Bott Periodicity and Hopf Rings

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Contents

Abstract

In this thesis I compute the Hopf rings in mod 2 and torsion-quotient homology of the spectra representing orthogonal, unitary, symplectic and self-conjugate K-theory. This gives a systematic way to understand the homology of all the Bott periodicity spaces and the effect in homology of the various maps between them. I make extensive use of formal power series methods. The submitted version contained an appendix listing programs which teach *Mathematica* some of the results of this thesis, but I have removed these as it makes more sense to email them to interested readers.

Declaration

No portion of this work has been submitted in support of an application for another degree or qualification of this or any other University or other institute of learning.

The Author

I received a first class honours degree (B.A.) in Mathematics from Cambridge University in 1988. I then completed part III of the Mathematical Tripos, leading in 1989 to a Certificate of Advanced Studies in Mathematics with distinction, still at Trinity College Cambridge. I started studying for a PhD in Mathematics at the University of Manchester in October 1989.

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Introduction

One of the oldest problems in algebraic topology is that of understanding the homology of the infinite classical groups and their coset spaces. The structure of the individual homology rings has long been known, but more remains to be said. One would like to understand the Hopf algebra structure of the homology and the action of the Steenrod and Kudo-Araki operations. One also needs to know the effect of various maps between the spaces, and the behaviour of spectral sequences arising from various fibrations. Ideally one would like to do this in a systematic and efficient way, giving canonical families of generators for the homology rings which are well related to each other via the maps in question. In this thesis, we achieve this with the help of two new tools: Hopf rings and formal power series methods.

It is well known that many of the spaces in question can be assembled into Ω-spectra, often equipped with a ring structure. The homology algebras of the spaces in such a spectrum fit together to form an algebraic object called a Hopf ring. Hopf rings have many different operations, so a small number of elements can generate a large structure and a short list of relations can have wide consequences. We shall find that the Hopf rings under consideration are all generated by a small number of generators modulo a small set of relations. The operations and maps which we are called upon to understand connect well with the Hopf ring structure, making the answers strikingly simple. This is reminiscent of the Hopf ring calculations of Miller and Ravenel in the complex oriented case – we discuss this further in section 2.1.

It has become clear that the most efficient and elegant way to state and manipulate the structural formulae in a large class of algebraic objects encountered in topology is to use the language of formal power series – see [**4**][**18**][**7**] for example. These methods are used extensively in this thesis, leading to many simplifications. I include a number of restatements of standard results in this language, which appear not to have been noticed before.

The structure which we shall reveal is rather beautiful, with many subtle and intricate effects conspiring in unexpected ways to give a consistent result.

Another important feature of this thesis is the use of the symbolic mathematics language *Mathematica*. I have written (and am happy to distribute by email) a *Mathematica* program which implements many of the results proved. After running this program one can use *Mathematica* to do computations in the Hopf rings under consideration. One can simply type in expressions in a notation similar to the usual one, and the program will attempt to evaluate them. I have used this to check various tables etc. in the thesis for consistency. Much of the code works for Hopf rings in general, rather than just the ones I compute.

The structure of this thesis is somewhat non-standard. It is designed to be a useful reference. It seemed to me that this was most easily achieved by splitting the thesis into three parts : first the results, then development of machinery, then proofs. Chapter 1 is mainly standard material about K-theory spectra, their homotopy groups and associated infinite loop spaces, and about maps between such spectra. We shall not give detailed proofs, but some of the background machinery is developed in section 6 and indications of proofs are given in chapter 7. In chapters 2 and 3 we set out various results about the mod 2 and torsion-quotient Hopf rings associated to K-theory spectra, and related matters. These we prove in detail, in sections 7.3 to 7.10, which broadly parallel 2.1 to 3.5. In addition, in sections 3.1 and 3.2 we develop a little general theory of homology mod torsion.

In chapter 4 we recall some standard facts about Hopf algebras and Hopf rings, and how they arise in topology. Next, in chapter 5, we build up the arsenal of spectral sequences which we need to prove the claims in earlier chapters. Some facts about the Serre spectral sequence with twisted coefficients appear to be new. I include an analysis of how this works in some typical cases involving K-theory spectra.

