(\mathcal{T})$, with boundary $\dot{H}(A;\mathcal{T})$.*

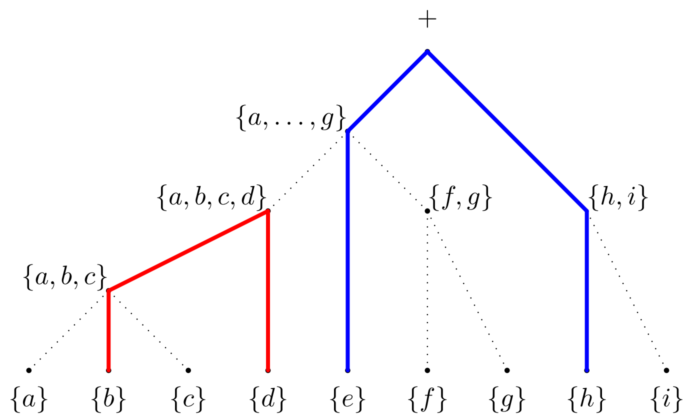
*Proof.* The restriction map $H(A;\mathcal{T}) \to \mathrm{Map}(\mathcal{T}', I)$ is injective, with image $C$ say. This is compact and convex. If we define $r\colon \mathcal{T}' \to I$ by $r(U) = |U|/(1+|A|)$ then $r$ lies in the interior of $C$. It is well-known that any compact convex set with nonempty interior in $\mathbb{R}^d$ is homeomorphic to the ball $B^d$. Moreover, a point $h$ lies on the boundary iff $h + \epsilon.(h-r)$ lies outside $H(A;\mathcal{T})$ for all $\epsilon > 0$.




This translates easily to the stated condition. □

**Definition 6.9.** Let $\mathcal{T}$ be an $A$-tree, and let $\mathcal{T}_+$ be the poset obtained by adjoining a largest element to $\mathcal{T}$. Given any height function $h \in H(A;\mathcal{T})$, we silently extend $h$ to $\mathcal{T}_+$ by putting $h(+) = 1$. For a nonempty set $U \subseteq A$, we note that the set $\mathcal{T}_+/U = \{V \in \mathcal{T}_+ \mid V \geq U\}$ is nonempty and linearly ordered by inclusion, so it has a smallest element; we denote this by $\pi_{\mathcal{T}}(U)$.

**Example 6.10.** If $\mathcal{T}$ is as in Remark 5.2, then $\pi_{\mathcal{T}}(\{b,d\}) = \{a,b,c,d\}$ and $\pi_{\mathcal{T}}(\{e,h\}) = +$, as we see from the diagram below.




**Proposition 6.11.** *If $h \in H(A;\mathcal{T})$ then $h \circ \pi_{\mathcal{T}} \in H(A)$. Moreover, the map $\pi_{\mathcal{T}}^*$ gives homeomorphisms*

$$H(A;\mathcal{T}) \to \{k \in H(A) \mid \tau(k) \subseteq \mathcal{T}\}$$
$$\dot{H}(A;\mathcal{T}) \to \{k \in H(A) \mid \tau(k) \subset \mathcal{T}\}.$$

*Proof.* Put $k = h \circ \pi_{\mathcal{T}}$. If $|U| = 1$ then $U$ is contained in some minimal element $U'$ of $\mathcal{T}$ (by Definition 5.1), and it is then clear that $\pi_{\mathcal{T}}(U) = U'$ and so $k(U) = h(U') = 0$. Suppose instead that we have sets $U, V \subseteq A$ with $U \subseteq V$. Then $\pi_{\mathcal{T}}(U) \subseteq \pi_{\mathcal{T}}(V)$ so $k(U) = h(\pi_{\mathcal{T}}(U)) \leq h(\pi_{\mathcal{T}}(V)) = k(V)$. Now suppose we have sets $U, V \subseteq A$ with $U \cap V \neq \emptyset$. Put $U' = \pi_{\mathcal{T}}(U)$ and $V' = \pi_{\mathcal{T}}(V)$. These sets overlap and they lie in $\mathcal{T}$ so they must be nested; we may assume that $U' \subseteq V'$. It follows from this that $\pi_{\mathcal{T}}(U \cup V) = V'$, and thus that

$$k(U \cup V) = k(V) = h(V') \geq h(U') = k(U),$$

so $k(U \cup V) = \max(k(U), k(V))$. This proves that $k$ is a height function, so we have a map $\pi_{\mathcal{T}}^* \colon H(A;\mathcal{T}) \to H(A)$. There is an evident restriction $\rho \colon H(A) \to H(A;\mathcal{T})$, which satisfies $\rho\pi_{\mathcal{T}}^* = 1$, so $\pi_{\mathcal{T}}^*$ is a closed embedding. All that is left is to identify the image.

Let $h$ and $k$ be as before. If $U \notin \mathcal{T}$ then the set $U' = \pi_{\mathcal{T}}(U)$ is a strict superset of $U$ with $k(U') = h(U') = k(U)$, so $U$ is not $k$-critical. This shows that $\tau(k) \subseteq \mathcal{T}$.

Conversely, suppose we start with a height function $k \in H(A)$ such that $\tau(k) \subseteq \mathcal{T}$. Put $h = \rho(k) \in H(A;\mathcal{T})$, so $h(U) = k(U)$ for $U \in \mathcal{T}$. Put $k' = \pi_{\mathcal{T}}^*(h) \in H(A)$. From the definitions it is clear that $k \leq k'$, and that $k'(U) = h(U) = k(U)$ for $U \in \mathcal{T}$. We claim that in fact $k = k'$. Indeed, suppose we have $U \in \mathcal{C}A$, with $k(U) = t$ say. Let $V$ be a maximal element of the set $\{V \mid U \subseteq V \subseteq A,\ k(V) = t\}$. Maximality implies that $V$ is $k$-critical, so $V \in \mathcal{T}$, so $k'(V) = k(V) = t$. We also have $k \leq k'$ so $t = k(U) \leq k'(U)$, and $U \subseteq V$ so $k'(U) \leq k'(V) = t$. This gives $k(U) = t = k'(U)$ as required. This in turn completes the proof that

$$\pi_{\mathcal{T}}^* H(A;\mathcal{T}) = \{k \in H(A) \mid \tau(k) \subseteq \mathcal{T}\}.$$

We leave the corresponding statement for $\dot{H}(A;\mathcal{T})$ to the reader. $\square$

**Notation .** From now on we identify $H(A;\mathcal{T})$ with $\pi_{\mathcal{T}}^* H(A;\mathcal{T}) \subseteq H(A)$, whenever this is convenient. From the proposition it is clear that $H(A;\mathcal{T}) \cap H(A;\mathcal{U}) = H(A;\mathcal{T} \cap \mathcal{U})$ and that $\partial H(A;\mathcal{T})$ is the union of the sets $H(A;\mathcal{U})$ for $\mathcal{U} \subset \mathcal{T}$. Thus, the sets $H(A;\mathcal{T})$ give a regular cell structure on $H(A)$.

**Example 6.12.** Take $A = \{a, b, c\}$, so $H(A)$ can be triangulated as in Example 6.3. It turns out that the cells $H(A;\mathcal{T})$ are just the simplices in this triangulation. Part of the correspondence is as follows:

| $\mathcal{T}$ | $H(A;\mathcal{T})$ |
|---|---|
| $\{\{a,b,c\}\}$ | $x_0$ |
| $\{\{a\},\{b,c\}\}$ | $a'$ |
| $\{\{a\},\{b\},\{c\}\}$ | $x_1$ |
| $\{\{a\},\{b,c\},\{a,b,c\}\}$ | $[x_0, a']$ |
| $\{\{a\},\{b\},\{c\},\{b,c\}\}$ | $[a', x_1]$ |
| $\{\{a\},\{b\},\{c\},\{a,b,c\}\}$ | $[x_0, x_1]$ |
| $\{\{a\},\{b\},\{c\},\{b,c\},\{a,b,c\}\}$ | $[x_0, a', x_1]$. |

The rest follows by symmetry.

**Definition 6.13.** Given a partition $\pi$ of $A$, we let $q_\pi\colon A \to \pi$ be the quotient map, and we define

$$h_\pi(U) = \begin{cases} 0 & \text{if } |q_\pi(U)| = 1 \\ 1 & \text{if } |q_\pi(U)| > 1. \end{cases}$$

This is easily seen to be a height function, so we have a map $\mathcal{P}(A) \to H(A) \subset \mathrm{Map}(\mathcal{C}(A), I)$. This extends in an obvious way to a map $\theta\colon |\mathcal{P}(A)| \to \mathrm{Map}(\mathcal{C}(A), I)$.

**Definition 6.14.** For any $A$-tree $\mathcal{T}$, we put $\mathcal{P}(A;\mathcal{T}) = \{\pi \in \mathcal{P}(A) \mid \pi \subseteq \mathcal{T}\}$ and $P(A;\mathcal{T}) = |\mathcal{P}(A;\mathcal{T})|$.

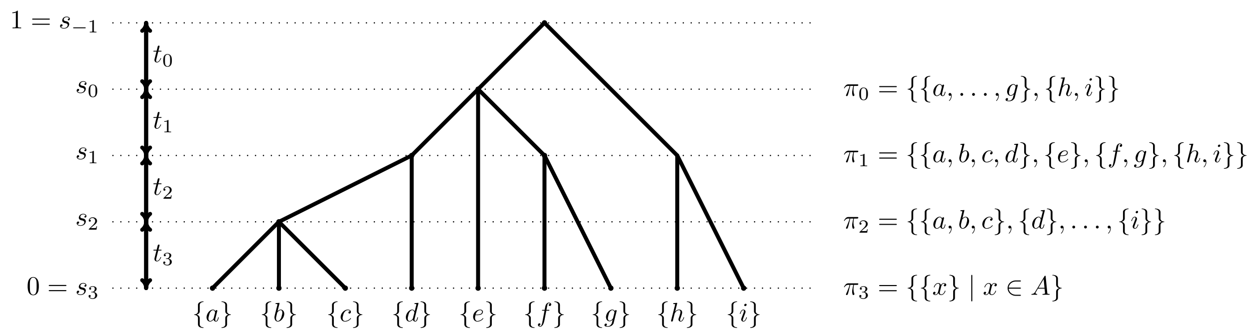
**Proposition 6.15.** *The map $\theta$ gives a homeomorphism $P(A) = |\mathcal{P}(A)| \to H(A)$, which restricts to give a homeomorphism $P(A;\mathcal{T}) \to H(A;\mathcal{T})$.*

*Proof.* First, we must show that the image of $\theta$ lies in $H(A)$. A point in $|\mathcal{P}(A)|$ is a map $x\colon \mathcal{P}(A) \to I$ whose support is a chain, such that $\sum_\pi x(\pi) = 1$. Suppose that the support consists of partitions $\pi_0 < \cdots < \pi_r$, and $x(\pi_i) = t_i$, so $t_i > 0$ and $\sum_i t_i = 1$. For any $U \in \mathcal{C}(A)$ there is an index $j(U) \in \{-1, 0, \ldots, r\}$ such that $|q_{\pi_i}(U)| > 1$ iff $i > j(U)$. We have $j(U) = r$ if $|U| = 1$, and $j(U) \geq j(V)$ whenever $U \subseteq V$, and $j(U \cup V) = \min(j(U), j(V))$ whenever $U \cap V \neq \emptyset$. Put $s_j = \sum_{i>j} t_i$, so $1 \geq s_0 > s_1 > \cdots > s_r = 0$. We have $\theta(x)(U) = s_{j(U)}$, and it follows easily that $\theta(x)$ is a height function. We also see that $U$ is $\theta(x)$-critical iff $U = A$ or $U$ is a singleton or a block of $\pi_i$ for some $i$. In other words, we have

$$\tau(\theta(x)) = \{A\} \cup \{\{a\} \mid a \in A\} \cup \bigcup_i \pi_i.$$

Given this, we see that $\theta(P(A;\mathcal{T})) \subseteq H(A;\mathcal{T})$.

Now suppose we start with a height function $h$, and let $\mathcal{T}$ be the tree of $h$-critical sets. Let the set of values of $h$ be $\{s_0, \ldots, s_r\}$, where $1 \geq s_0 > s_1 > \cdots > s_r = 0$. Put $\mathcal{T}_i = \{U \in \mathcal{T} \mid h(U) \leq s_i\}$, and let $\pi_i$ be the set of maximal elements in $\mathcal{T}_i$. One checks that $\pi_i \in \mathcal{P}(A)$ and $\pi_0 < \pi_1 < \ldots < \pi_r$. Put $t_i = s_{i-1} - s_i$ (where $s_{-1}$ is taken to be 1) and define $x\colon \mathcal{P}(A) \to I$ by $x(\pi_i) = t_i$ and $x(\pi) = 0$ for all $\pi \notin \{\pi_0, \ldots, \pi_r\}$.




We then have $x \in |\mathcal{P}(A;\mathcal{T})|$, so we can put $h' = \theta(x) \in H(A;\mathcal{T})$. We claim that $h' = h$. As $h', h \in H(A;\mathcal{T})$, it will suffice to show that $h(U) = h'(U)$ for $U \in \mathcal{T}$. If $|U| = 1$ we have $h(U) = 0 = \theta(x)(U)$, so we may assume that $|U| > 1$. Now consider the case $U = A$. Note that $h(A)$ is the largest value of $h$, namely $s_0$. Moreover, we can only have $|q_{\pi_i}(A)| = 1$ iff $\pi_i = \{\{A\}\}$ iff $A \in \mathcal{T}_i$ iff $h(A) \leq s_i$ iff $i = 0$. It follows that $h'(A) = t_1 + \cdots + t_r = s_0 = h(A)$. Finally, consider the case where $U \in \mathcal{T}$ but $|U| > 1$ and $U \neq A$. We have $h(U) = s_j$ for some $j$. For $i > j$, neither $U$ nor any superset of $U$ lies in $\mathcal{T}_i$, so $U$ cannot be contained in a block of $\pi_i$, so $|q_{\pi_i}(U)| > 1$. If $i \leq j$ then $U \in \mathcal{T}_i$ and so $|q_{\pi_i}(U)| = 1$. It follows that $h'(U) = \sum_{i>j} t_i = s_j = h(U)$, as required.

The above arguments show that $\theta$ gives a continuous bijection of compact Hausdorff spaces, and thus a homeomorphism. □

## 7. The operad structure for height functions

We next make $H$ into an operad. Fix a parameter $d$ with $1 \leq d \leq \infty$. Given $h \in H(A)$ and $a \in A$ we put

$$\phi(h, a) = \min\{h(\{a, a'\}) \mid a \neq a' \in A\}.$$

Next, suppose we have a map $p\colon A \to B$ and height functions $j \in H(B)$ and $h_b \in H(A_b)$ for all $b \in B$. We define a function $k\colon \mathcal{C}A \to I$ by

$$k(U) = \begin{cases} j(p(U)) & \text{if } |p(U)| > 1 \\ \phi(b,j) *_d h_b(U) & \text{if } p(U) = \{b\}. \end{cases}$$

**Lemma 7.1.** *This map $k\colon \mathcal{C}A \to I$ is a height function.*

*Proof.* If $|U| = 1$ then certainly $p(U) = \{b\}$ for some $b$ and $h_b(\{b\}) = 0$ (because $h_b$ is a height function) so $k(U) = 0$.

Next suppose we have $\emptyset \neq U \subseteq V \subseteq A$; we claim that $k(U) \leq k(V)$. If $|p(U)| > 1$ then also $|p(V)| > 1$ and the claim is clear. If $p(U) = \{b\}$ and $|p(V)| > 1$ then (from the definition of $\phi$) we have $\phi(b,j) *_d h_b(U) \leq \phi(b,j) \leq j(p(V))$, so $k(U) \leq k(V)$. If $p(U) = p(V) = \{b\}$ then $h_b(U) \leq h_b(V)$ and so again $k(U) \leq k(V)$, so the claim holds in all cases.

