AN INTRODUCTION TO THE CATEGORY OF SPECTRA

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1. Introduction

Early in the history of homotopy theory, people noticed a number of phenomena suggesting that it would be convenient to work in a context where one could make sense of negative-dimensional spheres. Let X be a finite pointed simplicial complex; some of the relevant phenomena are as follows.

- For most n, the homotopy sets $\pi_n X$ are abelian groups. The proof involves consideration of S^{n-2} and so breaks down for $n < 2$; this would be corrected if we had negative spheres.
- Calculation of homology groups is made much easier by the existence of the suspension isomorphism $\widetilde{H}_{n+k}\Sigma^k X = \widetilde{H}_n X$. This does not generally work for homotopy groups. However, a theorem of Freudenthal says that if X is a finite complex, we at least have a suspension isomorphism $\pi_{n+k}\Sigma^k X = \pi_{n+k+1}\Sigma^{k+1}X$ for large k. If could work in a context where S^{-k} makes sense, we could smash everything with S^{-k} to get a suspension isomorphism in homotopy parallel to the one in homology.
- We can embed X in S^{k+1} for large k, and let Y be the complement. Alexander duality says that $\widetilde{H}_n Y = \widetilde{H}^{k-n}X$, showing that X can be "turned upside-down", in a suitable sense. The shift by k is unpleasant, because the choice of k is not canonical, and the minimum possible k depends on X. Moreover, it is unsatisfactory that the homotopy type of Y is not determined by that of X (even after taking account of k). In a context where negative spheres exist, one can define $DX = S^{-k} \wedge Y$; one finds that $\widetilde{H}_n DX = \widetilde{H}^{-n}X$ and that DX is a well-defined functor of X, in a suitable sense.
- The Bott periodicity theorem says that the homotopy groups of the infinite orthogonal group $O(\infty)$ satisfy $\pi_{k+8}O(\infty) = \pi_k O(\infty)$ for all $k \geq 0$. It would be pleasant and natural to extend this pattern to negative values of k, which would again require negative spheres.

Considerations such as these led to the construction of the Spanier-Whitehead category $\mathcal{F}$ of finite spectra, which we briefly survey in Section 2. Although fairly straightforward, and very beautiful and interesting, this category has two defects.

- Many of the most important examples in homotopy theory are infinite complexes: Eilenberg-MacLane spaces, classifying spaces of finite groups, infinite-dimensional grassmannians and so on. The category $\mathcal{F}$ is strongly tied to finite complexes, so a wider framework is needed to capture these examples.
- Ordinary homotopy theory is made both easier and more interesting by its connections with geometry. However, $\mathcal{F}$ is essentially a homotopical category, with no geometric structure behind it. This also prevents a good theory of spectra with a group action, or of bundles of spectra over a space, or of diagrams of spectra.

The first problem was addressed by a number of people, but the definitive answer was provided by Boardman. He constructed a category $\mathcal{B}$ with excellent formal properties parallel to those of $\mathcal{F}$, whose subcategory of finite objects (suitably defined) is equivalent to $\mathcal{F}$. A popular exposition of this category is in Adams' book [1]. Margolis [15] gave a list of the main formal properties of $\mathcal{B}$ and its relationship with $\mathcal{F}$. He conjectured (with good evidence) that they characterise $\mathcal{B}$ up to equivalence. See [26] for some new evidence for this conjecture, and [27] for an investigation of some related systems of axioms.

The second problem has taken much longer to resolve. There have been a number of constructions of topological categories whose associated homotopy category (suitably defined) is equivalent to $\mathcal{B}$, with steadily improving formal properties [5, 6, 11, 14]. There is also a theorem of Lewis [13] which shows that it is impossible to have all the good properties that one might naively hope for. We will sketch one construction in Section 4.

2. The finite stable category

2.1. **Basics.** We first recall some basic definitions. In this section all spaces are assumed to be finite CW complexes with basepoints. (We could equally well use simplicial complexes instead, at the price of having to subdivide and simplicially approximate from time to time.) We write 0 for all basepoints, and we write $[A, B]$ for the set of based homotopy classes of maps from A to B. We define $A \vee B$ to be the quotient of the disjoint union of A and B in which the two basepoints are identified together. We also define $A \wedge B$ to be the quotient of $A \times B$ in which the subspace $A \times 0 \cup 0 \times B$ is identified with the single point $(0, 0)$. This is called the smash product of A and B; note that it is commutative and associative up to isomorphism and that $S^0 \wedge A = A$, where $S^0 = \{0, 1\}$. Moreover, $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$ and $[A \vee B, C] = [A, C] \times [B, C]$ (so $\vee$ is the coproduct in the homotopy category of pointed spaces). We let S^1 denote the quotient of $[0, 1]$ in which 0 is identified with 1, and we write $\Sigma A = S^1 \wedge A$. One can check that Σ sends the ball B^n to B^{n+1} and the sphere S^n to S^{n+1}, and the reduced homology of ΣA is just $\widetilde{H}_n(\Sigma A) = \widetilde{H}_{n-1}(A)$. Thus, we think of Σ as shifting all dimensions by one.

The quotient space of S^1 in which 1/2 is identified with 0 is evidently homeomorphic to $S^1 \vee S^1$, so we get a map $\delta \colon S^1 \to S^1 \vee S^1$ and thus a map $\delta \colon \Sigma A \to \Sigma A \vee \Sigma A$. It is well-known that the induced map $\delta^* \colon [\Sigma A, B] \times [\Sigma A, B] \to [\Sigma A, B]$ makes $[\Sigma A, B]$ into a group. There are apparently n different group structures on $[\Sigma^n A, B]$, but it is also well-known that they are all the same, and they are commutative when $n > 1$. We have an evident sequence of maps

$$[A, B] \xrightarrow{\Sigma} [\Sigma A, \Sigma B] \xrightarrow{\Sigma} [\Sigma^2 A, \Sigma^2 B] \to \dots.$$

Apart from the first two terms, it is a sequence of Abelian groups and homomorphisms. By a fundamental theorem of Freudenthal, after a finite number of terms, it becomes a sequence of isomorphisms. We define $[\Sigma^\infty A, \Sigma^\infty B]$ to be the group $[\Sigma^N A, \Sigma^N B]$ for large N, or if you prefer the colimit $\varinjlim_N [\Sigma^N A, \Sigma^N B]$. After doing a little point-set topology, one concludes that this is the same as the set $[A, QB]$, where $QB = \varinjlim_N \Omega^N \Sigma^N B$ and $\Omega^N C$ means the space of based continuous maps $S^N \to C$, with a suitable topology.