In chapter 6 we recall some ideas about Clifford modules and Atiyah's Real K-theory. Chapter 7 contains partial proofs for chapter 1 based on these ideas. It also contains full proofs for chapters 2 and 3. The input to these full proofs is the material in chapter 1 together with the machinery in chapters 4 and 5, a result on Kudo-Araki operations due to Priddy [**19**] and a few explicit geometrical constructions.

Notes on Sources and Originality

The study of the Bott periodicity spaces is well-trodden ground. As a consequence, this thesis contains a complex mixture of new results, old results presented in new ways, and standard exposition. These notes are an attempt to unravel this. They are placed at the front to avoid hiding anything, but they are probably best read after the main body of the thesis.

While there is little that is original in chapter 1, I know of no source in which all these related facts are collected together. The most useful references are [**3**] and [**2**]. The commutative diagram 1.3.1 draws on chapter 3 of [**1**]. The cofibre diagrams 1.3.2 and 1.3.3 are an elaboration of material in [**2**] and [**23**], perhaps somewhat disguised. I have not seen octahedral diagrams drawn in this way before; it seems to me to have much to recommend it. The entries in table 1.4.1 come from a variety of places. Most can be proven with less effort than it takes to find a source, particularly if one knows the descriptions of the relevant bundles in terms of Clifford modules [**3**]. The best reference for section 1.5 is [**8**]. Table 1.5.9 contains some folklore, none of it hard to check.

Chapter 2 contains more that is new. Most of what was previously known can be extracted (with some labour) from [**8**]. Other sources include [**25**] and [**24**]. For most of the spaces considered, the mod 2 homology is well known as a ring, often even as a Hopf algebra. However, the generators are not easy to relate to each other and the circle product is hard to understand from the classical description. In the self-conjugate case, the only information previously available [**23**] was the cohomology of the zeroth space $\mathbb{Z}\times BT$. Thus, a number of aspects of the Hopf ring structure are original, as is the whole programme of finding a minimal set of Hopf ring generators and relations. In order to show that the elements described by Hopf ring methods are generators, it appears to be necessary to recalculate most of the groups involved by Hopf ring methods. Some simplification could be achieved by appealing to facts already known. However, to import this information one would have to examine the structure maps of the various spectra and the unstable components of various stable maps in much greater detail than otherwise necessary. There would thus be little, if any, net saving. The Hopf ring proofs given in chapter 7 are mostly original. The formal power series equations (2.8.1), (2.8.10), (2.8.16), (2.8.17) and (2.8.18) are also new, although equivalent formulae involving binomial coefficients etc. are well known. The Bockstein homology calculations for $\mathrm{H}\mathbb{F}_* \underline{KO}_*$ stated in section 2.9 are extensions of those given in [**8**]. Those for $\mathrm{H}\mathbb{F}_* \underline{KT}_*$ are new.

It would surprise me if the theory in section 3.1 could not be found somewhere in the algebra literature. I found it independently and have not looked very hard for sources. The theory in section 3.2 is new. The description of the integral Hopf ring for KU in section 3.3, and the fact that it is freely generated by $\mathrm{H}_*\mathbb{C}P^\infty$ modulo formal group relations, is folklore. It is easy to deduce from the classical description given in (for example) [**24**], but I believe it has not been written down before. In the case of KO, rather less was previously known. The homology of all the spaces had been computed with coefficients $\mathbb{Z}[\frac{1}{2}]$, but in most cases the subring $\overline{\mathrm{H}}_*(X)$ had not. The Hopf ring structure is all original, as is everything in section 3.5 about the self conjugate case.

Chapter 4 is almost all standard – [**17**] is a good reference. The Penrose diagrams for Hopf ring identities are I think new in print, although Joyal has certainly drawn them before.

Chapter 5 is not quite so standard. The Bockstein spectral sequence in section 5.1 is folklore, but I do not know a good reference. The spectral sequence of the same name considered in [**6**] for example, is somewhat different. Section 5.2 about the Rothenberg-Steenrod spectral sequence is all standard [**21**][**26**]. Similarly, the discussion of the Eilenberg-Moore sequence in section 5.3 is based on [**16**]. There are any number of sources for the basic theory of the Serre spectral sequence, but the analysis of the local coefficients in the case of an infinite loop map (section 5.4) is new. The analysis of specific examples in section 5.5 is also original.