Now suppose we have sets $U, V \in \mathcal{C}A$ with $U \cap V \neq \emptyset$, say $a \in U \cap V$. Put $b = p(a)$ and $t = \phi(b,j)$. If neither $p(U)$ nor $p(V)$ is a singleton then

$$k(U \cup V) = j(p(U \cup V)) = j(p(U) \cup p(V)) = \max(j(p(U)), j(p(V))) = \max(k(U), k(V)).$$

If $p(U)$ and $p(V)$ are both singletons, then they must be equal to $\{b\}$, and we have $p(U \cup V) = \{b\}$ also, so

$$k(U \cup V) = t *_d h_b(U \cup V) = t *_d \max(h_b(U), h_b(V)) = \max(t *_d h_b(U), t *_d h_b(V)) = \max(k(U), k(V)).$$

By symmetry we now need only consider the case where $p(U) = \{b\}$ but $|p(V)| > 1$. As $b \in p(V)$ this implies that

$$k(V) = j(p(V)) \geq t \geq t *_d h_b(U) = k(U),$$

so $\max(k(U), k(V)) = j(p(V))$. We also find that $p(U \cup V) = p(U) \cup p(V) = p(V)$ and so $k(U \cup V) = j(p(V))$, so $k(U \cup V) = \max(k(U), k(V))$ as required. □

**Definition 7.2.** We define $\gamma_p^d\colon H(B) \times \prod_b H(A_b) \to H(A)$ by $\gamma_p^d(j; (h_b)_{b \in B}) = k$, where $k\colon \mathcal{C}A \to I$ is as described above.

**Proposition 7.3.** *This definition makes $H$ into an operad.*

*Proof.* Consider the following data:

- Finite sets $A$, $B$ and $C$.
- Maps $A \xrightarrow{p} B \xrightarrow{q} C$ (using which we define $A_b = p^{-1}\{b\}$ and $A_c = (qp)^{-1}\{c\}$ and $B_c = q^{-1}\{c\}$).
- A height function $k \in H(C)$.
- Height functions $j_c \in H(B_c)$ for all $c \in C$.
- Height functions $h_b \in H(A_b)$ for all $b \in B$.

These data can be combined in two different ways to give height functions $u, v \in H(A)$. The main axiom that we need to check is that $u = v$. In more detail, we put

$$\begin{aligned} m_c &= \gamma_p(j_c; (h_b)_{b \in B_c}) \in H(A_c) \\ n &= \gamma_q(k; (j_c)_{c \in C}) \in H(B) \\ u &= \gamma_p(n; (h_b)_{b \in B}) \in H(A) \\ v &= \gamma_{qp}(k; (m_c)_{c \in C}) \in H(A). \end{aligned}$$

From the definitions, we have

$$u(U) = v(U) = \begin{cases} k(qp(U)) & \text{if } |qp(U)| > 1 \\ \phi(k,c) *_d j_c(p(U)) & \text{if } |p(U)| > 1 \text{ and } qp(U) = \{c\} \\ \phi(k,c) *_d \phi(k_c, b) *_d h_b(U) & \text{if } p(U) = \{b\} \text{ and } q(b) = c. \end{cases}$$

□

We now take $d = \infty$. In this case, it is not hard to see that the maps $\gamma_p^\infty$ are piecewise linear. For future use, we will need to be more precise, and exhibit some subsets of $H(B) \times \prod_b H(A_b)$ on which the restriction of $\gamma_p^\infty$ is linear.

First put $X(p) = \mathcal{C}(B) \amalg \coprod_b \mathcal{C}(A_b)$. We can identify $H(A)$ with a subspace of $\mathrm{Map}(A, I)$, and thus identify $H(B) \times \prod_b H(A_b)$ with a subspace of $\mathrm{Map}(X(p), I)$.

**Definition 7.4.** A subset $T \subseteq H(B) \times \prod_b H(A_b) \subseteq \mathrm{Map}(X(p), I)$ is *unmixed* if there exists a total ordering of $X(p)$ such that every element $t \in T$ is nondecreasing as a map $X(p) \to I$. Equivalently, $T$ is unmixed if there do not exist $t, t' \in T$ and $x, x' \in X$ with $t(x) < t(x')$ and $t'(x) > t'(x')$.

**Lemma 7.5.** *If $T$ is unmixed then $\gamma_p^\infty|_T$ is (the restriction to $T$ of) a linear map.*

*Proof.* Fix a total order on $X(p)$ such that each map in $T$ is nondecreasing with respect to that order. Define a map $\xi\colon \mathcal{C}(A) \to X(p)$ as follows. If $U \in \mathcal{C}(A)$ and $|p(U)| > 1$ then we put $\xi(U) = p(U) \in \mathcal{C}(B) \subseteq X(p)$. If $p(U) = \{b\}$, we instead put

$$\Xi(U) = \{\{b, b'\} \mid b' \in B,\ b' \neq b\} \amalg \{U\} \subseteq \mathcal{C}(B) \amalg \mathcal{C}(A_b) \subseteq X(p).$$

We then let $\xi(U)$ be the smallest element of $\Xi(U)$, with respect to our chosen ordering. We now have a linear map $\xi^*\colon \mathrm{Map}(X(p), \mathbb{R}) \to \mathrm{Map}(\mathcal{C}(A), \mathbb{R})$, and one checks directly that $\gamma_p^\infty$ agrees with $\xi^*$ on $T$. ☐

**Proposition 7.6.** *If we take $d = \infty$ then the isomorphism $\theta\colon P \to H$ (from Proposition 6.15) respects operad structures.*

*Proof.* We first show that $\theta$ preserves operad structures on the 0-skeleton. Consider a map $p\colon A \to B$, a partition $\pi$ of $B$, and partitions $\omega_b$ of the fibres $A_b$. These give a partition

$$\sigma = \gamma_p(\pi; (\omega_b)_{b \in B}) = \{p^{-1}V \mid V \in \pi,\ |V| > 1,\ p^{-1}V \neq \emptyset\} \amalg \coprod_{\{b\} \in \pi} \omega_b \in \mathcal{P}(A),$$

and thus a height function $h_\sigma = \theta(\sigma) \in H(A)$. We also have height functions $h_\pi \in H(B)$ and $h_{\omega_b} \in H(A_b)$, and thus a height function $k = \gamma_p(h_\pi; (h_{\omega_b})_{b \in B}) \in H(A)$, which is easily seen to take only the values 0 and 1. We must show that $k = h_\sigma$, or equivalently that $k(U) = 0$ iff $U$ is contained in a single block of $\sigma$. We first consider the case where $|p(U)| > 2$, so $k(U) = h_\pi(p(U))$. In this case $U$ can only be contained in a block of the form $p^{-1}(V)$ for some non-singleton block $V \in \pi$, and this occurs iff $p(U)$ is contained in a block of $\pi$, or equivalently $k(U) = 0$. Now consider instead the case where $p(U) = \{b\}$, where $\{b\}$ is a block of $\pi$. This means that $\phi(h_\pi, b) = 1$ and so $k(U) = h_{\omega_b}(U)$, which is zero iff $U$ is contained in a block of $\omega_b$. In this case $\omega_b \subseteq \sigma$, and these are the only blocks of $\sigma$ that could possibly contain $U$, so again we have $k(U) = h_\sigma(U)$. Finally, consider the case where $p(U) = \{b\}$, and the block $V$ of $\pi$ containing $b$ has $|V| > 1$. This means that $p^{-1}(V)$ is a block of $\sigma$ and so $h_\sigma(U) = 0$. We also have $\phi(b, h_\pi) = 0$ (as $h_\pi(\{b, b'\}) = 0$ for any $b' \in V \setminus \{b\}$) and so $k(U) = 0$.

We now need to extend this to all of $P$. This amounts to the following: for any simplex $\sigma \in PB \times \prod_b PA_b$, the composite

$$\sigma \to PB \times \prod PA_b \xrightarrow{\theta \times \prod \theta} HB \times \prod HA_b \xrightarrow{\gamma} HA$$

is affine. By Lemma 7.5, it will suffice to show that the image of $\sigma$ in $HB \times \prod_b HA_b$ is unmixed.

Such a simplex $\sigma$ is determined by a chain $\pi_0 \leq \cdots \leq \pi_r$ in $\mathcal{P}(B)$, together with chains $\omega_{b0} \leq \cdots \leq \omega_{br}$ (of the same length) in $\mathcal{P}(A_b)$ for all $b$, such that for all $i < r$ we have either $\pi_i < \pi_{i+1}$ or $\omega_{bi} < \omega_{b,i+1}$ for some $b$. **unfinished** ☐

## 8. Spanier-Whitehead duality

Consider a vector space $V$ and an open subset $U \subseteq V \subseteq S^V$. Put $U^c = S^V \setminus U = (V \setminus U) \cup \{\infty\}$. We have a subtraction map $U \times S^V \to S^V$ which sends $U \times U^c$ to $S^V \setminus \{0\}$ (which is contractible). On passing to cofibres we get a map $U_+ \wedge (S^V/U^c) \to S^V$, which is adjoint to a map $\Sigma^{-V}(S^V/U^c) \to D(U_+)$. This can be shown to be an equivalence. Moreover, we have $S^V/U^c = U \cup \{\infty\}$.

Now specialise to the case where $V = NWA$ (for some finite set $A$ and $N > 0$), and $U$ is the subspace $\mathrm{Inj}_0(A, \mathbb{R}^N)$ of injective maps $x\colon A \to \mathbb{R}^N$ such that $\sum_a x(a) = 0$. The conclusion is

$$D(\mathrm{Inj}_0(A, \mathbb{R}^N)_+) = \Sigma^{-NWA} S^{NWA} / \mathrm{Inj}_0(A, \mathbb{R}^N)^c = \Sigma^{-NWA}\, \mathrm{Inj}_0(A, \mathbb{R}^N) \cup \{\infty\}.$$

It is known that $H_* \operatorname{Inj}_0(A, \mathbb{R}^N) = \operatorname{Poiss}_N(A)_*$, where $\operatorname{Poiss}_N$ is the operad (in graded abelian groups) for $N$-Poisson algebras. Moreover, this is concentrated in degrees $(N-1)i$ for $0 \leq i < a$, where $a = |A|$. We thus have

$$\widetilde{H}^*(\operatorname{Inj}_0(A, \mathbb{R}^N) \cup \{\infty\}) = \det(A)^N \otimes \operatorname{Poiss}_N(A)_{N(a-1)-*},$$

which is concentrated in degrees $(a-1) + (N-1)j$ for $0 \leq j < a$.

It is convenient to have simplicial versions of some of this.

**Lemma 8.1.** *Let $K$ be a simplicial complex with vertex set $U = V \amalg W$, and let $L$ and $M$ be the induced subcomplexes with vertex sets $V$ and $W$. Then $|L|$ is a strong deformation retract of $|K| \setminus |M|$ and $|M|$ is a strong deformation retract of $|K| \setminus |L|$.*

*Proof.* We have

$$\begin{aligned} |K| &= \{(x,y) \mid x \colon V \to I,\ y \colon W \to I,\ \textstyle\sum x + \sum y = 1,\ \operatorname{supp}(x) \cup \operatorname{supp}(y) \in \operatorname{simp}(K).\} \\ |L| &= \{(x,y) \in K \mid y = 0\} = \{(x,y) \in K \mid \textstyle\sum y = 0\} \\ |K| \setminus |M| &= \{(x,y) \in K \mid \textstyle\sum x > 0\}. \end{aligned}$$

We can thus define a retraction $p \colon |K| \setminus |M| \to |L|$ by $p(x,y) = (x/\sum x, 0)$ and a homotopy $h_t \colon |K| \setminus |M| \to |K| \setminus |M|$ by $h_t(x,y) = ((1 - t + t/\sum x)x, (1-t)y)$. This displays $|L|$ as a strong deformation retract of $|K| \setminus |M|$, and the other claim is of course symmetrical. ☐

Now suppose we have a finite set $A$ and a subcomplex $L \subset \partial\Delta_A$.

## 9. Filtered spheres

Given $\pi \in \mathcal{P}(A)$ we have a quotient map $q_\pi \colon A \to \pi$ and so inclusions $q_\pi^* \colon (BW\pi)^N \to (BWA)^N$ and $q_\pi^* \colon S^{NW\pi} \to S^{NWA}$. We filter $(BWA)^N$ by

$$F_k(BWA)^N = \{x \in BWA^N \subseteq \operatorname{Map}(A, I^N) \mid |x(A)| \leq k\} = \bigcup_{|\pi| \leq k} q_\pi^*(BW\pi)^N.$$

We use the induced filtration on $S^{NWA}$.

We next explain how this interacts with the (co)operad structure when $N = 1$. Suppose we have a map $p \colon A \to B$ as usual, giving a map

$$\gamma \colon S^{WB} \wedge \bigwedge_{b \in B} S^{WA_b} \to S^{WA}.$$

To avoid trivialities, assume that $B \neq \emptyset$ and $p$ is surjective. Suppose given $0 \leq j < |B|$ and $0 \leq i_b < |A_b|$ for all $b$, and put $i = j + \sum_b i_b$. We claim that

$$\gamma\left(F_{j+1}S^{WB} \wedge \bigwedge_b F_{i_b+1}S^{WA_b}\right) \subseteq F_{i+1}S^{WA}.$$

To see this, suppose that $y \in F_{j+1}BWB$ and $x_b \in F_{i_b+1}BWA_b$, and put $x = \gamma(y; (x_b)_{b \in B})$. We then have $y(B) = \{1\} \amalg Y$ and $x_b(A_b) = \{1\} \amalg X_b$, for some sets $Y$ and $X_b$ with $|Y| \leq j$ and $|X_b| \leq i_b$. If we put $X = Y \cup \bigcup_b y_b X_b$ we find that $|X| \leq i$ and $x(A) \subseteq \{1\} \amalg X$, so $x \in F_{i+1}S^{WA}$ as claimed.

This does not work when $N > 1$. For example, take $N = 2$ and $A = \{0,1,2\} \times \{3,4,5\}$ and $B = \{0,1,2\}$ and $p(i,j) = i$ (so $A_b = \{3,4,5\}$ for all $b$). Then put

$$\begin{aligned} y(0) &= [1, 1/2] & y(1) &= [1/2, 1] & y(2) &= [1, 1/2] \\ x_0(3) &= [1, 1/3] & x_0(4) &= [1/3, 1] & x_0(5) &= [1, 1/3] \\ x_1(3) &= [1, 1/5] & x_1(4) &= [1/5, 1] & x_1(5) &= [1, 1/5] \\ x_2(3) &= [1, 1/7] & x_2(4) &= [1/7, 1] & x_2(5) &= [1, 1/7], \end{aligned}$$

so

$$
\begin{array}{lll}
x(0,3) = [1,1/6] & x(0,4) = [1/3,1/2] & x(0,5) = [1,1/6] \\
x(1,3) = [1/2,1/3] & x(1,4) = [1/6,1] & x(1,5) = [1/2,1/3] \\
x(2,3) = [1,1/14] & x(2,4) = [1/7,1/2] & x(2,5) = [1,1/14].
\end{array}
$$

In this case we have $j = i_0 = i_1 = i_2 = 1$ so $i = 4$, but $|x(A)| - 1 = 5$ so $x \notin F_i S^{2WA}$.