2.2. **Finite spectra.** One can define a category with one object called $\Sigma^\infty A$ for each finite CW complex A, and morphisms $[\Sigma^\infty A, \Sigma^\infty B]$. It is easy to see that Σ induces a full and faithful endofunctor of this category. We prefer to arrange things so that Σ is actually an equivalence of categories. Accordingly, we define a category $\mathcal{F}$ whose objects are expressions of the form $\Sigma^{\infty+n} A$ where A is a finite CW complex and n is an integer. (If you prefer, you can take the objects to be pairs (n, A).) We refer to these objects as finite spectra. The maps are

$$[\Sigma^{\infty+n} A, \Sigma^{\infty+m} B] = \varinjlim_N [\Sigma^{N+n} A, \Sigma^{N+m} B].$$

Freudenthal's theorem again assures us that the limit is attained at a finite stage. The functor Σ induces a self-equivalence of the category $\mathcal{F}$. There are evident extensions of the functors $\vee$ and $\wedge$ to $\mathcal{F}$ such that $\Sigma^\infty A \vee \Sigma^\infty B = \Sigma^\infty (A \vee B)$ and $\Sigma^\infty A \wedge \Sigma^\infty B = \Sigma^\infty (A \wedge B)$ (although care is needed with signs when defining the smash product of morphisms). The category $\mathcal{F}$ is additive, with biproduct given by the functor $\vee$. The morphism sets $[X, Y]$ in $\mathcal{F}$ are finitely generated Abelian groups. One can define homology of finite spectra by $H_n \Sigma^{\infty+m} A = \widetilde{H}_{n-m} A$, and then the map

$$H_* \colon \mathbb{Q} \otimes [X, Y] \to \prod_n \mathrm{Hom}(H_n(X; \mathbb{Q}), H_n(Y; \mathbb{Q}))$$

is an isomorphism. The groups $[X, Y]$ themselves are known to be recursively computable, but the guaranteed algorithms are of totally infeasible complexity. Nonetheless, there are methods of computation which require more intelligence than the algorithms but have a reasonable chance of success.

2.3. **Stable homotopy groups of spheres.** Even the groups $\pi_n^S = [\Sigma^{\infty+n}S^0, \Sigma^\infty S^0]$ are hard, and are only known for $n \leq 60$ or so (they are zero when $n < 0$). The first few groups are as follows:

$$\begin{aligned}
\pi_0^S &= \mathbb{Z}\{\iota\} \\
\pi_1^S &= \mathbb{Z}/2\{\eta\} \\
\pi_2^S &= \mathbb{Z}/2\{\eta^2\} \\
\pi_3^S &= \mathbb{Z}/24\{\nu\} \\
\pi_4^S &= 0 \\
\pi_5^S &= 0 \\
\pi_6^S &= \mathbb{Z}/4\{\nu^2\}
\end{aligned}$$

Here ι is the identity map, and η comes from the map

$$\eta\colon S^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\} \to \mathbb{C} \cup \{\infty\} = S^2$$

defined by $\eta(z,w) = z/w$. Similarly, ν comes from division of quaternions. The expression η^2 really means $\eta \circ (\Sigma\eta)$, and ν^2 means $\nu \circ (\Sigma^3\nu)$.

Many general results are also known. For example, for any prime p, the p-torsion part of π_n^S is known for $n < 2p^3 - 2p$ and is zero for $n < 2p - 3$ (provided $n \neq 0$). Both the rank and the exponent are finite but unbounded as n tends to infinity. The group π_*^S is a graded ring, and is commutative in the graded sense. An important theorem of Nishida says that all elements of degree greater than zero are nilpotent.

2.4. **Triangulation.** The category $\mathcal{F}$ is not Abelian. Instead, it has a triangulated structure. This means that there is a distinguished class of diagrams of the shape $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ (called exact triangles) with certain properties to be listed below. In our case the exact triangles can be described as follows. Let A be a subcomplex of a finite CW complex B, and let C be obtained from B by attaching a cone $I \wedge A$ along the subspace $\{1\} \times A = A$. There is an evident copy of B in C, and if we collapse it to a point we get a copy of ΣA. We thus have a diagram of spaces $A \to B \to C \to \Sigma A$. We say that a diagram $X \to Y \to Z \to \Sigma X$ of finite spectra is an exact triangle if it is isomorphic to a diagram of the form $\Sigma^{\infty+n}A \to \Sigma^{\infty+n}B \to \Sigma^{\infty+n}C \to \Sigma^{\infty+n+1}A$ for some $n \in \mathbb{Z}$ and some A, B and C as above. Incidentally, one can show that C is homotopy equivalent to the space B/A obtained from B by identifying A with the basepoint.

The axioms for a triangulated category are as follows. In our case, they all follow from the theory of Puppe sequences in unstable homotopy theory.

(a) Any diagram isomorphic to an exact triangle is an exact triangle.

(b) Any diagram of the form $0 \to X \xrightarrow{1} X \to \Sigma 0 = 0$ is an exact triangle.

(c) Any diagram $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is an exact triangle if and only if the diagram $Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma X$ is an exact triangle.

(d) For any map $f\colon X \to Y$ there exists a spectrum Z and maps g, h such that $X \xrightarrow{f} Y \xrightarrow{g} Y \xrightarrow{h} \Sigma X$ is an exact triangle.

(e) Suppose we have a diagram as shown below (with h missing), in which the rows lie are exact triangles and the rectangles commute. Then there exists a (nonunique) map h making the whole diagram commutative.

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "f"'] & V \arrow[r] \arrow[d, "g"'] & W \arrow[r] \arrow[d, dotted, "h"] & \Sigma U \arrow[d, "\Sigma f"] \\
X \arrow[r] & Y \arrow[r] & Z \arrow[r] & \Sigma X
\end{tikzcd}$$

(f) Suppose we have maps $X \xrightarrow{v} Y \xrightarrow{u} Z$, and exact triangles (X,Y,U), (X,Z,V) and (Y,Z,W) as shown in the diagram. (A circled arrow $U \rightarrow X$ means a map $U \to \Sigma X$.) Then there exist maps r and s as shown, making (U,V,W) into an exact triangle, such that the following commutativities hold:

$$au = rd \qquad es = (\Sigma v)b \qquad sa = f \qquad br = c$$

$$
\begin{tikzcd}
 & & V \arrow[dl, "b"'] \arrow[drr, dotted, bend left, "s"] & & \\
 & X \arrow[rr, "uv"] \arrow[dr, "v"] & & Z \arrow[ul, "a"'] \arrow[dr, "f"] & \\
U \arrow[ur, "c"] \arrow[urr, dotted, bend left, "r"] & & Y \arrow[ll, "d"] \arrow[ur, "u"] & & W \arrow[ll, "e"]
\end{tikzcd}
$$

The last axiom is called the octahedral axiom (the diagram can be turned into an octahedron by lifting the outer vertices and drawing an extra line from W to U). In our case it basically just says that when we have inclusions $A \subseteq B \subseteq C$ of CW complexes we have $(C/A)/(B/A) = C/B$.

One of the most important consequences of the axioms is that whenever $X \to Y \to Z \to \Sigma X$ is an exact triangle and W is a finite spectrum, we have long exact sequences

$$\ldots \to [W, \Sigma^{-1}Z] \to [W, X] \to [W, Y] \to [W, Z] \to [W, \Sigma X] \to \ldots$$

and

$$\ldots \leftarrow [\Sigma^{-1}Z, W] \leftarrow [X, W] \leftarrow [Y, W] \leftarrow [Z, W] \leftarrow [\Sigma X, W] \leftarrow \ldots.$$

2.5. **Thom spectra.** Let X be a finite CW complex, and let V be a vector bundle over X. The Thom space X^V can be defined as the one-point compactification of the total space of V. This has many interesting properties, not least of which is the fact that when V is an oriented bundle of dimension n, the reduced cohomology $\widetilde{H}^*(X^V)$ is a free module over $H^*(X)$ on one generator in dimension n. This construction can be generalised to virtual bundles, in other words formal expressions of the form $V - W$, except that we now have a Thom spectrum X^{V-W} rather than a Thom space. The construction is to choose a surjective map from a trivial bundle $\mathbb{R}^n \times X$ onto W, with kernel U say, and define $X^{V-W} = \Sigma^{\infty-n} X^{V \oplus U}$.

2.6. **Duality.** For any finite spectrum X, there is an essentially unique spectrum DX (called the Spanier-Whitehead dual of X) equipped with a natural isomorphism $[W \wedge X, Y] = [W, DX \wedge Y]$. This can be constructed in a number of different ways. One way is to start with a simplicial complex A and embed it simplicially as a proper subcomplex of S^{N+1} for some $N > 0$. One can show that the complement of A has a deformation retract B which is a finite simplicial complex, and $D\Sigma^{\infty+n}A = \Sigma^{\infty-n-N}B$. Note that Alexander duality implies that $H_m X = H^{-m} DX$.

An important example arises when $X = \Sigma^\infty M_+$ for some smooth manifold M, with tangent bundle τ say. It is not hard to show geometrically that $D(\Sigma^\infty M_+)$ is $M^{-\tau}$, the Thom spectrum of the virtual bundle $-\tau$ over M; this phenomenon is called *Atiyah duality*.

We also write $F(X, Y) = DX \wedge Y$. This is a functor in both variables, it preserves cofibrations up to sign, and the defining property of DX can be rewritten as $[W, F(X, Y)] = [W \wedge X, Y]$.

2.7. **Splittings.** It often happens that we have a finite complex X that cannot be split into simpler pieces, but that the finite spectrum $\Sigma^\infty X$ does have a splitting. Group actions are one fruitful source of splittings. If a finite group G acts on X, then the map $G \to \mathrm{Aut}(\Sigma^\infty X)$ extends to a ring map $\mathbb{Z}[G] \to \mathrm{End}(\Sigma^\infty X)$. If $\widetilde{H}_* X$ is a p-torsion group, then this will factor through $(\mathbb{Z}/p^n)[G]$ for large n. Any idempotent element in $(\mathbb{Z}/p)[G]$ can be lifted uniquely to an idempotent in $(\mathbb{Z}/p^n)[G]$, which will give an idempotent in $\mathrm{End}(\Sigma^\infty X)$ and thus a splitting of X. The methods of modular representation theory give good information about idempotents in group rings, and thus a supply of interesting splittings. The Steinberg idempotent in $(\mathbb{Z}/p)[GL_n(\mathbb{Z}/p)]$ gives particularly important examples, as do various idempotents in $(\mathbb{Z}/p)[\Sigma_n]$.

Another common situation is to have a finite complex X and a filtration $F_0X \subseteq F_1X \subseteq \ldots \subseteq F_nX = X$ that splits stably, giving an equivalence $\Sigma^\infty X \simeq \bigvee_n \Sigma^\infty F_nX/F_{n-1}X$ of finite spectra. For example, one can take $X = U(m)$, and let F_nX be the space of matrices $A \in U(m)$ for which the rank of $A - I$ is at most n. A theorem of Miller says that the filtration splits stably, and that the quotient $F_nU(m)/F_{n-1}U(m)$ is the Thom space of a certain bundle over the Grassmannian of n-planes in $\mathbb{C}^m$. Later we will explain how to interpret $\Sigma^\infty X$ when X is an infinite complex; there are many examples in which X has a stably split filtration in which the quotients are finite spectra. This holds for $X = BU(n)$ or $X = \Omega U(n)$ or $X = \Omega^n S^{n+m}$, for example. The splitting of $\Omega^n S^{n+m}$ is due to Snaith; the Snaith summands in $\Omega^2 S^3$ are called *Brown-Gitler spectra*, and they have interesting homological properties with many applications.

3. Cobordism and Morava K-theory

We next outline the theory of complex cobordism, and the results of Hopkins, Devinatz and Smith showing how complex cobordism reveals an important part of the structure of $\mathcal{F}$.