Put $F_k = F_k S^{NWA}$ and $F_k^c = S^{NWA} \setminus F_k$. We then have

$$
\begin{aligned}
F_k S / F_{k-1} &= \bigvee_{|\pi|=k} \mathrm{Inj}_0(\pi, \mathbb{R}^N) \cup \{\infty\} = \bigvee_{|\pi|=k} \Sigma^{NW\pi} D(\mathrm{Inj}_0(\pi, \mathbb{R}^N)_+) \\
F_k \setminus F_{k-1} &= \coprod_{|\pi|=k} \mathrm{Inj}_0(\pi, \mathbb{R}^N) = F_{k-1}^c \setminus F_k^c \\
F_{k-1}^c / F_k^c &= \Sigma^{NWA} D(F_k / F_{k-1}) = \bigvee_{|\pi|=k} \Sigma^{NWA - NW\pi} \mathrm{Inj}_0(\pi, \mathbb{R}^N)_+ .
\end{aligned}
$$

Now put

$$
\widetilde{\mathcal{Q}}A = \{(\pi, U) \in \mathcal{P}A \times \mathcal{C}A \mid U \text{ is a union of blocks of } \pi\} \simeq \coprod_{\pi \in \mathcal{P}A} \mathcal{C}\pi.
$$

We consider this as a subposet of $\mathcal{P}A \times \mathcal{C}A$, so we have a geometric realisation $\widetilde{Q}A = |\widetilde{\mathcal{Q}}A| \subseteq PA \times BWA$. This can be identified as

$$
\widetilde{Q}A = \underset{\pi \in \mathcal{P}A}{\underrightarrow{\mathrm{hocolim}}}\, |\mathcal{C}\pi| = \underset{\pi \in \mathcal{P}A}{\underrightarrow{\mathrm{hocolim}}}\, BW\pi.
$$

This space is contractible, as $\widetilde{\mathcal{Q}}A$ has a largest object. However, certain associated complexes will have interesting homotopy. Firstly, we put

$$
\begin{aligned}
\dot{\mathcal{Q}}A &= \{(\pi, U) \in \widetilde{\mathcal{Q}}A \mid U \neq A\} \\
\dot{Q}A &= |\dot{\mathcal{Q}}A| = \underset{\mathcal{P}A}{\underrightarrow{\mathrm{hocolim}}}\, SW\pi \\
QA &= \widetilde{Q}A / \dot{Q}A = \underset{\mathcal{P}A}{\underrightarrow{\mathrm{hocolim}}}^{*}\, S^{W\pi}.
\end{aligned}
$$

Here we write $\underrightarrow{\mathrm{hocolim}}$ for the homotopy colimit in the unbased category, and $\underrightarrow{\mathrm{hocolim}}^*$ for the based analogue. For a diagram $X$ indexed by a category $\mathcal{J}$ we have

$$
\underset{\mathcal{J}}{\underrightarrow{\mathrm{hocolim}}}^{*}\, X = \underset{\mathcal{J}}{\underrightarrow{\mathrm{hocolim}}}\, X / \underset{\mathcal{J}}{\underrightarrow{\mathrm{hocolim}}} * = (\underset{\mathcal{J}}{\underrightarrow{\mathrm{hocolim}}}\, X)/|\mathcal{J}|.
$$

In particular, if $|\mathcal{J}|$ is contractible then the two versions of the homotopy colimit are homotopy equivalent.

Note that the above constructions still give little new: as $\mathcal{P}A$ has a largest element, we find that the evident maps $SWA \to \dot{Q}A$ and $S^{WA} \to QA$ are homotopy equivalences.

Next, recall that we have subposets $\mathcal{P}_k A, \mathcal{P}'A, \mathcal{P}'_k \subseteq \mathcal{P}A$. We write $\widetilde{\mathcal{Q}}_k A$, $\widetilde{\mathcal{Q}}'A$ and $\widetilde{\mathcal{Q}}'_k A$ for the preimages of these posets in $\widetilde{\mathcal{Q}}A$. We then write $\widetilde{Q}_k A$, $\widetilde{Q}'A$ and $\widetilde{Q}'_k A$ for the geometric realisations, and put $Q_k A = \widetilde{Q}_k A / (\widetilde{Q}_k A \cap \dot{Q}A)$ and so on.

Note that $Q_k A = \underrightarrow{\mathrm{hocolim}}_{\mathcal{P}_k A}\, S^{W\pi}$, whereas $\underrightarrow{\mathrm{colim}}_{\mathcal{P}_k A}\, S^{W\pi}$ is easily identified with $F_k S^{WA}$. We claim that the map $Q_k A \to F_k S^{WA}$ is a homotopy equivalence. **Prove this.**

Next, note that we can apply $\underrightarrow{\mathrm{hocolim}}^*_{\mathcal{P}'_k A}$ and $\underrightarrow{\mathrm{hocolim}}^*_{\mathcal{P}_k A}$ to the cofibrations $SW\pi_+ \to S^0 \to S^{W\pi}$ give a diagram

$$
\begin{array}{ccccc}
\dot{Q}'_k A & \longrightarrow & \widetilde{Q}'_k A & \longrightarrow & Q'_k A \\
\downarrow & & \downarrow & & \downarrow \\
\dot{Q}_k A & \longrightarrow & \widetilde{Q}_k A & \longrightarrow & Q_k A,
\end{array}
$$

in which the rows are cofibrations. As $SW1 = \emptyset$, we see directly that $\dot{Q}'_k A = \dot{Q}_k A$, so the right hand square is a homotopy pushout. Taking vertical cofibres shows that the map $\overline{P}_k A \to \overline{Q}_k A$ is an equivalence.

On the other hand, we have a map $\tau\colon s\mathcal{C}A \to \mathcal{P}A$ sending a chain $(U_0 \subset \cdots \subset U_d)$ to the partition

$$\pi = \{U_0, U_1 \setminus U_0, \dots, U_d \setminus U_{d-1}, A \setminus U_d\} \setminus \{\emptyset\}$$

This gives a map $\tau\colon BWA \to PA$. In this picture $SWA$ is the subcomplex generated by vertices $U = (U_0 \subset \cdots \subset U_d)$ as above for which $U_d \neq A$. Implicitly here $U_0 \neq \emptyset$, and $U_0$ and $A \setminus U_d$ are distinct blocks of $\pi$ so $\pi \in \mathcal{P}'A$. This shows that $\tau(SWA) \subseteq P'A$, so $\tau$ induces a map $\tau\colon S^{WA} \to \overline{P}A$. Essentially by definition we have $F_k S^{WA} = \tau^{-1}\overline{P}_k A$. We thus have a map

$$Q_k A \to F_k S^{WA} \xrightarrow{\tau} \overline{P}_k A.$$

It is natural to hope that this map sends $Q'_k A$ to the basepoint, and that the resulting map $\overline{Q}_k A \to \overline{P}_k A$ is inverse to the equivalence considered above. However, we have not yet proved this.

Most of the above should also work with $S^{WA}$ replaced by $S^{NWA}$ for any $N > 0$. **Supply details**

## 10. Chained preorders

**Definition 10.1.** A *linear preorder* on a finite set $A$ is a relation $R$ such that

(a) For all $a, b \in A$ we have either $aRb$ or $bRa$.
(b) If $aRb$ and $bRc$ then $aRc$.

We write $LP(A)$ for the set of linear preorders on $A$.

Note that the first axiom implies $aRa$. We write $a \equiv_R b$ if $aRb$ and $bRa$; this is an equivalence relation, and $R$ induces a total ordering of the quotient set $A/\equiv_R$.

If $R'$ is a linear preorder, we write $R \mid R'$ (and say that $R'$ refines $R$) if $aR'b \implies aRb$, or $R \supseteq R'$ as subsets of $A^2$. This gives a partial order on $LP(A)$.

We say that a linear preorder $R$ is *separated* if $(a \equiv_R b \implies a = b)$, so it is just a linear ordering of $A$. We write $\mathrm{Ord}(A)$ for the set of linear orderings.

**Remark 10.2.** Let $R$ be a linear preorder, and let $A_1, \dots, A_r$ be the equivalence classes for $\equiv_R$. It is not hard to see that there is a bijection $\{\text{refinements of } R\} = \prod_i LP(A_i)$.

**Definition 10.3.** An $n$-chain of linear preorderings is a sequence $(R_1, \dots, R_n) \in LP(A)^n$ with $R_i \mid R_{i+1}$. We say that the chain is separated if $R_n$ is separated. We write $CL_n(A)$ for the set of separated $n$-chains. We give this a partial order as follows: we have $R \leq Q$ if $R_1 \subseteq Q_1$ and $R_i \cap R_{i-1}^{\mathrm{op}} \subseteq Q_i \cap Q_{i-1}^{\mathrm{op}}$ for all $i > 1$.

TODO:

- $|CL_n(A)| \simeq \mathrm{Inj}(A, \mathbb{R}^n)$
- Operad structure, Faddell-Neuwirth fibrations?
- Comparison with Salvetti, Kashiwabara, Smith, Getzler

## 11. The Smith operad

Let $E$ be the usual functor from sets to simplicial sets given by $(EX)_n = \mathrm{Map}([n], X)$. It is well-known that this is contractible, and $E(X \times Y) = EX \times EY$. We thus have an operad $E\,\mathrm{Ord}$ in simplicial sets, known as the Barratt-Eccles operad. It is known that this is $E_\infty$, so its algebras are essentially infinite loop spaces.

Jeff Smith defined suboperads $C_k$ of $E\,\mathrm{Ord}$ whose algebras are essentially $k$-fold loop spaces. We recall these operads here.

If $|A| \leq 1$ then $E\,\mathrm{Ord}(A)$ is a point, and $C_k(A) = E\,\mathrm{Ord}(A)$. If $|A| = 2$ then $C_k(A)$ is the $k-1$-skeleton of $E\,\mathrm{Ord}(A)$. To understand this in more detail, let $\tau\colon A \to A$ be the map that exchanges the two elements of $A$, and let $\rho(a)$ be the ordering for which $a < \tau(a)$. Define $i_m\colon A \to \mathrm{Map}([m], \mathrm{Ord}(A))$ by $i_m(a)(j) = \rho(\tau^j(a))$. Then $i_m$ gives a bijection from $A$ to the set of nondegenerate $m$-simplices of $E\,\mathrm{Ord}(A)$. It follows that $C_k(A)$ is generated by $\{i_m(a) \mid m < k, a \in A\}$. I think that $|C_k(A)|$ is homeomorphic to $S((WA)^k)$, with the two $m$-cells corresponding to the hemispheres in $S((WA)^{m+1})$ obtained by cutting along $S((WA)^m)$.

Now consider a set $A$ with $|A|>2$. For any $B\subseteq A$ we have a restriction map $\mathrm{res}_B\colon \mathrm{Ord}(A)\to\mathrm{Ord}(B)$, which induces $\mathrm{res}_B\colon E\,\mathrm{Ord}(A)\to E\,\mathrm{Ord}(B)$. We put

$$C_k(A)=\bigcap_{|B|=2}\mathrm{res}_B^{-1}C_k(B).$$

Here are some thoughts about counting cells in simplicial sets. Put $b_k(X)=|X_k|$, and let $a_k(X)$ be the number of nondegenerate $k$-simplices. Each nondegenerate $k$-simplex gives $\left(\begin{array}{c}n\\k\end{array}\right)$ $n$-simplices, so $b_k=\sum_{j\leq k}\left(\begin{array}{c}k\\j\end{array}\right)a_j$. If we put

$$\begin{aligned}f(X,t)&=\sum_k a_k(X)t^k/k!\\ g(X,t)&=\sum_k b_k(X)t^k\end{aligned}$$

we find that

$$g(X,t)=\frac{1}{1-t}\sum_k a_k(X)\left(\frac{t}{1-t}\right)^k.$$

We have $b_k(X\times Y)=b_k(X)b_k(Y)$, and

$$a_m(X\times Y)=\sum_{m=i+j+k}\frac{m!}{i!j!k!}a_{i+j}(X)a_{i+k}(Y).$$

The point here is that $m!/(i!j!k!)$ is the number of strictly increasing maps $u=(v,w)\colon[m]\to[i+j]\times[i+k]$ for which $v$ and $w$ are surjective. I do not know a useful formula for $f(X\times Y,t)$ or $g(X\times Y,t)$ based on these.

In the case $X=EN$ with $|N|=n$, we have

$$\begin{aligned}a_k(EN)&=n(n-1)^k\\ b_k(EN)&=n^{k+1}\\ f(EN,t)&=ne^{(n-1)t}\\ g(EN,t)&=n/(1-nt)\end{aligned}$$

- The spaces $C_k(A)$ give a suboperad of $E\,\mathrm{Ord}$
- Are there simplicial analogs of the Faddell-Neuwirth fibrations? There are certainly maps $\rho^A_B\colon C_k(A)\to C_k(B)$ for $B\subseteq A$; these come from $\gamma_i$, where $i\colon B\to A$ is the inclusion. We also have $\gamma_p$ for $p\colon A\to A/B$, which gives rise to a map $C_k(B)\to C_k(A)$, which should be a section for $\rho^A_B$. When $|A|=3$ and $|B|=2=k$, the numerology is as follows. In $C_2(A)$, the numbers of nondegenerate simplices in dimensions $0,\dots,3$ are $6,30,36,12$ (and there are none in higher dimensions). In $C_2(B)$ there are two nondegenerate 0-simplices, two nondegenerate 1-simplices, and no nondegenerate simplices in higher dimension. Experimentally, it seems that for $x\in C_2(B)_d$, the size of the preimage $\pi^{-1}\{x\}\subseteq C_2(A)_d$ is independent of $x$. Everything is consistent with the idea that $C_2(A)$ is a twisted cartesian product of $C_2(B)$ with a complex $F$ with nondegenerate cell count $(3,6,2)$, which should be homotopy equivalent to $S^1\vee S^1$. Moreover, it is not hard to see that for any $A$ the preimage in $C_2(A_+)$ of any vertex in $C_2(A)$ is a copy of $(\Delta_{|A|}\amalg\Delta_{|A|})/\sim$, where the equivalence relation identifies corresponding vertices in the two simplices. This has the homotopy type of $\mathbb{R}^2\setminus A$, as expected. To obtain this description, consider the set $A'$ of gaps in $A$ (with the order given by the chosen vertex); this can be identified with $\{0,\dots,|A|\}$. One simplex corresponds to strictly increasing maps $[k]\to A'$, and the other to strictly decreasing maps. When $k=0$ any map is both increasing and decreasing which is why the vertices get identified.

  However, things seem to go wrong for $A=\{0,1,2,3\}$ and $B=\{0,1,2\}$. Here the 2-simplex $[[0,1,2],[0,1,2],[0,1,2]]$ has 36 preimages, whereas $[[0,1,2],[0,2,1],[2,0,1]]$ has 38 and $[[0,1,2],[0,2,1],[2,1,0]]$ has 35.

- Define a map from the linear preorder model to the Smith model. A vertex of $LP_k(A)$ is a $k$-flag of preorders, the finest of which is a linear order, so there is an evident projection $LP_k(A) \to \mathrm{Ord}(A) = E\,\mathrm{Ord}(A)_0$. An $n$-simplex of $LP_k(A)$ is a chain of $n+1$ such $k$-flags, giving a map $LP_k(A)_n \to E\,\mathrm{Ord}(A)_n$. We claim that this lands in $C_k(A)_n$. Indeed, suppose we have $a \neq b \in A$, and they become separated at the $r_i$'th stage in the $i$'th flag of preorders, so $r_1 \geq 1$ and $r_i \leq k$. If the relative ordering of $a$ and $b$ changes between the $i$'th and $i+1$'st flags, then we must have $r_{i+1} > r_i$. It follows that there can be at most $k-1$ such swaps.
- Does the linear preorder model admit a compatible operad structure?
- Prove that $C_k(A)$ is a finite complex.
- Note that $C_1(A) = \mathrm{Ord}(A)$, so $C_1$ is just the associative operad. One can check that this does not map to the little 1-cubes operad, so there is no map $C_k \to \mathrm{Cub}_k$ in general. I do not know of a good map from $C_k(A)$ to $\mathrm{Inj}(A, \mathbb{R}^k)$ either, although probably one exists.

## 12. Configuration spaces

Here we analyse the homotopy type and cohomology of the spaces $F_0(A) = \mathrm{Inj}_0(A, V)$ for $|A| \leq 3$, where $V$ is an inner product space of dimension $d > 1$ say. It is clear that the space $F(A) = \mathrm{Inj}(A, V)$ is homeomorphic to $V \times F_0(A)$, which is homotopy equivalent to $F_0(A)$, so we will work with whichever is more convenient. We choose a generator $u \in \widetilde{H}^{d-1} S(V)$.

For $a, b \in A$ with $a \neq b$ we define $\delta_{ab} \colon F(A) \to S(V)$ by $\delta_{ab}(v) = (v_b - v_a)/\|v_b - v_a\|$. We then put $u_{ab} = \delta_{ab}^* u \in H^{d-1} F(A)$, and note that $u_{ba} = (-1)^d u_{ab}$ and $u_{ab}^2 = 0$. Given $a, b, c \in A$ (all distinct), we put

$$r_{abc} = u_{ab} u_{bc} + u_{bc} u_{ca} + u_{ca} u_{ab}.$$

It will turn out that this is always zero. For the moment we note that $r_{abc} = r_{bca}$ and

$$\begin{aligned} r_{bac} &= u_{ba} u_{ac} + u_{ac} u_{cb} + u_{cb} u_{ba} \\ &= (-1)^{2d} (u_{ab} u_{ca} + u_{ca} u_{bc} + u_{bc} u_{ab}) \\ &= (-1)^{2d + (d-1)^2} (u_{ca} u_{ab} + u_{bc} u_{ca} + u_{ab} u_{bc}) \\ &= (-1)^{d-1} r_{abc}. \end{aligned}$$

It follows that, up to sign, $r_{abc}$ depends only on $\{a, b, c\}$.

It is clear that $F(A)$ is contractible for $|A| \leq 1$.

If $A = \{0, 1\}$ we see that $\delta_{01} \colon F(A) \to S(V)$ is a homotopy equivalence. Thus, if $A = \{a, b\}$ then

$$H^* F(A) = \mathbb{Z}\{1, u_{ab}, u_{ba}\} / (u_{ba} = (-1)^d u_{ab}).$$

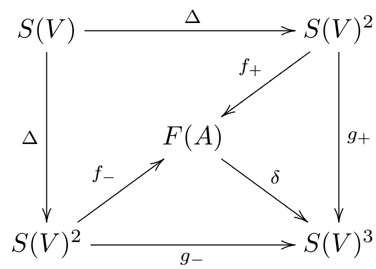
Now consider $A = \{0, 1, 2\}$. Define

$$\delta = (\delta_{01}, \delta_{12}, \delta_{20}) \colon F(A) \to S(V)^3.$$

Define $\Delta \colon S(V) \to S(V)^2$ and $f_+, f_- \colon S(V)^2 \to F(A)$ and $g_+, g_- \colon S(V)^2 \to S(V)^3$ by

$$\begin{aligned} \Delta(v) &= (v, v) \\ f_\pm(v, w) &= (-v, v, \pm(v - w)) \\ g_+(v, w) &= (v, -w, -v) \\ g_-(v, w) &= (v, -v, -w) \end{aligned}$$

It is straightforward to check that we have a diagram as follows, in which the top left triangle commutes on the nose, and the rest commutes up to homotopy:




**Proposition 12.1.** *The top left triangle in the above diagram is a homotopy pushout.*

*Proof.* We introduce the spaces

$$\begin{aligned} W &= \text{ pushout of } S(V)^2 \xleftarrow{\Delta} S(V) \xrightarrow{\Delta} S(V)^2 \\ X &= \{(x,y,z) \in F(A) \mid x+y=0,\ \|x\| = \|y\| = 1\} \\ Y &= \{(x,y,z) \in X \mid \|z-x\| = 1 \text{ or } \|z-y\| = 1\} \end{aligned}$$

It is straightforward to check that $f_+$ and $f_-$ give a homeomorphism $W \to Y$, so we need only show that $Y$ is a deformation retract of $X$, and $X$ is a deformation retract of $F(A)$. The second statement follows using the homeomorphism $(0,\infty) \times V \times X \to F(A)$ given by

$$(t,v,x,y,z) \mapsto (tx+v, ty+v, tz+v).$$

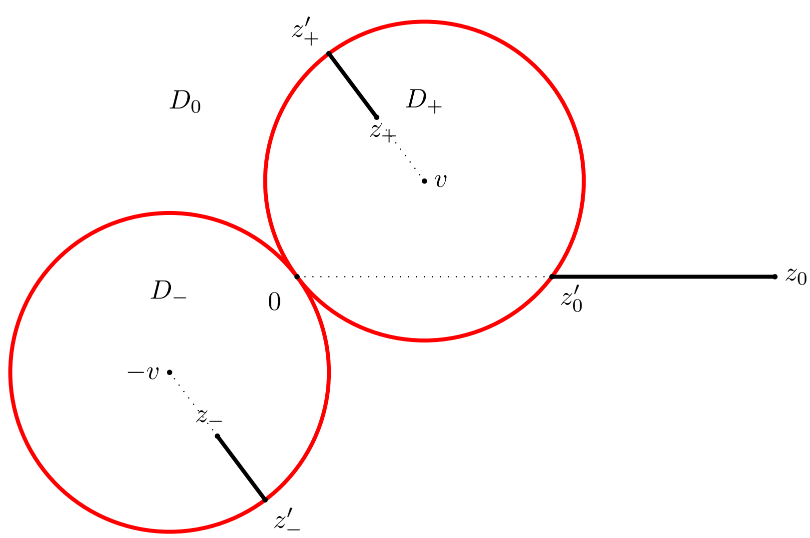
For the former, we put

$$\begin{aligned} D_\pm &= \{(-v,v,z) \in X \mid 0 < \|z - (\pm v)\| \leq 1\} \\ D_0 &= \{(-v,v,z) \in X \mid \|z-v\| \geq 1 \text{ and } \|z+v\| \geq 1\} \\ &= \{(-v,v,z) \in X \mid 2|z.v| \leq \|z\|^2\}, \end{aligned}$$

so $X = D_- \cup D_0 \cup D_+$. We then define maps $q_\pm \colon D_\pm \to V$ and $q_0 \colon D_0 \to V$ by

$$\begin{aligned} q_\pm(-v,v,z) &= v + (z - (\pm v))/\|z - (\pm v)\| \\ q_0(-v,v,z) &= \begin{cases} 0 & \text{if } z = 0 \\ 2|z.v|z/\|z\|^2 & \text{otherwise} \end{cases} \end{aligned}$$

The picture below shows $v$ and $z_+$ such that $(-v,v,z_+) \in D_+$, together with $z'_+ = q_+(-v,v,z_+)$. The numbers $z_-$ and $z'_-$ are related in the analogous way, as are $z_0$ and $z'_0$.



It is clear that $q_+$ and $q_-$ are continuous. Using the fact that $2|z.v| \leq \|z\|^2$ on $D_0$, we see that $q_0$ is also continuous. At any point where $q_i$ and $q_j$ are both defined, we see that $q_i(-v,v,z) = q_j(-v,v,z) = z$. It follows that we get a well-defined continuous map $q\colon X \to V$ with $q|_{D_i} = q_i$. We then define $r(-v,v,z) = (-v,v,q(-v,v,z))$.

By a check of cases, we find that

- If $(-v,v,z) \in D_+$ then $\|q(-v,v,z) - v\| = 1$
- If $(-v,v,z) \in D_-$ then $\|q(-v,v,z) + v\| = 1$
- If $(-v,v,z) \in D_0$ and $z.v \geq 0$ then $\|q(-v,v,z) - v\| = 1$
- If $(-v,v,z) \in D_0$ and $z.v \leq 0$ then $\|q(-v,v,z) + v\| = 1$.

It follows that $r$ defines a map $X \to Y$, and one checks that $r|_Y = 1_Y$, so $r$ is a retraction. Moreover, the line segment joining $x$ to $r(x)$ lies wholly in $X$, so we have a deformation retraction, as required. □

**Proposition 12.2.** *$r_{012} = 0$ in $H^{2d-2}F(A)$.*

*Proof.* Our homotopy pushout now gives an exact sequence

$$0 = H^{2d-3}S(V) \to H^{2d-2}F(A) \xrightarrow{(f_+^*,f_-^*)} H^{2d-2}S(V)^2 \oplus H^{2d-2}S(V)^2.$$

It will thus suffice to prove that $f_\pm^*(r_{012}) = 0$ in $H^{2d-2}S(V)^2 \simeq \mathbb{Z}$. Let $a$ and $b$ be the obvious two generators of $H^{d-1}S(V)^2$. Using $\delta f_+ = g_+$ we find that

$$\begin{aligned} f_+^*(u_{01}) &= a \\ f_+^*(u_{12}) &= (-1)^d b \\ f_+^*(u_{20}) &= (-1)^d a \\ f_+^*(r_{012}) &= (-1)^d ab + ba + (-1)^d a^2 = (-1)^d ab + (-1)^{d-1} ab = 0. \end{aligned}$$

□

**Corollary 12.3.** *For any finite set $A$ and any three distinct elements $a,b,c \in A$ we have $r_{abc} = 0$ in $H^{2d-2}F(A)$.*

*Proof.* Define $j\colon \{0,1,2\} \to A$ by $j(0) = a$ and $j(1) = b$ and $j(2) = c$. This induces a map $j^*\colon F(A) \to F(\{0,1,2\})$ and thus a ring homomorphism $j^{**}\colon H^*F(\{0,1,2\}) \to H^*F(A)$ with $r_{abc} = j^{**}r_{012} = 0$. □

## 13. The Stasheff operad

In this section we introduce the Stasheff operad, which we call $K$. Some key properties are as follows.

(a) There is an operad $\mathcal{K}$ in posets and an isomorphism $K(A) = |\mathcal{K}(A)|$ respecting the operad structure.
(b) There is a natural map of operads from $K$ to the little one-cubes operad (as in Section 2).
(c) If $|A| = n$ then $K(A)$ is a union of $n!$ balls of dimension $n-2$; in particular, $K(A)$ is compact.

The first two of these are discussed in theis section; property (c) is covered in Section 14.

We now start work on the definitions.

**Definition 13.1.** A tree $A \in \mathrm{Trees}(A)$ is *full* if every singleton lies in $\mathcal{T}$, and also $A \in \mathcal{T}$. We write $\mathrm{Trees}_1(A)$ for the poset of full trees, ordered by inclusion. We observe (or define) that $\mathrm{Trees}_1(\emptyset) = \emptyset$. Note that $\mathrm{Trees}_1(A)$ has a smallest element, namely the tree $\mathcal{T}_0 = \{\{a\} \mid a \in A\} \cup \{A\}$ (sometimes called the *corolla*). One can check that $\mathrm{Trees}_1$ is a suboperad of Trees.

**Definition 13.2.** A *Stasheff tree* on an finite, totally ordered set $A$ is a full tree $\mathcal{T} \in \mathrm{Trees}_1(A)$ such that every set in $\mathcal{T}$ is an interval. We write $\mathcal{K}^+(A)$ for the poset of Stasheff trees, ordered by inclusion. Given an unordered set $A$, we also put

$$\mathcal{K}(A) = \{(R,\mathcal{T}) \mid R \in \mathrm{Ord}(A) \text{ and } \mathcal{T} \in \mathcal{K}^+(A,R)\}.$$

We write $K^+(A)$ and $K(A)$ for the geometric realisations of $\mathcal{K}^+(A)$ and $\mathcal{K}(A)$.

Note that the corolla lies in $\mathcal{K}^+(A)$ and is the smallest element there. It follows that $K^+(A)$ is contractible, and $K(A)$ is homotopy equivalent to the discrete set $\mathrm{Ord}(A)$. We will show later that $K^+(A)$ is homeomorphic to a ball of dimension $|A| - 2$. It is called a *Stasheff cell* or *associahedron*.

**Proposition 13.3.** *$\mathcal{K}$ is a suboperad of* Ord × Trees.

*Proof.* Left to the reader. □

It follows by geometric realisation that $K$ is an operad in spaces. To proceed further, we need a slightly different picture of this structure.

**Definition 13.4.** Let $A$ be a finite, totally ordered set of size $n$ say. We write $\mathcal{J} = \mathcal{J}(A)$ for the set of intervals $J \subseteq A$. (Such intervals are determined by their endpoints, so there are $(n^2+n)/2$ of them.) We let $K_*^+(A)$ denote the set of functions $t\colon \mathcal{J}(A) \to [0,1]$ such that

(a) $t(\{a\}) = 1$ for all $a \in A$, and also $t(A) = 1$

(b) The support $\text{supp}(t) = \{J \in \mathcal{J}(A) \mid t(J) > 0\}$ is a tree (necessarily full).

More generally, if $A$ is any finite set we put

$$K_*(A) = \coprod_{R \in \text{Ord}(A)} K_*^+(A, R).$$

If $\mathcal{T}$ is a Stasheff tree, we define $t_{\mathcal{T}} \in K_*^+(A)$ by

$$t_{\mathcal{T}}(J) = \begin{cases} 1 & \text{if } J \in \mathcal{T} \\ 0 & \text{otherwise.} \end{cases}$$

**Proposition 13.5.** *The map $\mathcal{T} \mapsto t_{\mathcal{T}}$ extends to give natural homeomorphisms $K^+(A) = K_*^+(A)$ and $K(A) = K_*(A)$.*

*Proof.* Fix an order on $A$ and a Stasheff tree $\mathcal{T}$. Let $\mathcal{X}$ be the set of Stasheff trees that are contained in $\mathcal{T}$, and put $X = |\mathcal{X}|$. Let $X_*$ be the subspace of those $t \in K_*^+(A)$ for which $\text{supp}(t) \subseteq \mathcal{T}$. By a straightforward reduction, it will suffice to show that our map gives a homeomorphism $X \to X_*$. To see this, put

$$\mathcal{T}' = \{J \in \mathcal{T} \mid 1 < |J| < |A|\} = \mathcal{T} \setminus (\text{ the corolla }).$$

Then $X_*$ can be identified with $\text{Map}(\mathcal{T}', [0,1])$. On the other hand, $\mathcal{X}$ can be identified with the set of subsets of $\mathcal{T}'$, and thus with $\text{Map}(\mathcal{T}', \{0,1\})$. We have $|\{0,1\}| = [0,1]$ and geometric realisation preserves finite products so

$$X = |\mathcal{X}| = |\text{Map}(\mathcal{T}', \{0,1\})| = \text{Map}(\mathcal{T}', [0,1]) = X_*.$$

We leave it to the reader to check that the map implicit in this argument is the one considered earlier. □

The proposition tells us that the spaces $K_*(A)$ form an operad. It is not hard to give explicit formulae for this.

**Proposition 13.6.** *Suppose given the following data:*

(a) *A map $p\colon A \to B$ of finite sets, giving a splitting $A = \coprod_b A_b$*

(b) *A point $S \in \text{Ord}(B)$ and $s \in K_*^+(B, S)$*

(c) *Points $R_b \in \text{Ord}(A_b)$ and $r_b \in K_*^+(A_b, R_b)$.*

*The resulting point $(R, r) = \gamma_p((S, s); (R_b, r_b)_{b \in B}) \in K_*(A)$ is then given by*

(i) *$a <_R a'$ iff $(p(a) <_S p(a')$ or $(p(a) = p(a') = b$ and $a <_{R_b} a'))$*

(ii) $$r(J) = \begin{cases} s(p(J)) & \text{if } J = p^{-1}p(J) \\ r_b(J) & \text{if } p(J) = \{b\} \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Part (i) is just the fact that the projection $(R, r) \mapsto R$ is a map of operads $K_* \to \text{Ord}$. Part (ii) is left to the reader. □

We next construct a map $f\colon K \to C_1$ of operads, where $C_1$ is the little 1-cubes operad as in Section 2. Recall that we defined $C_1$ in terms of the half-open interval $[0,1)$; this will be convenient for our treatment here. Our map $f$ will come as the composite of maps $K \to K_* \xrightarrow{g} \text{Ord} \times \mathring{\Delta} \xrightarrow{h} C_1$, where Ord and $\mathring{\Delta}$ come from Examples 1.2 and 1.4 respectively.

**Definition 13.7.** Let $C_1'(A)$ be the subspace of $C_1(A)$ consisting of maps $f\colon A\times J\to J$ in $C_1(A)$ that are bijections. (For example, if $A=\{0,\dots,n-1\}$ we can define $u\in C_1'(A)$ by $u(i,t)=i/n+t$.) This clearly gives a suboperad of $C_1$.