Given a space X and an integer $n \geq 0$ we define a *geometric n-chain* in X to be a compact smooth manifold M (possibly with boundary) equipped with a continuous map $f\colon M \to X$. We regard (M_0, f_0) and (M_1, f_1) as equivalent if there is a differomorphism $g\colon M_0 \to M_1$ with $f_1 g = f_0$. We write $GC_n X$ for the set of equivalence classes, which is a commutative semigroup under disjoint union. We define a differential $\partial\colon GC_n X \to GC_{n-1} X$ by $\partial[M, f] = [\partial M, f|_{\partial M}]$. One can make sense of the homology $MO_* X = H(GC_* X, \partial)$, and (because $\partial(M \times I) = M \amalg M$) one finds that $MO_* X$ is a vector space over $\mathbb{Z}/2$. In the case where X is a point, one can use cartesian products to make $MO_*(\text{point})$ into a graded ring, which is completely described by a remarkable theorem of Thom: it is a polynomial algebra over $\mathbb{Z}/2$ with one generator x_n in degree n for each integer $n > 0$ not of the form $2^k - 1$. New perspectives on this answer and the underlying algebra were provided by Quillen [17, 18] and Mitchell [16]. One can also show that $MO_* X = MO_*(\text{point}) \otimes_{\mathbb{Z}/2} H_*(X; \mathbb{Z}/2)$, so this construction does not yield new invariants of spaces. This isomorphism gives an obvious way to define $MO_* X$ when X is a finite spectrum.

The story changes however, if we work with oriented manifolds. This gives groups $MSO_* X$ with a richer structure; in particular, they are not annihilated by 2. There are various hints that complex manifolds would give still more interesting invariants, but there are technical problems, not least the lack of a good theory of complex manifolds with boundary. It turns out to be appropriate to generalize and consider manifolds with a specified complex structure on the stable normal bundle, known as "stably complex manifolds". The precise definitions are delicate; details are explained in [17], and Buchstaber and Ray [2] have provided naturally occuring examples where the details are important. In any case, one ends up with a ring $MU_* = MU_*(\text{point})$, and groups $MU_* X$ for all spaces X that are modules over it. It is not the case that $MU_* X = MU_* \otimes_{\mathbb{Z}} H_* X$, but there is still a suspension isomorphism, which allows one to define $MU_* X$ when X is a finite spectrum. One finds that this is a generalized homology theory (known as *complex cobordism*), so it converts cofibre seqences of spectra to long exact sequences of modules. The nilpotence theory of Devinatz, Hopkins and Smith [3, 12, 21] (which will be outlined below) shows that $MU_* X$ is an extremely powerful invariant of X.

The ring MU_* turns out to be a polynomial algebra over $\mathbb{Z}$ with one generator in each positive even degree. There is no canonical system of generators, but nonetheless, Quillen showed that MU_* is canonically isomorphic to an algebraically defined object: Lazard's classifying ring for formal group laws [1, 20]. This was the start of an extensive relationship between stable homotopy and formal group theory. The algebra provides many natural examples of graded rings A_* equipped with a formal group law and thus a map $MU_* \to A_*$. It is natural to ask whether there is a generalised homology theory $A_*(X)$ whose value on a point is the ring A_*. Satisfactory answers for a broad class of rings A_* are given in [25], which surveys and consolidates a great deal of much older literature and extends newer ideas from [5]. In particular, we can consider the rings $K(p, n)_* = \mathbb{F}_p[v_n, v_n^{-1}]$, where p is prime and $n > 0$ and v_n has degree $2(p^n - 1)$. This is made into an algebra over MU_* using a well-known formal group law, and by old or new methods, one can construct an associated generalised homology theory $K(p, n)_* X$, known as *Morava K-theory.* It is convenient to extend the definition by putting $K(p, 0)_* X = H_*(X; \mathbb{Q})$.

A key theorem of Hopkins, Devinatz and Smith says that if $f\colon \Sigma^d X \to X$ is a self-map of a finite spectrum X, and $K(p, n)_*(f) = 0$ for all primes p and all $n \geq 0$, then the iterated composite $f^m\colon \Sigma^{md} X \to X$ is zero for large m, or in other words f is composition-nilpotent. They also show that Morava K-theory detects nilpotence in a number of other senses, and give formulations involving the single theory MU instead of the collection of theories $K(p, n)$.

If R is a commutative ring, it is well-known that the ideal of nilpotent elements is the intersection of all the prime ideals, so the Zariski spectrum is unchanged if we take the quotient by this ideal. One can deduce that the classification of certain types of subcategories of the abelian category of R-modules is again insensitive to the ideal of nilpotents. Our category $\mathcal{F}$ of finite spectra is triangulated rather than abelian, but nonetheless Hopkins and Smith developed an analogous theory and deduced a classification of the thick subcategories of $\mathcal{F}$, with many important consequences.

4. Boardman's category $\mathcal{B}$

It is clearly desirable to have a category $\mathcal{C}$ analogous to $\mathcal{F}$ but without finiteness conditions. There are various obvious candidates: one could take the Ind-completion of $\mathcal{F}$, or just follow the definition of $\mathcal{F}$ but allow infinite CW complexes instead of finite ones. Unfortunately, these categories turn out to have unsatisfactory technical properties. The requirements were first assembled in axiomatic form by Margolis [15]; in outline, they are as follows:

- $\mathcal{C}$ should be a triangulated category
- Every family $\{X_\alpha\}$ of objects in $\mathcal{C}$ should have a coproduct, written $\bigvee_\alpha X_\alpha$
- For any $X, Y \in \mathcal{C}$ there should be functorially associated objects $X \wedge Y$ and $F(X,Y)$ making $\mathcal{C}$ a closed symmetric monoidal category.
- If we let small($\mathcal{C}$) be the subcategory of objects W for which the natural map $\bigoplus_\alpha [W, X_\alpha] \to [W, \bigvee_\alpha X_\alpha]$ is always an isomorphism, then small($\mathcal{C}$) should be equivalent to $\mathcal{F}$.

Historically, the work of Margolis came after Boardman's construction of a category $\mathcal{B}$ satisfying the axioms, and Adams's explanation [1] of a slightly different way to approach the construction. Margolis conjectured that if $\mathcal{C}$ satisfies the axioms then $\mathcal{C}$ is equivalent to $\mathcal{B}$. Schwede and Shipley [22] have proved that this is true, provided that $\mathcal{C}$ is the homotopy category of a closed model category in the sense of Quillen [4, 19] satisfying suitable axioms. There is also good evidence for the conjecture without this additional assumption. The objects of $\mathcal{B}$ are generally called *spectra*, although in some contexts one introduces different words to distinguish between objects in different underlying geometric categories.

Probably the best approach to constructing $\mathcal{B}$ is via the theory of *orthogonal spectra*, as we now describe. Let $\mathcal{V}$ denote the category of finite-dimensional vector spaces over $\mathbb{R}$ equipped with an inner product. The morphisms are linear isomorphisms that preserve inner products. For any $V \in \mathcal{V}$ (with $\dim(V) = n$ say) we write S^V for the one-point compactification of V; this is homeomorphic to S^n. An orthogonal spectrum X consists of a functor $\mathcal{V} \to \{\text{based spaces}\}$ together with maps $S^U \wedge X(V) \to X(U \oplus V)$ satisfying various continuity and compatibility conditions that we will not spell out. We write $\mathcal{S}$ for the category of orthogonal spectra.

For orthogonal spectra X and Y, the morphism set $\mathcal{S}(X,Y)$ has a natural topology, and we could define an associated homotopy category by the rule $[X,Y] = \pi_0\mathcal{S}(X,Y)$. Unfortunately, the resulting category is not the one that we want. Instead, we define the homotopy groups of X by the rule $\pi_k(X) = \varinjlim_N \pi_{k+N}(X(\mathbb{R}^N))$. We then say that a map $f\colon X \to Y$ is a *weak equivalence* if $\pi_*(f)\colon \pi_*(X) \to \pi_*(Y)$ is an isomorphism. We now construct a new category $\mathrm{Ho}(\mathcal{S})$ by starting with $\mathcal{S}$ and adjoining formal inverses for all weak equivalences. It can be shown that this is equivalent to $\mathcal{B}$ (or can be taken as the definition of $\mathcal{B}$).

This process of adjoining formal inverses can be subtle. To manage the subtleties, we need the theory of model categories in the sense of Quillen [4, 10]. In particular, this will show that $\mathrm{Ho}(\mathcal{S})(X,Y) = \pi_0\mathcal{S}(X,Y)$ for certain classes of spectra X and Y; this is enough to get started with computations and prove that Margolis's axioms are satisfied.

Given orthogonal spectra X, Y and Z, a *pairing* from X and Y to Z consists of maps $\alpha_{U,V}\colon X(U) \wedge Y(V) \to Z(U \oplus V)$ satisfying some obvious compatibility conditions. One can show that there is an orthogonal spectrum $X \wedge Y$ such that pairings from X and Y to Z biject with morphisms from $X \wedge Y$ to Z. This construction gives a symmetric monoidal structure on $\mathcal{S}$. This in turn gives rise to a symmetric monoidal structure on $\mathrm{Ho}(\mathcal{S})$; however, there are some hidden subtleties in this step, which again are best handled by the general theory of model categories. One consequence is that the topology of the classifying space of the symmetric group Σ_k is mixed in to the structure of the k-fold smash product $X^{(k)} = X \wedge \cdots \wedge X$.

A *ring spectrum* is an object $R \in \mathrm{Ho}(\mathcal{S})$ equipped with a unit map $\eta\colon S^0 \to R$ and a multiplication map $\mu\colon R \wedge R \to R$ such that the following diagrams in $\mathrm{Ho}(\mathcal{S})$ commute:

$$\begin{tikzcd}
R \wedge R \wedge R \arrow[r, "\mu\wedge 1"] \arrow[d, "1\wedge\mu"'] & R \wedge R \arrow[d, "\mu"] \\
R \wedge R \arrow[r, "\mu"'] & R
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "\eta\wedge 1"] \arrow[dr, "1"'] & R \wedge R \arrow[d, "\mu"] & R \arrow[l, "1\wedge\eta"'] \arrow[dl, "1"] \\
& R &
\end{tikzcd}$$

Because we now have a good underlying geometric category $\mathcal{S}$, we can formulate a more precise notion: a *strict ring spectrum* is an object $R \in \mathcal{S}$ equipped with morphisms $S^0 \xrightarrow{\eta} R \xleftarrow{\mu} R \wedge R$ such that the above diagrams commute in $\mathcal{S}$ (not just in $\mathrm{Ho}(\mathcal{S})$).

The symmetric monoidal structure on $\mathcal{S}$ includes a natural map $\tau_{XY} \colon X \wedge Y \to Y \wedge X$. We say that a strict ring spectrum R is *strictly commutative* if $\mu \circ \tau_{RR} = \mu$. This is a surprisingly stringent condition, with extensive computational consequences.

5. Examples of spectra

Some important functors that construct objects of $\mathcal{B}$ are as follows:

(a) For any based space X there is a suspension spectrum $\Sigma^\infty X \in \mathcal{B}$, whose homotopy groups are given by $\pi_n \Sigma^\infty X = \pi_n^S X = \varinjlim_k \pi_{n+k} \Sigma^k X$. The relevant orthogonal spectrum is just $(\Sigma^\infty X)(V) = S^V \wedge X$.

We will mention one important example of infinite complexes X and Y for which $[\Sigma^\infty X, \Sigma^\infty Y]$ is well-understood. Let G be a finite group, with classifying space BG. Let AG^+ be the set of isomorphism classes of finite sets with a G-action. We can define addition and multiplication on AG^+ by $[X] + [Y] = [X \amalg Y]$ and $[X][Y] = [X \times Y]$. There are no additive inverses, but we can formally adjoin them to get a ring called AG, the Burnside ring of G; this is not hard to work with explicitly. There is a ring map $\epsilon \colon AG \to \mathbb{Z}$ defined by $\epsilon([X] - [Y]) = |X| - |Y|$, with kernel I say. We then have a completed ring $\widehat{A}G = \varprojlim_n AG/I^n$. The Segal conjecture (which was proved by Carlsson) gives an isomorphism $\widehat{A}G \simeq [\Sigma^\infty BG_+, \Sigma^\infty S^0]$. One can deduce a description of $[\Sigma^\infty BG_+, \Sigma^\infty BH_+]$ in similar terms for any finite group H.