**Definition 13.8.** Given $(R,x)\in \operatorname{Ord}(A)\times\mathring{\Delta}(A)$ we define $h(R,x)\colon A\times J\to J$ by

$$h(R,x)(a,t)=tx(a)+\sum_{b<_R a}x(b).$$

**Proposition 13.9.** *This defines an operad isomorphism* $h\colon \operatorname{Ord}\times\mathring{\Delta}\to C_1'$.

*Proof.* Given $(R,x)$ as above, we can list the elements of $A$ as $a_1<_R\cdots<_R a_n$, and put $y_i=\sum_{j<i}x(a_i)$, so $0=y_1<y_2<\cdots<y_n<1$. We find that $h(R,x)$ gives a bijection $\{a_i\}\times J\to[y_i,y_{i+1})$ (where we interpret $y_{n+1}$ as 1). It follows that $h(R,x)\colon A\times J\to J$ is bijective, so $h(R,x)\in C_1'(A)$. It is not hard to see that $h\colon \operatorname{Ord}(A)\times\mathring{\Delta}(A)\to C_1'(A)$ is a homeomorphism. However, we still need to check that it respects the operad structure.

Consider a map $p\colon A\to B$, together with points $(R_b,x_b)\in\operatorname{Ord}(A_b)\times\mathring{\Delta}(A_b)$ and $(S,y)\in\operatorname{Ord}(B)\times\mathring{\Delta}(B)$. Applying $\gamma_p$ gives $(R,x)\in\operatorname{Ord}(A)\times\mathring{\Delta}(A)$, where

- $a<_R a'$ iff $p(a)<_S p(a')$ or $p(a)=p(a')=b$ and $a<_{R_b}a'$
- $x(a)=x_{p(a)}(a)y(p(a))$.

Put $u_b=h(R_b,x_b)\colon A_b\times J\to J$ and $v=h(S,y)\colon B\times J\to J$. Applying $\gamma_p$ gives $w\colon A\times J\to J$, given by

$$\begin{aligned} w(a,t)&=v(p(a),u_{p(a)}(a,t))\\ &=u_{p(a)}(a,t)y(p(a))+\sum_{b<_S p(a)}y(b) \end{aligned}$$

We must show that $w=h(R,x)$.

Using $\sum_{a\in A_b}x_b(a)=1$ we see that

$$\sum_{b<_S p(a)}y(b)=\sum_{p(a')<p(a)}y(p(a'))x_{p(a')}(a')=\sum_{p(a')<p(a)}x(a').$$

We also have

$$u_{p(a)}(a,t)y(p(a))=\sum_{p(a')=p(a),a'<a}x_{p(a')}y(p(a'))+tx_{p(a)}y(p(a))=\sum_{p(a')=p(a),a'<a}x(a')+tx(a),$$

so $w(a,t)=\sum_{a'<_R a}x(a')+tx(a)$, as required. □

We now start working on the construction of a map $K_*^+(A)\to\mathring{\Delta}(A)$.

**Definition 13.10.** Suppose we have a Stasheff tree $\mathcal{T}\in\mathcal{K}^+(A)$, and a point $t\in K_*^+(A)$ with $\operatorname{supp}(t)\subseteq\mathcal{T}$. Given $J\in\mathcal{T}$, we let $c(\mathcal{T},J)$ be the set of children of $J$, or in other words the maximal elements of the set $\{J'\in\mathcal{T}\mid J'\subset J\}$. We then define numbers $\lambda(\mathcal{T},t,J)>0$ recursively by

$$\lambda(\mathcal{T},t,J)=\begin{cases}1 & \text{if } |J|=1\\ \sum_{J'\in c(\mathcal{T},J)}\lambda(\mathcal{T},t,J')^{1-t(J')} & \text{if } |J|>1.\end{cases}$$

Then, for $a\in A$ we put

$$\mu(\mathcal{T},t,a)=\prod_{a\in J\in\mathcal{T}}\lambda(\mathcal{T},t,J)^{-t(J)}.$$

**Lemma 13.11.** *Suppose we have another Stasheff tree $\mathcal{T}'$ that also contains* $\operatorname{supp}(t)$*; then $\lambda(\mathcal{T}',t,J)=\lambda(\mathcal{T},t,J)$ for all $J\in\mathcal{T}\cap\mathcal{T}'$, and $\mu(\mathcal{T}',t,a)=\mu(\mathcal{T},t,a)$ for all $a\in A$.*

*Proof.* By considering the inclusions $\mathcal{T} \supseteq \mathcal{T} \cap \mathcal{T}' \subseteq \mathcal{T}'$, we reduce to the case $\mathcal{T}' \subseteq \mathcal{T}$. By an evident induction, we then reduce to the case where $\mathcal{T} = \mathcal{T}' \amalg \{J_0\}$ for some $J_0$ with $1 < |J_0| < |A|$. Let $J_1$ be the parent of $J_0$ in $\mathcal{T}$, and put $\mathcal{C} = c(\mathcal{T}, J_0)$. We then have $c(\mathcal{T}, J_1) = \mathcal{C}' \amalg \{J_0\}$ for some set $\mathcal{C}'$, and $c(\mathcal{T}', J_1) = \mathcal{C}' \amalg \mathcal{C}$.

From the definitions we see that $\lambda(\mathcal{T}', t, J) = \lambda(\mathcal{T}, t, J)$ whenever $J$ is a proper subset of $J_0$, and similarly when $J \cap J_0 = \emptyset$. Next, as $\mathrm{supp}(t) \subseteq \mathcal{T}'$ we have $t(J_0) = 0$ and so

$$\begin{aligned}\lambda(\mathcal{T}, t, J_1) &= \sum_{J \in \mathcal{C}' \amalg \{J_0\}} \lambda(\mathcal{T}, t, J)^{1-t(J)} \\ &= \lambda(\mathcal{T}, t, J_0) + \sum_{J \in \mathcal{C}} \lambda(\mathcal{T}, t, J)^{1-t(J)} \\ &= \sum_{J \in \mathcal{C}'} \lambda(\mathcal{T}, t, J)^{1-t(J)} + \sum_{J \in \mathcal{C}} \lambda(\mathcal{T}, t, J)^{1-t(J)} \\ &= \sum_{J \in \mathcal{C}' \amalg \mathcal{C}} \lambda(\mathcal{T}', t, J)^{1-t(J)} \\ &= \lambda(\mathcal{T}', t, J_1).\end{aligned}$$

It is now easy to see that $\lambda(\mathcal{T}, t, J) = \lambda(\mathcal{T}', t, J)$ for the remaining intervals $J \in \mathcal{T}'$ (those that contain $J_1$). Given this, we see that $\mu(\mathcal{T}, t, a)$ is the same as $\mu(\mathcal{T}', t, a)$ apart from a factor $\lambda(\mathcal{T}, t, J_0)^{-t(J_0)}$, which can be ignored as $t(J_0) = 0$. □

**Lemma 13.12.** *With $A$, $\mathcal{T}$ and $t$ as above, we have $\sum_{a \in A} \mu(\mathcal{T}, t, a) = 1$.*

*Proof.* Let the children of $A$ in $\mathcal{T}$ be $B_1, \dots, B_r$, so

$$\lambda(\mathcal{T}, t, A) = \sum_i \lambda(\mathcal{T}, t, B_i)^{1-t(B_i)}.$$

Put $\mathcal{T}_i = \{J \in \mathcal{T} \mid J \subseteq B_i\} \in \mathcal{K}^+(B_i)$. Define $t_i \colon \mathcal{J}(B_i) \to [0,1]$ by $t_i(B_i) = 1$ and $t_i(J) = t(J)$ for all other $J$, so $t_i \in K_*(B_i)$, with $\mathrm{supp}(t_i) \subseteq \mathcal{T}_i$. By induction on $|A|$ we may assume that $\sum_{a \in B_i} \mu(\mathcal{T}_i, t_i, a) = 1$. Now note that when $a \in B_i$, the number $\mu(\mathcal{T}, t, a)$ involves almost the same terms as $\mu(\mathcal{T}_i, t_i, a)$ except that the expression $\lambda(\mathcal{T}, t, B_i) = \lambda(\mathcal{T}_i, t_i, B_i)$ occurs with exponent $-t(B_i)$ rather than $-1$, and there is an extra term of $\lambda(\mathcal{T}, t, A)^{-1}$. We thus have

$$\mu(\mathcal{T}, t, a) = \mu(\mathcal{T}_i, t_i, a) \lambda(\mathcal{T}, t, B_i)^{1-t(B_i)} \lambda(\mathcal{T}, t, A)^{-1}.$$

Taking the sum over all $a$ in $B_i$, we get

$$\sum_{a \in B_i} \mu(\mathcal{T}, t, a) = \lambda(\mathcal{T}, t, A)^{-1} \lambda(\mathcal{T}, t, B_i)^{1-t(B_i)}.$$

Now take the sum over $i$ and use the definition of $\lambda(\mathcal{T}, t, A)$ to get $\sum_{a \in A} \mu(\mathcal{T}, t, a) = 1$. □

**Definition 13.13.** We define $g \colon K_*^+(A) \to \mathring{\Delta}(A)$ by $g(t)(a) = \mu(\mathrm{supp}(t), t, a)$. (This defines a point of $\mathring{\Delta}(A)$ by Lemma 13.12.) We then extend this to a map $K_*(A) \to \mathrm{Ord}(A) \times \mathring{\Delta}(A)$ by $g(R, t) = (R, g(t))$.

**Lemma 13.14.** *The map $g$ is continuous.*

*Proof.* For any Stasheff tree $\mathcal{T}$, put $X(\mathcal{T}) = \{t \in K_*^+(A) \mid \mathrm{supp}(t) \subseteq \mathcal{T}\}$. These sets form a finite closed cover of $X(\mathcal{T})$, so it suffices to check continuity on $X(\mathcal{T})$. Lemma 13.11 tells us that on $X(\mathcal{T})$ we have $g(t)(a) = \mu(\mathcal{T}, t, a)$, and this is visibly continuous. □

**Proposition 13.15.** *The map $g \colon K_*(A) \to \mathrm{Ord}(A) \times \mathring{\Delta}(A)$ is a morphism of operads.*

*Proof.* Suppose given the following data:

(a) A map $p \colon A \to B$ of finite sets, giving a splitting $A = \coprod_b A_b$
(b) A point $S \in \mathrm{Ord}(B)$ and $s \in K_*^+(B, S)$
(c) Points $R_b \in \mathrm{Ord}(A_b)$ and $r_b \in K_*^+(A_b, R_b)$.

We then get a point $(R,r) \in K_*^+(A)$ as in Proposition 13.6. We regard $A$ and $B$ as ordered sets using the orders $R$ and $S$; then $p$ preserves orders, and the induced order on $A_b$ is just $R_b$. We also have

$$r(J) = \begin{cases} s(p(J)) & \text{if } J = p^{-1}p(J) \\ r_b(J) & \text{if } p(J) = \{b\} \\ 0 & \text{otherwise.} \end{cases}$$

(In particular, we have $r(A_b) = s(\{b\}) = r_b(A_b) = 1$.) Thus

$$\operatorname{supp}(r) = \{p^{-1}(K) \mid K \in \operatorname{supp}(s)\} \cup \bigcup_{b \in B} \operatorname{supp}(r_b).$$

From the definitions, we find that $\lambda(r,J) = \lambda(r_b,J)$ whenever $J \in \operatorname{supp}(r_b)$. (Here we suppress the first argument in $\lambda$, which is reasonable by Lemma 13.11.) We next claim that for $K \in \operatorname{supp}(s)$ with $|K| > 1$ we have $\lambda(r, p^{-1}(K)) = \lambda(s,K)$. To see this, suppose that the children of $K$ are $K_1, \dots, K_m$, so the children of $p^{-1}(K)$ are the sets $p^{-1}(K_i)$, so

$$\lambda(r, p^{-1}(K)) = \sum_i \lambda(r, p^{-1}(K_i))^{1-s(K_i)}.$$

If $|K_i| > 1$ then we may assume inductively that $\lambda(r, p^{-1}(K_i)) = \lambda(s, K_i)$ so $\lambda(r, p^{-1}(K_i))^{1-s(K_i)} = \lambda(s,K_i)^{1-s(K_i)}$. If $|K_i| = 1$ then $s(K_i) = 1$ so $\lambda(r, p^{-1}(K_i))^{1-s(K_i)} = 1 = \lambda(s,K_i)^{1-s(K_i)}$ irrespective of the values of $\lambda(r, p^{-1}(K_i))$ and $\lambda(s, K_i)$. We can thus deduce that $\lambda(r, p^{-1}(K)) = \lambda(s,K)$ as claimed.

We now claim that for all $a \in A$ with $p(a) = b$ we have $g(r)(a) = g(r_b)(a)\, g(s)(b)$. Indeed, we have

$$\{J \in \operatorname{supp}(r) \mid a \in J\} = \{J \in \operatorname{supp}(r_b) \mid a \in J\} \amalg \{p^{-1}(K) \mid K \in \operatorname{supp}(s),\ |K| > 1,\ b \in K\}.$$

If we take the product of the terms $\lambda(r,J)^{-r(J)}$ for $J \in \operatorname{supp}(r_b)$ containing $a$, we get $\mu(r_b, a) = g(r_b)(a)$. Similarly, recall that $g(s)(b)$ is the product of the terms $\lambda(s,K)^{-s(K)}$ for $K \in \operatorname{supp}(s)$ containing $b$. We can omit the factor corresponding to $K = \{b\}$, because $\lambda(s, \{b\}) = 1$. For the remaining factors we have $\lambda(s,K)^{-s(K)} = \lambda(r, p^{-1}(K))^{-r(K)}$, and these are the remaining terms in $g(r)(a)$, which proves the claim.

Comparing this with the definition in Example 1.4, we see that $g(r) = \gamma_p(g(s); (g(r_b))_{b \in B})$, so $g$ is a morphism of operads. □

## 14. The Loday embedding

Another interesting feature of the operad $K$ is that $K^+(A)$ is homeomorphic to a ball of dimension $n-2$, where $n = |A|$. In fact, Loday has given an embedding of $K^+(A)$ as a convex polytope in $\mathbb{R}^{n-2}$, as we will explain in this section. We should warn the reader, however, that this embedding is not very compatible with the operad structure. **Cross-reference for details?**

We will work with $K_*^+(A)$, which was proved in Proposition 13.5 to be homeomorphic to $K^+(A)$. We will embed this in a vector space $V$, described in the following definition:

**Definition 14.1.** Given a finite linearly ordered set $A$ with $|A| = n$, we let $G = \operatorname{Gaps}(A)$ denote the set of intervals of length two in $A$. (This is naturally ordered and has $|G| = n-1$.) We then put $V = \{x \colon G \to \mathbb{R} \mid \sum_g x(g) = 0\}$, so $\dim(V) = n-2$. Note that $V^*$ is naturally identified with $\operatorname{Map}(G, \mathbb{R})/\mathbb{R}$.

**Remark 14.2.** Let $J$ be an interval in $A$, and let $A/J$ denote the quotient set in which $J$ is collapsed to a point. We give this the total order such that the quotient map $q \colon A \to A/J$ is nondecreasing. If $g \in \operatorname{Gaps}(A)$ and $g \not\subseteq J$ then $q(g) \in \operatorname{Gaps}(A/J)$. This gives a bijection $\{g \in \operatorname{Gaps}(A) \mid g \not\subseteq J\} \to \operatorname{Gaps}(A/J)$, and of course $\{g \in \operatorname{Gaps}(A) \mid g \subseteq J\} = \operatorname{Gaps}(J)$, so we have a splitting $\operatorname{Gaps}(A) \simeq \operatorname{Gaps}(J) \amalg \operatorname{Gaps}(A/J)$.