(b) For any virtual vector bundle V over any space X, there is a Thom spectrum $X^V \in \mathcal{B}$. In particular, if V is the tautological virtual bundle (of virtual dimension zero) over the classifying space BU, then there is an associated Thom spectrum, normally denoted by MU. This has the property that the groups $MU_* X$ (as in Section 3) are given by $\pi_*(MU \wedge \Sigma^\infty X_+)$ (this is proved by a geometric argument, and is essentially the first step in the calculation of MU_*). One can construct MU (and also MO and MSO) as strictly commutative ring spectra.

Bott periodicity gives an equivalence $BU \simeq \Omega SU$, and the filtration of ΩSU by the subspaces $\Omega SU(k)$ gives a filtration of MU by subspectra $X(m)$, which are important in the proof of the Hopkins-Devinatz-Smith nilpotence theorem. There are models of these homotopy types that are strict ring spectra, but they cannot be made commutative.

(c) For any generalized cohomology theory A^*, there is an essentially unique spectrum $A \in \mathcal{B}$ with $A^n X = [\Sigma^\infty X_+, \Sigma^n A]$ for all spaces X and $n \in \mathbb{Z}$. Similarly, for any generalized homology theory B_*, there is an essentially unique spectrum $B \in \mathcal{B}$ with $B_n X = \pi_n(B \wedge \Sigma^\infty X_+)$ for all spaces X and $n \in \mathbb{Z}$. These facts are known as *Brown representability*; the word "essentially" hides some subtleties.

(d) In particular, for any abelian group A there is an Eilenberg-MacLane spectrum $HA \in \mathcal{B}$ such that $[\Sigma^\infty X_+, \Sigma^n HA] = H^n(X; A)$ and $\pi_n(HA \wedge \Sigma^\infty X_+) = H_n(X; A)$. If A is a commutative ring, then HA is a strictly commutative ring spectrum. It is common to consider the case $A = \mathbb{Z}/2$. Here it can be shown that

$$\pi_*((H\mathbb{Z}/2) \wedge (H\mathbb{Z}/2)) = \mathbb{Z}/2[\xi_1, \xi_2, \xi_3, \dots],$$

with $|\xi_k| = 2^k - 1$. This is known as the *dual Steenrod algebra*, and denoted by $\mathcal{A}_*$. The dual group $\mathcal{A}^k = \mathrm{Hom}(\mathcal{A}_k, \mathbb{Z}/2)$ can be identified with $[H\mathbb{Z}/2, \Sigma^k H\mathbb{Z}/2]$, so these groups again form a graded ring (under composition), called the *Steenrod algebra*. This ring is noncommutative, but its structure can be described quite explicitly. It is important, because the mod 2 cohomology of any space (or spectrum) has a natural structure as an $\mathcal{A}^*$-module. There is a similar story for mod p cohomology when p is an odd prime, but the details are a little more complicated.

(e) Another consequence of Brown representability is that there is a spectrum $I \in \mathcal{B}$ such that $[X, I] \simeq \mathrm{Hom}(\pi_0 X, \mathbb{Q}/\mathbb{Z})$ for all $X \in \mathcal{B}$. This is called the *Brown-Comenetz dual* of S^0; it is geometrically mysterious, and a fertile source of counterexamples.

(f) If M_* is a flat module over MU_*, then the functor $X \mapsto M_* \otimes_{MU_*} MU_*X$ is a homology theory, so there is a representing spectrum M with $\pi_*(M \wedge X) = M_* \otimes_{MU_*} MU_*X$. The Landweber exact functor theorem shows that flatness is not actually necessary: a weaker condition called *Landweber exactness* will suffice. This condition is formulated in terms of formal group theory, and is often easy to check in practice.

Often M_* is a ring, and the MU_*-module structure arises from a ring map $MU_* \to M_*$, which corresponds (by Quillen's description of MU_*) to a formal group law over M_*.

Important examples include the Johnson-Wilson spectra $E(p,n)$, with $E(p,n)_* = \mathbb{Z}_{(p)}[v_1,\ldots,v_n][v_n^{-1}]$ (where $|v_k| = 2(p^k-1)$). This has a canonical formal group law, which we will not describe here. The ring $K(p,n)_* = \mathbb{Z}/p[v_n,v_n^{-1}]$ is naturally a quotient $E(p,n)_*/I_n$, where $I_n = (p,v_1,\ldots,v_{n-1})$. It it is not Landweber exact, but a corresponding spectrum $K(p,n)$ can be constructed by other means. It is also useful to consider the completed spectra $\widehat{E}(p,n)$ with

$$\pi_*(\widehat{E}(p,n)) = (E(p,n)_*)^\wedge_{I_n} = \mathbb{Z}_p[\![v_1,\ldots,v_{n-1}]\!][v_n^{\pm 1}].$$

This is again Landweber exact.

When M_* is a ring, one might hope to find a model of this homotopy type that is actually a strict ring spectrum, preferably strictly commutative. Unfortunately, this does not work very well. Often there will be uncountably many different ways to make M into a strict ring spectrum, with no way to pick out a preferred choice. Moreover, there will often not be any choice that is strictly commutative. However, by a theorem of Hopkins and Miller, there is an essentially unique strictly commutative model for $\widehat{E}(p,n)$. The reason why this case is special involves quite deep aspects of the algebraic theory of formal groups.