**Definition 14.3.** Consider an interval $J \in \mathcal{J}(A)$ with $|J| = k$, so $1 < k \le n$. If $k < n$ we define $v_J \in V$ by

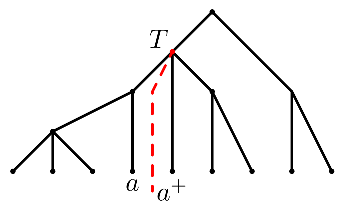
$$v_J(g) = \begin{cases} -2/((n-1)(k-1)) & \text{if } g \subseteq J \\ 2/((n-1)(n-k)) & \text{if } g \not\subseteq J \end{cases}$$

We also define $v_A = 0$. We then define $\phi \colon K_*^+(A) \to V$ to be the unique map that is linear on simplices and sends $J$ to $v_J$ on vertices.

We will show later that $\phi$ is injective, and the image is the convex hull of the points $v_J$ for $J \neq A$, and thus is a convex polytope (which is dual to the polytope constructed by Loday). To prove all this we need to find elements of $V^*$ that define the walls of our polytope. These elements will be indexed by binary Stasheff trees, as we now explain.

**Definition 14.4.** Given an interval $J \in \mathcal{J}(A)$ and a Stasheff tree $\mathcal{T} \in \mathcal{K}_0'(A)$ we put $\mathcal{T}' = \{T \in \mathcal{T} \mid |T| > 1\}$, and we let $\pi_{\mathcal{T}}(J)$ denote the smallest set in $\mathcal{T}$ that contains $J$, so $\pi_{\mathcal{T}} \colon \mathcal{J}(A) \to \mathcal{T}'$. Note that $G \subseteq \mathcal{J}(A)$ so we get a restricted map $\pi_{\mathcal{T}} \colon G \to \mathcal{T}'$.

**Remark 14.5.** If $g = \{a, a^+\} \in G$ then $T = \pi_{\mathcal{T}}(P)$ is the highest vertex that we can reach by moving upwards from a point between $a$ and $a^+$ without crossing any lines:




**Lemma 14.6.** *The map $\pi_{\mathcal{T}} \colon G \to \mathcal{T}'$ is surjective. More precisely, if $T \in \mathcal{T}'$ and $T = \pi_{\mathcal{T}}(J)$ for some interval $J$ then there exists $g \in G$ such that $g \subseteq J$ and $\pi_{\mathcal{T}}(g) = T$. Moreover, if $\mathcal{T}$ is a binary tree then $\pi_{\mathcal{T}} \colon G \to \mathcal{T}'$ is a bijection.*

*Proof.* This is more or less clear from the picture, or we can argue as follows. Suppose that $T = \pi_{\mathcal{T}}(J)$ for some interval $J = [a, b]$. (If no such $J$ is given in advance, we can just take $J = T$.) Let $U$ be the child of $T$ containing $a$ (or in other words, the largest proper subset of $T$ that contains $a$ and lies in $\mathcal{T}$). This must have the form $[u, v]$ for some $u, v \in A$ with $u \leq a \leq v$. We must have $v < b$ (otherwise $\pi_{\mathcal{T}}(J)$ would be $U$ rather than $T$). Put $g = \{v, v^+\} \in G$ and $T' = \pi_{\mathcal{T}}(g)$. Then $T'$ lies in $\mathcal{T}$ and meets $U$ but is not contained in $U$, so it must be strictly larger than $U$. Moreover $g \subseteq J$ so $T' \subseteq \pi_{\mathcal{T}}(J) = T$. As $U$ is a child of $T$ it follows that $T' = T$, as required.

If $\mathcal{T}$ is a binary tree then $\#\mathcal{T}' = n - 1 = \#G$, so $\pi_{\mathcal{T}} \colon G \to \mathcal{T}'$ must be a bijection. □

**Definition 14.7.** We write $\mathcal{B}_0(A)$ for the set of binary Stasheff trees (which are just the maximal elements in $\mathcal{K}_0'(A)$). Given $\mathcal{T} \in \mathcal{B}_0(A)$ we define $w_{\mathcal{T}} \colon G \to \mathbb{R}$ by

$$w_{\mathcal{T}}(g) = |\{J \in \mathcal{J}(A) \mid \pi_{\mathcal{T}}(J) = \pi_{\mathcal{T}}(g)\}|.$$

We call this the *Loday vector* for $\mathcal{T}$. We write $\overline{w}_{\mathcal{T}}$ for the image of $w_{\mathcal{T}}$ in $V^* = \mathrm{Map}(G, \mathbb{R})/\mathbb{R}$.

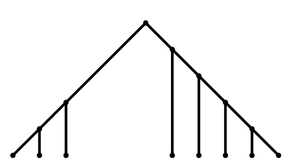
**Remark 14.8.** This can be reformulated slightly as follows. Put $T = \pi_{\mathcal{T}}(g)$, and let $T_0$ and $T_1$ be the two children of $T$. For an interval $J = [a, b]$, we have $\pi_{\mathcal{T}}(J) = T$ iff $a \in T_0$ and $b \in T_1$. It follows that $w_{\mathcal{T}}(g) = |T_0||T_1|$, which is Loday's definition.

**Lemma 14.9.** $\{w_{\mathcal{T}} \mid \mathcal{T} \in \mathcal{B}_0(A)\}$ *spans* $\mathrm{Map}(G, \mathbb{R})$.

*Proof.* We may assume that $n = |A| > 1$, otherwise the statement is vacuous. We may also assume for simplicity that $A = \{1, \dots, n\}$. For $1 \leq i \leq n - 1$ we put

$$\mathcal{T}_i' = \{[1, j] \mid 1 < j \leq i\} \amalg \{[j, n] \mid i < j < n\}$$
$$\mathcal{T}_i = \{\{i\} \mid i \in A\} \amalg \mathcal{T}_i' \amalg \{A\}$$

For example, when $n = 9$ and $i = 3$, the tree $\mathcal{T}_3$ looks like this:



In general, there are $i$ leaves in the left hand part, and $n-i-1$ in the right hand part. Put

$$t_i = w_{\mathcal{T}_i} = (1, 2, \dots, i-1, i(n-i), n-i-1, n-i-2, \dots, 1).$$

If $i < n-1$ we put $m_i = i(n-i-1) > 0$ and

$$s_i = (t_i - t_{i+1})/m_i = (0, \dots, 0, 1, -1, 0, \dots, 0).$$

Here the 1 is in the $i$'th place, and the $-1$ in the $(i+1)$st. The elements $s_i$ form a basis for the hyperplane $H = \{y \in \mathbb{R}^{n-1} \mid \sum_j y_j = 0\}$, and $t_1 \notin H$, so the elements $t_i$ give a basis for $\mathbb{R}^{n-1}$, as required. □

**Definition 14.10.** Let $J$ be an interval of size $k$, where $1 < k < n = |A|$. We then put $\mathcal{K}^+(A; J) = \{\mathcal{T} \in \mathcal{K}^+(A) \mid J \in \mathcal{T}\}$.

**Proposition 14.11.** *There is a natural isomorphism of posets $\mathcal{K}^+(A; J) = \mathcal{K}^+(J) \times \mathcal{K}^+(A/J)$.*

*Proof.* Let $q\colon A \to A/J$ be the projection. Given $\mathcal{T} \in \mathcal{K}^+(A; J)$ we put

$$\mathcal{U} = \{T \in \mathcal{T} \mid T \subseteq J\}$$
$$\mathcal{V} = \{q(T) \mid T \in \mathcal{T} \text{ and } T \not\subseteq J\}.$$

It is clear that $\mathcal{U} \in \mathcal{K}^+(J)$. Next, observe that if $T \in \mathcal{T}$ and $T \not\subseteq J$ then either $T \cap J = \emptyset$ or $T \supseteq J$. Either way, we have $T = q^{-1}(q(T))$. Given this, we see that $\mathcal{V} \in \mathcal{K}^+(A/J)$.

Conversely, suppose we start with Stasheff trees $\mathcal{U} \in \mathcal{K}^+(J)$ and $\mathcal{V} \in \mathcal{K}^+(A/J)$. We then put $\mathcal{T} = \mathcal{U} \cup \{q^{-1}(V) \mid V \in \mathcal{V}\}$, and we check that this is a Stasheff tree on $A$. (It is obtained by grafting the tree $\mathcal{U}$ to the basepoint leaf in $\mathcal{V}$.) These two procedures are mutually inverse, as required. □

**Proposition 14.12.** *If $J \in \partial\mathcal{J}(A)$ then $\{w_{\mathcal{T}} \mid \mathcal{T} \in \mathcal{K}^+(A; J)\}$ spans* $\mathrm{Map}(G, \mathbb{R})$.

*Proof.* **prove this** □

**Proposition 14.13.** *For $J \in \partial\mathcal{J}(A)$ and $\mathcal{T} \in \mathcal{B}_0(A)$ we have $\langle v_J, \overline{w}_{\mathcal{T}} \rangle \leq 1$, with equality iff $J \in \mathcal{T}$. Moreover:*

- *If we fix $\mathcal{T} \in \mathcal{B}_0(A)$ then $\{v_J \mid J \in \mathcal{T}'\}$ is a basis for $V$.*
- *If we fix $J \in \partial\mathcal{J}(A)$ then $\{\overline{w}_{\mathcal{T}} \mid J \in \mathcal{T}\}$ spans $V^*$.*

*Proof.* As $\pi_{\mathcal{T}}\colon G \to \mathcal{T}'$ is a bijection, we can put $\mathcal{T}' = \mathcal{A} \amalg \mathcal{B}$, where $\mathcal{A} = \{\pi_{\mathcal{T}}(g) \mid g \subseteq J\}$ and $\mathcal{B} = \{\pi_{\mathcal{T}}(g) \mid g \not\subseteq J\}$. Now put $X = \{I \in \mathcal{K}^+(A) \mid \pi_{\mathcal{T}}(I) \in \mathcal{A}\}$ and $Y = \{I \in \mathcal{K}^+(A) \mid \pi_{\mathcal{T}}(I) \in \mathcal{B}\}$, and $x = |X|$ and $y = |Y|$. It is clear that $x + y = |\mathcal{K}^+(A)| = n(n-1)/2$. We also have $x = \sum_{g \subseteq J} w_{\mathcal{T}}(g)$ and $y = \sum_{g \not\subseteq J} w_{\mathcal{T}}(g)$, so

$$\langle v_J, \overline{w}_{\mathcal{T}} \rangle = \frac{2}{n-1}\left(\frac{y}{n-k} - \frac{x}{k-1}\right) = \frac{2}{n-1}\left(\frac{(n^2-n)/2 - x}{n-k} - \frac{x}{k-1}\right) = 1 + \frac{k^2-k-2x}{(n-k)(k-1)}.$$

We must thus show that $x \geq (k^2-k)/2$, with equality iff $J \in \mathcal{T}$.

Put $X_0 = \{I \in \mathcal{K}^+(A) \mid I \subseteq J\}$, so $|X_0| = (k^2-k)/2$. Using Lemma 14.6 we see that $X_0 \subseteq X$, so $x \geq (k^2-k)/2$. If $J \in \mathcal{T}$ then one checks that $\mathcal{A} = \{T \in \mathcal{T} \mid T \subseteq J\}$ and thus that $X = X_0$, so $x = (k^2-k)/2$.

Now suppose instead that $J \notin \mathcal{T}$, so the interval $T = \pi_{\mathcal{T}}(J)$ is strictly larger than $J$. Lemma 14.6 tells us that $T = \pi_{\mathcal{T}}(g)$ for some $g \subseteq J$, but also $T = \pi_{\mathcal{T}}(T)$, so $T \in X \setminus X_0$, so $x > (k^2-k)/2$. This completes the first claim of the proposition.

Now fix a binary Stasheff tree $\mathcal{T}$, so $|\mathcal{T}'| = n-1$. Let $U_0$ be the span of the vectors $v_J$ for $J \in \mathcal{T}'$, so $U_0 \leq V < \mathrm{Map}(G, \mathbb{R}) = V \oplus \mathbb{R}$; we must show that $U_0 = V$. Put $U = U_0 \oplus \mathbb{R} \leq \mathrm{Map}(G, \mathbb{R})$; it will suffice to show that $U = \mathrm{Map}(G, \mathbb{R})$, or that the orthogonal complement of $U$ is zero. It is easy to see that $U$ is spanned by the vectors $u_J$ for $J \in \mathcal{T}'$, where

$$u_J(g) = \begin{cases} 1 & \text{if } g \subseteq J \\ 0 & \text{otherwise} \end{cases}.$$

We must therefore show that if $f\colon G \to \mathbb{R}$ and $\sum_{g \subseteq J} f(g) = 0$ for all $J \in \mathcal{T}'$ then $f = 0$. Let the children of the root in $\mathcal{T}$ be $A_0$ and $A_1$, and let $\mathcal{T}_0$ and $\mathcal{T}_1$ be the corresponding subtrees. By induction we may assume the claim is valid for $\mathcal{T}_i$, and thus that $f(g) = 0$ whenever $g \subseteq A_0$ or $g \subseteq A_1$. This applies to all $g \in G$ except for the element $g_0 = \{\max(A_0), \min(A_1)\}$. However, we also have $A \in \mathcal{T}$ so $\sum_{g\in G} f(g) = 0$ so we must have $f(g_0) = 0$ also.

Now instead fix an interval $J \in \partial\mathcal{J}(A)$. **Unfinished** □

**Remark 14.14.** It is natural to try to make contact with some kind of space of height functions. One attempt would be to consider maps $h\colon \mathcal{J} \to I$ such that $h(A) = 1$ and $h(J \cup K) = \max(h(J), h(K))$ whenever $J \cap K \neq \emptyset$. Such a map satisfies $h(J) = \max\{h(g) \mid g \in \mathrm{Gaps}(J)\}$ and so we find that the space of such $h$ is just $BWG$. We say that an interval $J$ is $h$-critical if any strictly larger interval $J'$ has $h(J') > h(J)$. The critical sets then form a Stasheff tree.

TODO:

- There is a natural map to the little 1-cubes operad.
- $K^+(A)$ is a ball, and has a cubical structure.
- There is some kind of duality with the space of height functions defined on intervals.
- Pictures for $n = 3, 4$ and maybe $n = 5$.

## 15. The Fulton-Macpherson compactification

Let $N$ be a finite-dimensional vector space. For any finite set $A$ we put $F(A) = F_N(A) = \mathrm{Inj}(A, N)/\sim$, where $f \sim g$ iff $g = uf + n$ for some $u \in \mathbb{R}^+$ and $n \in N$. After choosing an inner product we get a homeomorphism $\mathrm{Inj}(A, N) = N \times \mathbb{R}^+ \times F(A)$, so $F(A)$ is homotopy equivalent to $\mathrm{Inj}(A, N)$ and thus to the little cube space $C_d(A)$ (where $d = \dim(N)$). It follows that $F$ is an operad in the homotopy category. In this section, we construct compactifications $\overline{F}(A)$ that form an operad on the nose, such that the inclusion $F(A) \to \overline{F}(A)$ is a homotopy equivalence for all $A$.

Our compactifications are homeomorphic to those constructed by Axelrod, who adapted a procedure used by Fulton and Macpherson to compactify configuration spaces in complex manifolds. The operad structure was probably first considered by Getzler. A more concrete treatment was given by Sinha. We believe that our treatment has certain technical and conceptual advantages, however. Our approach was also first considered in the context of complex geometry, in the unpublished PhD thesis of Daniel Singh.

**Definition 15.1.** In this section we identify $WA$ with $\mathrm{Map}(A, \mathbb{R})/\mathbb{R}$, where $\mathbb{R}$ is embedded as the set of constant maps $A \to \mathbb{R}$. With this model, there is an obvious way to make $WA$ a contravariant functor of $A$. We also put $NWA = N \otimes WA = \mathrm{Map}(A, N)/N$.

**Definition 15.2.** For any finite-dimensional vector space $V$, a *ray* in $V$ means a subset of the form $\{tv \mid t \geq 0\}$ for some $v \neq 0$. We write $S(V)$ for the set of rays. Of course, if $V$ has an inner product then this can be identified with the unit sphere. Next, suppose we have a quotient map $f\colon V \to W = V/U$. We put

$$S(V, W) = \{(x, y) \in S(V) \times S(W) \mid x \subseteq f^{-1}(y)\}.$$

**Proposition 15.3.** *The set $S(V, W)$ is closed in $S(V) \times S(W)$. The projection $q\colon S(V, W) \to S(W)$ is a trivialisable fibre bundle, whose fibre is a ball. The projection $p\colon S(V, W) \mid S(V)$ induces a homeomorphism $p^{-1}(S(V) \setminus S(U)) \to S(V) \setminus S(U)$. We also have $p^{-1}S(U) = S(U) \times S(W)$.*

*Proof.* If $x \in S(U)$ then $f(x) = \{0\}$ so $x \subseteq f^{-1}(y)$ for any ray $y \in S(W)$. It follows that $p^{-1}S(U) = S(U) \times W$. On the other hand, if $x \in S(V) \setminus S(U)$ then the set $y = f(x)$ is a ray in $W$, and it is the unique ray such that $x \subseteq f^{-1}y$. It follows that $p\colon p^{-1}(S(V) \setminus S(U)) \to S(V) \setminus S(U)$ is a homeomorphism. For the rest, choose an inner product on $V$ and thus identify $f$ with an orthogonal projection $V = U \oplus W \to W$. Put $B(U) = \{u \in U \mid \|u\| \leq 1\}$. We then have

$$\begin{aligned} S(V, W) &= \{(u, v, w) \in U \times W \times W \mid \|u\|^2 + \|v\|^2 = \|w\|^2 = 1 \text{ and } v = tw \text{ for some } t \geq 0\} \\ &= \{(u, v, w) \in U \times W \times W \mid \|u\|^2 + \|v\|^2 = \|w\|^2 = 1 \text{ and } v = |v.w|w\} \\ &= \{(u, \sqrt{1 - \|u\|^2}w, w) \mid (u, w) \in B(U) \times S(W)\}. \end{aligned}$$

The second description shows that $S(V, W)$ is closed in $S(V) \times S(W)$, and the third description shows that $q$ gives a trivialisable bundle with fibre $B(U)$. □

**Definition 15.4.** Consider a point $x \in \prod_B S(NWB)$, where the product runs over subsets $B \subseteq A$ with $|B| > 1$. Note that when $C \subseteq B$ we have a restriction map $\rho = \rho^B_C \colon NWB \to NWC$, which is surjective, so $S(NWB, NWC)$ is defined. We say that $x$ is *coherent* if for all such $B$ and $C$ we have $x_B \subseteq (\rho^B_C)^{-1} x_C$, so $(x_B, x_C) \in S(NWB, NWC)$. We define $\overline{F}(A) = \overline{F}_N(A)$ to be the set of all coherent $x$.

We next give a stratification of $\overline{F}(A)$.

**Definition 15.5.** Let $x$ be a point in $\overline{F}(A)$. We say that a set $U \subseteq A$ is *$x$-critical* if for all $T$ with $U \subset T \subseteq A$ we have $\rho^T_U(x_T) = 0$. Note that singletons are $x$-critical (because $NW\{a\} = 0$), and $A$ itself is vacuously $x$-critical. Note also that if $f \colon T \to N$ represents $x_T$, then $\rho^T_U(x_T) = 0$ iff $f|_U$ is constant. We write $t(x)$ for the set of $x$-critical sets.

**Lemma 15.6.** *$t(x)$ is a tree.*

*Proof.* Suppose that $U, V \in t(x)$ and $U \cap V \neq \emptyset$. We must show that either $U \subseteq V$ or $V \subseteq U$. If not, then the set $T = U \cup V$ is a strict superset of both $U$ and $V$. As $U$ and $V$ are both critical, this means that $\rho^T_U(x_T) = 0$ and $\rho^T_V(x_T) = 0$. If $f \colon T \to N$ represents $x_T$, this means that $f|_U$ and $f|_V$ are constant. As $U$ and $V$ overlap, this means that $f$ is constant, so $x_T = 0$, contrary to the assumption that $x_T \in S(NWT)$. □

**Lemma 15.7.** *For any $x \in \overline{F}(A)$ and any $B \subseteq A$ (with $|B| > 1$) there exists a unique $x$-critical set $T \supseteq B$ with $x_B = x_T|_B$. Moreover, this is the smallest $x$-critical set containing $B$.*

*Proof.* Let $T$ be a set of largest possible size such that $x_T|_B \neq 0$ (so $x_T|_B = x_B$ by the coherence condition). We claim that $T$ is $x$-critical. Indeed, if $U \supset T$ then $x_U|_T|_B = x_U|_B = 0$ by the maximality of $T$. However, we have $x_T|_B = x_B$, so $x_U|_T \neq x_T$, so by coherence we must have $x_U|_T = 0$. This proves that $T$ is $x$-critical. If $T'$ is any other critical set containing $B$ then Lemma 15.6 tells us that either $T' \subset T$ or $T \subseteq T'$. If the former then $x_T|_{T'} = 0$ (by the criticality of $T'$), so $x_T|_B = x_T|_{T'}|_B = 0$, contrary to hypothesis. Thus $T \subseteq T'$, showing that $T$ is indeed the smallest $x$-critical set containing $B$. □

**Definition 15.8.** Given a full tree $\mathcal{T}$ we put

$$\begin{aligned} \overline{F}(A; = \mathcal{T}) &= \{x \in \overline{F}(A) \mid t(x) = \mathcal{T}\} \\ \overline{F}(A; \subseteq \mathcal{T}) &= \{x \in \overline{F}(A) \mid t(x) \subseteq \mathcal{T}\} \\ \overline{F}(A; \supseteq \mathcal{T}) &= \{x \in \overline{F}(A) \mid t(x) \supseteq \mathcal{T}\}. \end{aligned}$$

We also define a space $\overline{F}(A; \mathcal{T}) \subseteq \prod_{T \in \mathcal{T}'} S(NWT)$ as follows: a point $x \in \prod_{T \in \mathcal{T}} S(NWT)$ lies in $\overline{F}(A; \mathcal{T}')$ if it satisfies the coherence condition $x_T \subseteq (\rho^T_U)^{-1} x_U$ whenever $U \subseteq T$ and $U, T \in \mathcal{T}'$. Here, as usual, $\mathcal{T}' = \{T \in \mathcal{T} \mid |T| > 1\}$. There is an evident projection $\tau \colon \overline{F}(A) \to \overline{F}(A; \mathcal{T})$ (by forgetting $x_B$ for all $B \notin \mathcal{T}$).

**Remark 15.9.** For any $T \subseteq A$, it is easy to see that $\{x \mid T \text{ is } x\text{-critical }\}$ is closed in $\overline{F}(A)$. It follows that $\overline{F}(A; \supseteq \mathcal{T})$ is closed, $\overline{F}(A; \subseteq \mathcal{T})$ is open, and $\overline{F}(A; = \mathcal{T})$ is locally closed.

**Proposition 15.10.** *The projection $\tau \colon \overline{F}(A) \to \overline{F}(A; \mathcal{T})$ restricts to give an open inclusion $\overline{F}(A; \subseteq \mathcal{T}) \to \overline{F}(A; \mathcal{T})$. The image is the set $U \subseteq \overline{F}(A; \mathcal{T})$ defined by the following condition: for all $B$, we have $x_{\pi_{\mathcal{T}}(B)}|_B \neq 0$ in $NWB$. (Here, as usual, $\pi_{\mathcal{T}}(B)$ is the smallest set in $\mathcal{T}$ containing $B$.)*

*Proof.* We first claim that $\overline{F}(A; \subseteq \mathcal{T}) = \tau^{-1} U$. In one direction, suppose that $\tau(x) \in U$. Then for $B \notin \mathcal{T}$, the set $T = \pi_{\mathcal{T}}(B)$ is a strict superset of $T$ with $x_T|_B \neq 0$, so $B$ is not critical. This gives $t(x) \subseteq \mathcal{T}$, so we see that $\tau^{-1}(U) \subseteq \overline{F}(A; \subseteq \mathcal{T})$.

Conversely, if $x \in \overline{F}(A; \subseteq \mathcal{T})$ we claim that $\tau(x) \in U$. To see this, put $T = \pi_{\mathcal{T}}(B)$ and $U = \pi_{t(x)}(B)$. As $x \in \overline{F}(A; \subseteq \mathcal{T})$ we have $t(x) \subseteq \mathcal{T}$ and so $B \subseteq T \subseteq U$. Lemma 15.7 tells us that $x_U|_T|_B = x_U|_B = x_B \neq 0$. This implies that $x_U|_T \neq 0$, so by coherence $x_U|_T = x_T$, so $x_T|_B = x_B$. We therefore have $\overline{F}(A; \subseteq \mathcal{T}) = \tau^{-1} U$.

We now define a map $\sigma \colon U \to \prod_B S(NWB)$ by $\sigma(x)_B = x_{\pi_{\mathcal{T}}(B)}|_B$. Clearly, if $x \in \overline{F}(A; \mathcal{T})$ we have $\sigma(\tau(x)) = x$.

We claim that for any $x \in U$, the element $\sigma(x)$ is coherent. To see this, consider sets $B \subseteq C$ and put $T = \pi_{\mathcal{T}}(B)$ and $U = \pi_{\mathcal{T}}(C)$, so we have inclusions

$$\begin{array}{ccc} B & \rightarrowtail & C \\ \downarrow & & \downarrow \\ T & \rightarrowtail & U \end{array}$$

We must show that $\sigma(x)_C|_B \in \{0, \sigma(x)_B\}$, or in other words $x_U|_B = x_U|_C|_B \in \{0, x_T|_B\}$. As $x$ is coherent, we have $x_U|_T \in \{0, x_T\}$, and restricting this to $B$ gives the desired conclusion. We have thus defined $\sigma\colon U \to \overline{F}(A)$, and clearly $\tau\sigma = 1$. As $\overline{F}(A; \subseteq \mathcal{T}) = \tau^{-1}(U)$ it follows that $\sigma(U) \subseteq \overline{F}(A; \subseteq \mathcal{T})$ and thus that $\tau$ and $\sigma$ give mutually inverse homeomorphisms $\overline{F}(A; \subseteq \mathcal{T}) \simeq U$. □

**Corollary 15.11.** *$\overline{F}(A; = \mathcal{T})$ is homeomorphic to $\prod_{T \in \mathcal{T}'} F(\delta T)$ (where $\delta T$ is the set of children of $T$, as in Definition 5.3).*

*Proof.* We have a canonical projection $q_T\colon T \to \delta T$, sending $a \in T$ to the unique child of $T$ containing $a$. This gives an embedding $q_T^*\colon F(\delta T) \subseteq S(NW\delta T) \to S(NWT)$. Together, these give a map $j\colon \prod_T F(\delta T) \to \prod_T S(NWT)$. We claim that this lands in the set $U \subseteq \overline{F}(A; \mathcal{T})$. Indeed, for $U, T \in \mathcal{T}$ with $U \subset T$, the map $q_T$ is constant on $U$, so $j(x)_T|_U = 0$; this shows that $j(x)$ is always coherent. Moreover, if $|B| > 1$ and $T = \pi_{\mathcal{T}}(B)$ then $q_T(B)$ is not a singleton (by the minimality of $T$). As $x_T \in F(\delta T)$ is represented by an injective map $\delta T \to N$, it follows that $j(x_T)|_B \neq 0$, so $j(x) \in U$. We therefore have a point $y = \sigma(j(x)) \in \overline{F}(A; \subseteq \mathcal{T})$. We next claim that all sets in $\mathcal{T}$ are $y$-critical. Indeed, suppose that $U \in \mathcal{T}$ and $B \supset U$, and put $T = \pi_{\mathcal{T}}(B) \supset U$. We then have $y_B = (q_T^* x_T)|_B$ so $y_B|_U = (q_T^* x_T)|_U$, which is 0 because $q_T$ is constant on $U$. We thus have $y \in \overline{F}(A; = \mathcal{T})$. We leave it to the reader to check that $\sigma j\colon \prod_T F(\delta T) \to \overline{F}(A; = \mathcal{T})$ is a homeomorphism. □

**Corollary 15.12.** *The permutation group $\Sigma_A$ acts freely on $\overline{F}(A)$.*

*Proof.* Suppose that $x \in \overline{F}(A)$ and $\sigma \in \Sigma_A$ and $\sigma.x = x$. We must show that $\sigma = 1$. The definition of the tree $\mathcal{T} = t(x)$ is natural, so we must have $\sigma\mathcal{T} = \mathcal{T}$, so for $T \in \mathcal{T}$ we have $\sigma(T) \in \mathcal{T}$. We claim that in fact $\sigma(T) = T$. This is clear for $T = A$, so we can work by downwards induction on $|A|$. We may thus assume that $T \in \delta U$ for some $U \in \mathcal{T}$ with $\sigma(U) = U$. It follows that $\sigma$ permutes $U$ and $\delta U$ and preserves $x_U \in S(NWU)$. As in the previous corollary we have $x_U = q_T^* y$, where $y \in S(NW\delta U)$ is represented by an injective map $\delta U \to N$. As $\sigma$ preserves $x_U$ we see that it must act as the identity on $\delta U$, so $\sigma(T) = T$ as claimed. At the end of the induction we see that $\sigma$ preserves all singletons, so $\sigma = 1$. □

Corollary 15.11 can be extended as follows. Recall that $\delta' T$ is the set of children $U \subset T$ for which $|T| > 1$. For any set $A$ we put $\Delta(A) = \{t\colon A \to [0,1] \mid \sum_a t(a) = 1\}$, as before. We also let $X * Y$ denote the join of $X$ and $Y$, which is the quotient of the space $X \amalg (X \times I \times Y) \amalg Y$ by the relation generated by $(x, 0, y) \sim x$ and $(x, 1, y) \sim y$. Note in particular that $X * \emptyset = X$.

**Proposition 15.13.** *An inner product on $N$ gives a natural homeomorphism*

$$\prod_{T \in \mathcal{T}'} \Delta(\delta' T) * S(NW\delta T) \to \overline{F}(A; \mathcal{T}).$$

**Remark 15.14.** The proposition is equivalent to the claim that

$$\overline{F}(A; \mathcal{T}) = \Delta(\delta' A) * S(NW\delta A) \times \prod_{U \in \delta' A} \overline{F}(U; \mathcal{T}|_U),$$

where $\mathcal{T}|_U = \{V \in \mathcal{T} \mid V \subseteq U\}$. We will give a proof of the proposition in a single step, but it is also possible to give a recursive proof based on the above reformulation.

*Proof.* First, put $\mathcal{T}'' = \{T \in \mathcal{T}' \mid \delta' T \neq \emptyset\}$. Then let $X$ be the space of triples $(t, p, q)$, where

- $t \in \prod_{T \in \mathcal{T}' \setminus \{A\}} [0,1]$
- $p, q \in \prod_{T \in \mathcal{T}''} [0,1]$
- $p_T^2 + q_T^2 = 1$ for all $T$ (so $p$ determines $q$)
- If $T \in \mathcal{T}''$ then $\sum_{W \in \delta' T} t_W^2 = 1$.

Then put $Y = X \times \prod_{T \in \mathcal{T}'} S(NW\delta T)$. We can define

$$\theta \colon Y \to \prod_{T \in \mathcal{T}'} \Delta(\delta' T) * S(NW\delta T)$$

as follows. If $T \in \mathcal{T}'$ and $\delta' T \neq \emptyset$ we put

$$\theta(t, p, q, x)_T = [(t_W^2)_{W \in \delta' T}, q_T^2, x_T] \in \Delta(\delta'_T) * S(NW\delta T).$$

If $T \in \mathcal{T}'$ and $\delta' T = \emptyset$ then we have $\Delta(\delta' T) * S(NW\delta T) = S(NW\delta T) = S(NWT)$ and we put

$$\theta(t, p, q, x)_T = x_T.$$

Now let $\sim$ be the equivalence relation on $X$ generated by the following rules:

- If $T \in \mathcal{T}''$ and $p_T = 1$ (or $q_T = 1$) then the equivalence class $[t, p, q, x]$ is independent of $x_T$.
- If $T \in \mathcal{T}''$ and $p_T = 0$ (or $q_T = 1$) then the equivalence class $[t, p, q, x]$ is independent of $t_W$ for all $W \in \delta' T$.