(g) For any small symmetric monoidal category category $\mathcal{A}$, there is a K-theory spectrum $K(\mathcal{A}) \in \mathcal{B}$. Computationally, this is very mysterious, apart from the fact that it is always connective (ie $\pi_n K(\mathcal{A}) = 0$ for $n < 0$) and $\pi_0 K(\mathcal{A})$ is the group completion of the monoid of connected components in $\mathcal{A}$. Thomason has shown [23] that for every connective spectrum X there exists $\mathcal{A}$ with $K(\mathcal{A}) \simeq X$.
   - If $\mathcal{A}$ is the category of finite sets and isomorphisms, then $K(\mathcal{A}) = \Sigma^\infty S^0$.
   - Let G be a finite group, and let $\mathcal{A}$ be the category of finite G-sets and isomorphisms. Let $\mathcal{A}_f$ (resp $\mathcal{A}_t$) be the subcategory of free (resp. transitive) G-sets. Then $K(\mathcal{A}) = \Sigma^\infty B(\mathcal{A}_t)_+$, which can also be described as the wedge over the conjugacy classes of subgroups $H \leq G$ of the spectra $\Sigma^\infty BW_G H_+$, or as the fixed point spectrum of the G-equivariant sphere spectrum in the sense of Lewis-May-Steinberger [14]. On the other hand, $K(\mathcal{A}_f) = \Sigma^\infty BG_+$.
   - Work of Kathryn Lesh can be interpreted as exhibiting symmetric monoidal categories $\mathcal{M}_n$ (of "finite multisets with multiplicities at most n") whose K-theory is the n'th symmetric power of $\Sigma^\infty S^0$.
   - If $\mathcal{A}$ is the symmetric monoidal category with object set $\mathbb{N}$ and only identity morphisms, then $K(\mathcal{A}) = H\mathbb{Z}$.
   - One can set up a category $\mathcal{A}$, whose objects are smooth compact closed 1-manifolds, and whose morphisms are cobordisms between them. With the right choice of details, the K-theory spectrum $K(\mathcal{A})$ is then closely related to the classifying space of the stable mapping class group, and an important theorem of Madsen and Weiss can be interpreted as saying that $K(\mathcal{A})$ is the Thom spectrum of the negative of the tautological bundle over $\mathbb{C}P^\infty$, up to adjustment of π_{-2}.
   - If R is a commutative ring and $\mathcal{A}$ is the category of finitely generated projective R-modules, then $K(\mathcal{A})$ is the algebraic K-theory spectrum usually denoted by $K(R)$. Even in the case $R = \mathbb{Z}$, this contains a great deal of arithmetic information. By rather different methods one can construct spectra called $THH(R)$ and $TC(R)$ (topological Hochschild homology and topological cyclic homology) that approximate $K(R)$; there is an extensive literature on these approximations. The definitions can be set up in such a way that the spectra $K(R)$, $THH(R)$ and $TC(R)$ are all strictly commutative ring spectra.

(h) The above construction can be modified slightly to take account of a topology on the morphism sets of $\mathcal{A}$. We can then feed in the category of finite-dimensional complex vector spaces and isomorphisms (or a skeleton thereof) to get a spectrum known as kU, the connective complex K-theory spectrum, with a homotopy element $u \in \pi_2 kU$ such that $\pi_* kU = \mathbb{Z}[u]$. This has the property that for finite complexes

X, the group $kU^0X = [\Sigma^\infty X_+, kU]$ is the group completion of the monoid of isomorphism classes of complex vector bundles on X, or in other words the K-theory of X as defined by Grothendieck. By a colimit construction, one can build a periodized version called KU with $\pi_*KU = \mathbb{Z}[u, u^{-1}]$. There are direct constructions of kU and KU using Bott periodicity rather than symmetric monoidal categories. There are also constructions with a more analytic flavour, based on spaces of Fredholm operators and so on. Both kU and KU are strictly commutative ring spectra.

The infinite complex projective space $\mathbb{C}P^\infty$ is well-known to be a commutative group up to homotopy. Using this, one can make the spectrum $R = \Sigma^\infty(\mathbb{C}P^\infty)_+$ into a ring spectrum. The standard identification $\mathbb{C}P^1 = S^2$ gives rise to an element $v \in \pi_2(R)$, and we can use a homotopy colimit construction to invert this element, giving a new ring spectrum $R[v^{-1}]$. It is a theorem of Snaith that $R[v^{-1}]$ is homotopy equivalent to KU.

(i) Let F be a functor from based spaces to based spaces. Under mild conditions, we can use the homeomorphism $S^1 \wedge S^n \to S^{n+1}$ to get a map

$$S^1 \to \mathrm{Map}(S^n, S^{n+1}) \xrightarrow{F} \mathrm{Map}(FS^n, FS^{n+1}),$$

and thus an adjoint map $\Sigma FS^n \to FS^{n+1}$. This gives a sequence of spectra $\Sigma^{-n}\Sigma^\infty FS^n$, whose homotopy colimit (in a suitable sense) is denoted by D_1F. This is called the *linearization* or *first Goodwillie derivative* of F. Goodwillie [7–9] has set up a "calculus of functors" in which the higher derivatives are spectra D_nF with an action of Σ_n. (The slogan is that where the ordinary calculus of functions has a denominator of $n!$, the calculus of functors will take coinvariants under an action of Σ_n.) Even the derivatives of the identity functor are interesting; they fit in an intricate web of relationships with partition complexes, symmetric powers of the sphere spectrum, Steinberg modules and so on. There are other versions of calculus for functors from other categories to spaces, with applications to embeddings of manifolds, for example.

(j) A *Moore spectrum* is a spectrum X for which $\pi_n(X) = 0$ when $n < 0$ and $H_n(X) = 0$ when $n > 0$. Let $\mathcal{H}$ be the category of Moore spectra. The functor $H_0 \colon \mathcal{H} \to \mathrm{Ab}$ is then close to being an equivalence: for any $X, Y \in \mathcal{H}$ there is a natural short exact sequence

$$\mathrm{Ext}(H_0(X), H_0(Y))/2 \to [X, Y] \to \mathrm{Hom}(\pi_0(X), \pi_0(Y)).$$

Moreover, given any abelian group A there is a Moore spectrum SA (unique up to non-canoncal isomorphism) with $H_0(SA) \simeq A$.

(k) Let C be an elliptic curve over a ring k. (Number theorists are often interested in the case where k is a small ring like $\mathbb{Z}$, but it is also useful to consider larger rings like $\mathbb{Z}[\frac{1}{6}, c_4, c_6][(c_6^2 - c_4^3)^{-1}]$ that have various universal properties in the theory of elliptic curves.) From this we obtain a formal group $\widehat{C}$, which can be thought of as the part of C infinitesimally close to zero. It often happens that there is a spectrum E that corresponds to $\widehat{C}$ under the standard dictionary relating formal groups to cohomology theories. Spectra arising in this way are called *elliptic spectra*. The details are usually adjusted so that $\pi_*(E) = k[u, u^{-1}]$ with $k = \pi_0(E)$ and $u \in \pi_2(E)$. In many cases E can be constructed using the Landweber Exact Functor Theorem, as in (f).