Put $\overline{X} = X/\sim$. One checks that $\theta$ induces a homeomorphism

$$\theta \colon \overline{X} \to \prod_{T \in \mathcal{T}'} \Delta(\delta' T) * S(NW\delta T).$$

Now fix a point $(t, p, q, x) \in X$. Given $U, T \in \mathcal{T}'$ with $U \subseteq T$ we note that $\{V \in \mathcal{T} \mid U \subseteq V \subseteq T\}$ is a chain, say $U = V_0 \subset V_1 \subset \cdots \subset V_r = T$. Put

$$s_{TU} = q_U \prod_{U \subset V \subseteq T} p_V \prod_{U \subseteq V \subset T} t_V,$$

where $q_U$ is taken to be 1 if $\delta' U = \emptyset$. Next, the projection $r_T \colon T \to \delta T$ gives an inclusion $r_T^* \colon S(NW\delta T) \to S(NWT)$, and we let $y_T \colon T \to N$ be the unique representative of $r_T^* x_T$ that satisfies $\sum_{a \in T} y_T(a) = 0$ and $\sum_{a \in T} \|y_T(a)\|^2 = 1$. We then extend this to a map $y_T \colon A \to N$ by $y_T(a) = 0$ for $a \notin T$. Note that for $T \neq U$, the vectors $y_T$ and $y_U$ are orthogonal. Indeed, we have either $T \cap U = \emptyset$ or $T \subset U$ or $U \subset T$. The claim is clear in the first case, as $y_V$ is supported on $V$. In the second case, it follows from the fact that $y_U$ is constant on $T$ and $\sum_{a \in T} y_T(a) = 0$. The third case is similar.

We now define $z_T \colon T \to N$ by

$$z_T = \sum_{U \subseteq T} s_{TU} y_U|_T.$$

It is easy to see that $\sum_{a \in T} z_T(a) = 0$. Using the orthogonality of the $y$'s, we also have $\|z_T\|^2 = \sum_{U \subseteq T} s_{TU}^2$. We claim that this is just 1. Indeed, if $T$ is minimal in $\mathcal{T}'$ then $\delta' T = \emptyset$ and $s_{TT} = q_T = 1$ and the claim is clear. If $T$ is not minimal, then

$$\sum_{U \subseteq T} s_{TU}^2 = s_{TT}^2 + \sum_{W \in \delta' T} \sum_{U \subseteq W} s_{TU}^2.$$

Here $s_{TT} = q_T$ and $s_{TU} = p_T t_W s_{WU}$ and we may assume inductively that $\sum_{U \subseteq W} s_{WU}^2 = 1$, so

$$\sum_{U \subseteq T} s_{TU}^2 = q_T^2 + \sum_{W \in \delta' T} p_T^2 t_W^2 = p_T^2 + q_T^2 = 1,$$

as claimed. We thus have $z \in \prod_{T \in \mathcal{T}'} S(NWT)$. We claim that this is coherent. To see this, suppose that $T' \subseteq T$, and put

$$\lambda = \prod_{T' \subseteq V \subset T} t_V \prod_{T' \subset V \subseteq T} p_V.$$

We find that for $U \subseteq T'$ we have $s_{TU} = \lambda s_{T'U}$ for all $U \subseteq T'$. Moreover, if $U \subseteq T$ but $U \not\subseteq T'$ we note that $y_U$ is constant on $T'$. It follows that $z_T|_{T'} = \lambda z_{T'}$ (mod constants), which proves coherence. We can thus define a map $\phi \colon X \to \overline{F}(A; \mathcal{T})$ by $(t, p, q, x) \mapsto z$. Note that if $q_U = 0$ then $s_{TU} = 0$ for all $T \supseteq U$ and so $\theta(t, p, q, x)$ does not depend on $x_U$. On the other hand, if $t_V$ is a factor in $s_{TU}$, then so is $p_W$, where $W$ is the parent of $V$. It follows that if $p_W = 0$ then $\theta(t, p, q, x)$ does not depend on $t_V$ for all $V \in \delta' W$. This means that $\theta$ induces a map $\theta \colon \overline{X} \to \overline{F}(A; \mathcal{T})$. As $\overline{X}$ is compact and $\overline{F}(A; \mathcal{T})$ is Hausdorff, it will suffice to show that this map has a possibly discontinuous inverse; continuity will then be automatic.

Suppose now that we start with a point $z \in \overline{F}(A;\mathcal{T})$. We may assume that $z_T$ is normalised, so $\sum_{a\in T} z_T(a) = 0$ and $\sum_{a\in T} \|z_T(a)\|^2 = 1$; this fixes $z_T$ uniquely as a map $T \to N$. There are then unique maps $z'_T : \delta T \to N$ and $z''_U : U \to N$ (for $U \in \delta' T$) such that

- $z_T = r_T^* z'_T + \sum_U z''_U$
- $z''_U$ is supported on $U$
- $\sum_{a\in U} z''_U(a) = 0$
- $\sum_U |U| z'_T(U) = 0$.

Note that each $U \in \mathcal{T}' \setminus \{A\}$ lies in $\delta' T$ for a unique $T$ (viz. the parent of $U$) so this is a valid definition of $z''_U$ for all such $U$.

The above decomposition is orthogonal, so

$$\|r_T^* z'_T\|^2 + \sum_{U\in\delta' T} \|z''_U\|^2 = \|z_T\|^2 = 1.$$

We put $q_T^2 = \|r_T^* z'_T\|^2$ and $p_T^2 = \sum_{U\in\delta' T} \|z''_U\|^2$, so $p_T^2 + q_T^2 = 1$. If $p_T > 0$ then we put $t_U = \|z''_U\|/p_T$; otherwise we take $t_U = |\delta' T|^{-1/2}$ for definiteness. Note that $\|z''_U\| = p_T t_U$ even when $p_T = 0$, and that $\sum_{U\in\delta' T} t_U^2 = 1$. Note also that coherence means that $z''_U$ is a nonnegative multiple of $z_U$, and $z_U$ is normalised so $z''_U = p_T t_U z_U$.

If $q_T > 0$ then $z'_T \neq 0$ and we let $x_T$ denote the class of $z'_T$ in $S(NW\delta T)$. If $q_T = 0$ then we take $x_T$ to be an arbitrary point in $S(NW\delta T)$. We can thus define a (possibly discontinuous) map $\eta : \overline{F}(A;\mathcal{T}) \to \overline{X}$ by $\eta(z) = [t, p, q, x]$. Given our earlier manipulations, it is straightforward to check that $\eta\theta = 1_{\overline{X}}$, so $\theta$ is injective.

Finally, we need to check that $\theta\eta = 1$. Let $z$, $z'$, $z''$, $p$, $q$, $t$ and $x$ be as above. As in our definition of $\theta$, we let $y_T$ be the normalised and extended version of $x_T$. It then follows that $q_T y_T|_T = r_T^* z'_T$. We then put

$$s_{TU} = q_U \prod_{U\subset V\subseteq T} p_V \prod_{U\subseteq V\subset T} t_V$$

as before, and $\overline{z}_T = \sum_{U\subseteq T} s_{TU} y_U|_T$, so $\overline{z} = \theta\eta(z)$. We will show by induction on $|T|$ that $\overline{z}_T = z_T$. If $T$ is minimal then $\delta' T = \emptyset$ so $z_T = r_T^* z'_T = q_T y_T = \overline{z}_T$ as required. If $T$ is not minimal then

$$\overline{z}_T = q_T y_T|_T + \sum_{U\in\delta' T} \sum_{W\subseteq U} s_{TW} y_W|_T = r_T^* z'_T + \sum_{U\in\delta' T} p_T t_U \overline{z}_U$$

(where $\overline{z}_U$ has implicitly been extended by zero over $T \setminus U$). By induction we have $\overline{z}_U = z_U$ for $U \in \delta' T$, so $p_T t_U \overline{z}_U = z''_U$. The equation displayed above therefore gives $\overline{z}_T = z_T$, as required. □

**Corollary 15.15.** *$\overline{F}(A)$ and $\overline{F}(A;\mathcal{T})$ are naturally manifolds with corners.*

*Proof.* For $\overline{F}(A;\mathcal{T})$, this is clear from the proposition. For $\overline{F}(A)$, note that any point $x \in \overline{F}(A)$ has a neighbourhood $\overline{F}(A; \subseteq t(x))$ that is homeomorphic to an open subset of the manifold-with-corners $\overline{F}(A; t(x))$. □

**Corollary 15.16.** *$\overline{F}(A;\mathcal{T})$ is homotopy equivalent to $\prod_T S(NWT)$, where $T$ runs over the minimal sets in $\mathcal{T}'$.*

*Proof.* If $T$ is not minimal then $\delta' T \neq \emptyset$ and $\Delta(\delta' T) * S(NW\delta T)$ is contractible. If $T$ is minimal then $\delta T \simeq T$ and $\delta' T = \emptyset$ so $\Delta(\delta' T) * S(NW\delta T) = S(NWT)$. The claim now follows from Proposition 15.13. □

We now make $\overline{F}$ into an operad. Consider a map $p : A \to B$ of finite sets, together with points $y \in \overline{F}(B)$ and $x_b \in \overline{F}(A_b)$ for $b \in B$. Consider a set $T \subseteq A$ with $|T| > 1$. If $|p(T)| > 1$ then the map $p : T \to p(T)$ induces $p^* : S(NWp(T)) \to S(NWT)$, and we put $z_T = p^* y_{p(T)}$. Otherwise, $p(T) = \{b\}$ for some $b$, so $T \subseteq A_b$, and we put $z_T = x_{bT} \in S(NWT)$. This gives a point $z \in \prod_{T\in\mathcal{T}'} S(NWT)$, which we claim is coherent. To see this, suppose we have $U \in \mathcal{T}'$ with $U \subseteq T$. If $|p(U)| > 1$ then certainly also $|p(T)| > 1$, so $z_T = p^* y_{p(T)}$ and $z_U = p^* y_{p(U)}$. As $y \in \overline{F}(B)$ we have $y_{p(T)}|_{p(U)} \in \{0, y_{p(U)}\}$, and it follows that $z_T|_U \in \{0, z_U\}$ as required. Suppose instead that $p(T) = \{b\}$, in which case we also have $p(U) = \{b\}$. This means that $z_T = x_{bT}$ and $z_U = x_{bU}$ so $z_T|_U \in \{0, z_U\}$ by the coherence of $x_b$. This just leaves the case where $p(U) = \{b\}$ but $|p(T)| > 1$. We then have $z_T = p^* y_{p(T)}$ and $p$ is constant on $U$ so $z_T|_U = 0$. Thus $z \in \overline{F}(A)$, and we can define

$$\gamma_p \colon \overline{F}(B) \times \overline{F}(p) \to \overline{F}(A)$$

by $\gamma_p(y; (x_b)_{b\in B}) = z$.

**Proposition 15.17.** *This definition makes $\overline{F}$ an operad.*

*Proof.* Consider maps $A \xrightarrow{p} B \xrightarrow{q} C$, together with elements $z \in \overline{F}(C)$, and $y_c \in \overline{F}(B_c)$ for $c \in C$, and $x_b \in \overline{F}(A_b)$ for $b \in B$. Using these we define

$$\begin{aligned} v &= \gamma_q(z; (y_c)_{c\in C}) \in \overline{F}(B) \\ u_c &= \gamma_p(y_c; (x_b)_{b\in B_c}) \in \overline{F}(A_c) \\ w &= \gamma_p(v; (x_b)_{b\in B}) \\ w' &= \gamma_q(z; (u_c)_{c\in C}). \end{aligned}$$

We must show that $w = w'$. In fact, one checks from the definitions that

$$w_T = w'_T = \begin{cases} (qp)^* z_{qp(T)} & \text{if } |qp(T)| > 1 \\ p^* y_{c,p(T)} & \text{if } |p(T)| > 1 \text{ but } qp(T) = \{c\} \\ x_{b,T} & \text{if } p(T) = \{b\}. \end{cases}$$

**Re-check this** ☐

TODO:

- $F(A)$ is dense in $\overline{F}(A)$.
- The inclusion $F(A) \to \overline{F}(A)$ is a homotopy equivalence.
- The operad structure on $\overline{F}$ gives the standard operad structure on $H_*\overline{F} \simeq \text{Poiss}$
- Comparison with Dev's model.

## 16. Fulton-Macpherson and Stasheff

We now take $N = \mathbb{R}$, and show that this makes $F$ the same as the Stasheff operad.

**Definition 16.1.** Let $A$ be a totally ordered set. We let $W_+A$ be the set of nondecreasing maps $A \to \mathbb{R}$ mod constants, and we let $S(W_+A)$ denote the image of $W_+A$ in $S(WA)$. This can be identified with the set of nondecreasing maps $f \colon A \to \mathbb{R}$ such that $\min(f) = 0$ and $\max(f) = 1$ (although this makes it harder to see the restriction maps $S(W_+A) \to S(W_+B)$ for $B \subseteq A$). Next, let $\mathcal{J} = \mathcal{J}(A)$ denote the set of intervals in $A$ of length greater than one. Given an element $x \in \prod_{J\in\mathcal{J}} S(W_+A)$, we say that $x$ is *coherent* if for all $I \subseteq J$ we have $\rho^J_I(x_J) \subseteq x_I$. We let $\overline{F}_+(A)$ denote the set of all coherent $x$.

**Proposition 16.2.** *For any finite set $A$ we have $\overline{F}(A) \simeq \coprod_{R\in\text{Ord}(A)} \overline{F}_+(A, R)$.*

*Proof.* Given a set $T = \{a, b\}$ of size two, and an element $x \in S(WT)$, we can represent $x$ by a nonconstant map $f \colon T \to \mathbb{R}$, and we can ask whether $f(a) < f(b)$. We will abusively write $x(a) < x(b)$ for this, rather than explicitly taking a representative.

Given $x \in \overline{F}(A)$, we define an order on $A$ as follows. For $a, b \in A$ with $a \neq b$ we have $x_{\{a,b\}} \in S(W\{a, b\})$, and we put $a <_x b$ iff $x_{\{a,b\}}(a) < x_{\{a,b\}}(b)$. If so, it is easy to see that $x_T(a) \leq x_T(b)$ for all $T \subseteq A$ with $a, b \in T$. Conversely, if there exists $T$ such that $x_T(a) < x_T(b)$ then $a <_x b$. Given this (and the fact that $x_{\{a,b,c\}}$ cannot be constant) we see that our relation is transitive and thus is a total order on $A$. We have therefore defined a map $\omega \colon \overline{F}(A) \to \text{Ord}(A)$. The fibres are easily seen to be open, and they give a disjoint cover of $\overline{F}(A)$, so they are also closed.

Now fix a point $R \in \text{Ord}(A)$, and put $X = \omega^{-1}\{R\}$. It is easy to see that

$$X = \{x \in \overline{F}(A) \mid x_T \in S(W_+T) \text{ for all } T\}.$$

We must identify this with $\overline{F}_+(A)$. There is an evident projection $\pi \colon X \to \overline{F}_+(A)$ (by forgetting the coordinates $x_T$ when $T$ is not an interval). In the other direction, suppose we have $x \in \overline{F}_+(A)$ and a subset $T \subseteq A$ with $|T| > 1$. We then put $\sigma(x)_T = x_J|_T$, where $J = [\min(T), \max(T)]$ is the smallest interval containing $T$. This is valid because $x_J \colon J \to \mathbb{R}$ is nondecreasing and nonconstant, so $x_J(\min(T)) < x_J(\max(T))$, so $x_J|_T$ is nonconstant. One can check directly that $\sigma$ is inverse to $\rho$. $\square$

## References