The spectrum TMF (standing for *topological modular forms*) "wants to be" the universal example of an elliptic spectrum. It is not in fact an elliptic spectrum, but it is close to being one, and it admits a canonical map to every elliptic spectrum. If we let MF_* denote the group of integral modular forms as defined by number theorists (graded so that forms of weight k appear in degree $2k$) then we have

$$\pi_*(TMF)[\tfrac{1}{6}] = \mathbb{Z}[\tfrac{1}{6}, c_4, c_6][(c_6^2 - c_4^3)^{-1}] = MF_*[\tfrac{1}{6}].$$

The significance of the number 6 here is that 2 and 3 are the only primes that can divide $|\,\mathrm{Aut}(C)|$, for any elliptic curve C. If we do not invert 6 then the homotopy groups $\pi_*(TMF)$ are completely known, but different from MF_* and too complex to describe here.

There is a dense network of partially understood interactions between elliptic spectra, conformal field theories, vertex operator algebras, chiral differential operators and mathematical models of string theory. It seems likely that some central aspects of this picture remain to be discovered.

Under (g) we discussed the algebraic K-theory spectrum $K(R)$ associated to to a commutative ring R. The construction can be generalised to define $K(R)$ when R is a commutative ring spectrum; in particular, we can define $K(kU)$. Rognes and Ausoni have found evidence of a relationship between $K(kU)$ and TMF, but many features of this remain obscure.

(l) Let R be a strictly commutative ring spectrum, so for any space X we have a ring $R^0(X) = [\Sigma^\infty X_+, R]$, and a group of units $R^0(X)^\times$. It can be shown that there is a spectrum $gl_1(R)$ such that $R^0(X)^\times = [\Sigma^\infty X_+, gl_1(R)]$ for all X. By taking $X = S^n$ we see that $\pi_n(gl_1(R)) = \pi_n(R)$ for $n > 0$, but $H_*(gl_1(R))$ is not closely related to $H_*(R)$, and many aspects of the topology of $gl_1(R)$ are mysterious, even in the case $R = S^0$.

(m) The notion of Bousfield localisation provides an important way to construct new spectra from old. The simplest kinds of Bousfield localisations are the arithmetic ones: if X is a spectrum and p is a prime number then there are spectra and maps $X[\frac{1}{p}] \xleftarrow{i} X \xrightarrow{j} X_{(p)}$ such that i induces an isomorphism

$$\pi_*(X)[\tfrac{1}{p}] = \pi_*(X) \otimes \mathbb{Z}[\tfrac{1}{p}] \to \pi_*(X[\tfrac{1}{p}])$$

and j induces an isomorphism

$$\pi_*(X)_{(p)} = \pi_*(X) \otimes \mathbb{Z}_{(p)} \to \pi_*(X_{(p)}).$$

Similarly, there is a map $X \to X\mathbb{Q}$ inducing $\pi_*(X) \otimes \mathbb{Q} \simeq \pi_*(X\mathbb{Q})$. These properties characterise $X[\frac{1}{p}]$, $X_{(p)}$ and $X\mathbb{Q}$ up to canonical homotopy equivalence. Along similar lines, there is a p-adic completion map $X \to X_p^\wedge$. If each homotopy group $\pi_k(X)$ is finitely generated, then we just have $\pi_k(X_p^\wedge) = \pi_k(X)_p^\wedge = \pi_k(X) \otimes \mathbb{Z}_p$. In the infinitely generated case the picture is more complicated, but still well-understood.

Next, for any spectrum E there is a functor $L_E \colon \mathcal{B} \to \mathcal{B}$ and a natural map $i \colon X \to L_E X$ characterised as follows: the induced map $E_* i \colon E_* X \to E_* L_E X$ is an isomorphism, and if $f \colon X \to Y$ is such that $E_* f$ is an isomorphism, then there is a unique map $g \colon Y \to L_E X$ with $gf = i$. The spectrum $L_E X$ is called the *Bousfield localisation* of X with respect to E, and we say that X is *E-local* if the map $X \to L_E X$ is a homotopy equivalence. The slogan is that the category $\mathcal{B}_E$ of E-local spectra is the part of stable homotopy theory that is visible to E. Apart from the arithmetic completions and localisations, the most important cases are the *chromatic localisations* $L_{E(p,n)}$ and $L_{K(p,n)}$, which have been studied intensively [24], building on the Nilpotence Theorem and its consequences.

(n) The theory of surgery aims to understand compact smooth manifolds by cutting them into simpler pieces and reassembling the pieces. A key ingredient is as follows: if we have an n-dimensional manifold with an embedded copy of $S^{i-1} \times B^j$ (where $i+j = n+1$), then we can remove the interior to leave a manifold with boundary $S^{i-1} \times S^{j-1}$, then glue on a copy of $B^i \times S^{j-1}$ to obtain a new closed manifold M'. If the original copy of $S^{i-1} \times B^j$ is chosen appropriately, then the cohomology of M' will be smaller than that of M. By iterating this process, we hope to convert M to S^n. There are various obstructions to completing this process (and similar processes for related problems), and it turns out that these can be encoded as problems in stable homotopy theory. Cobordism spectra such as MSO play a role, as do the spectra kO and $gl_1(S^0)$. When the dimension n is even and M is oriented, the multiplication map $H^{n/2}(M) \otimes H^{n/2}(M) \to H^n(M) = \mathbb{Z}$ gives a bilinear form on $H^{n/2}(M)$, which is symmetric or antisymmetric depending on the parity of $n/2$. Because of this, it turns out that we need to consider a kind of K-theory of abelian groups equipped with a bilinear form. This is known as L-theory. There are various different versions, depending on details that we have skipped over. One of them gives a strictly commutative ring spectrum L with $\pi_*(L) = \mathbb{Z}[x,y]/(2x, x^2)$, where $|x| = 2$ and $|y| = 4$ (so for $k \geq 0$ we have $\pi_{4k}(L) = \mathbb{Z}.y^k$ and $\pi_{4k+2}(L) = (\mathbb{Z}/2).y^k x$). There is a canonical map $\sigma \colon MSO \to L$ of strictly commutative ring spectra. For an oriented manifold M of dimension $4k$, the cobordism class $[M]$ gives an element of $\pi_{4k}(MSO)$, so we must have $\sigma_*([M]) = d.y^k$ for some integer d. This integer is just the signature of the symmetric bilinear form on $H^{2k}(M; \mathbb{R})$.

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