Chapter 6 is essentially extracted from [**3**] and [**2**]. Some signs, choices of generators etc. have been changed. This is part of a programme I have to tidy up the theory and make the conventions more transparent and consistent. I have written *Mathematica* code to implement the various Clifford algebras etc. involved, but this is still in progress.

The *Mathematica* programs referred to above are all new.

Notation

For convenience, we collect in this section some remarks about notation used elsewhere in this thesis.

Miscellaneous Rings etc.:

$$\begin{aligned} \mathbb{F} &= \mathbb{Z}/(2) \\ \mathbb{N} &= \text{Natural numbers } \ni 0 \\ \mathbb{H} &= \text{Quaternions} \\ \mathbb{O} &= \text{Octonions} \\ \mathbb{K}, \mathbb{L} & \quad \text{variables taking the value } \mathbb{R}, \mathbb{C}, \mathbb{H} \text{ or } T \end{aligned}$$

(see section 1.1)

Types of Homology:

$$\begin{aligned} \mathrm{H}\mathbb{F}_*(X) &= \mathrm{H}_*(X;\mathbb{F}) \\ \overline{\mathrm{H}}_*(X) &= \mathrm{H}_*(X;\mathbb{Z})/\text{torsion} \\ \tilde{\mathrm{H}}_*(X) &= \text{reduced integral homology of } X \\ \mathrm{H}[\tfrac{1}{2}]_*(X) &= \mathrm{H}_*(X;\mathbb{Z}[\tfrac{1}{2}]) \end{aligned}$$

Building Blocks for Hopf Algebras:

$$\begin{aligned} P[x] &= \text{Polynomial algebra on } x \\ E[x] &= \text{Exterior algebra on } x \\ D[x] &= \text{Divided power algebra on } x \\ k[G] &= \text{Group algebra of a group } G \text{ over } k \\ k\{a_0, \dots a_n\} &= \text{Free } k\text{-module on } \{a_0, \dots a_n\} \end{aligned}$$

The base ring is to be understood from the context. Where we have an infinite family $\{x_k\}$ of variables, we almost always index them so that x_k lies in dimension k. In an expression like $P[x_{2k}]$ or $E[y_{4k+5}]$ the index k is supposed to be interpreted as a free variable running over all nonnegative integers, so we have an algebra with infinitely many generators. There are a few exceptions to this, which should be clear in context.

Infinite Loop Spaces:

$$\begin{aligned} \underline{E}_k &= k\text{'th space in the } \Omega\text{-spectrum } E \\ \underline{E}'_k &= \text{base component in } \underline{E}_k \end{aligned}$$

(sections 1.5,4.3).

Structure Maps for Hopf Rings:

$$\begin{aligned} \epsilon &= \text{augmentation} \\ \psi &= \text{coproduct} \\ \eta &= \text{unit map for } *\text{-product } (\eta_k(1) = [0_k] = 1_k) \\ \sigma &= *\text{-product } (ab = a * b = \sigma(a \otimes b)) \\ \chi &= \text{antipode} \\ \theta &= \text{unit map for } \circ\text{-product } (\theta(1_0) = [1]) \\ \mu &= \circ\text{-product } (a \circ b = \mu(a \otimes b)) \end{aligned}$$

(see chapter 4)

Elements of Hopf Rings

$$\begin{aligned} e &= \text{suspension class} \\ [a] &= \text{element of } \mathrm{H}_0\, \underline{E}_k = \mathbb{Z}[\pi_{-k}E] \text{ corresponding to } a \in \pi_{-k}E \end{aligned}$$

(see chapter 4)

The following homotopy elements are defined in section 1.1:

$$\alpha \;\; \beta \;\; \gamma \;\; \theta \;\; \lambda \;\; \mu \;\; \nu$$

Finally, we list various series and the sections in which they are defined.

$$\begin{array}{rl} z(t), \overline{z}(t) & 2.1 \\ \overline{z}'(t), p(t_1, t_2, \dots), q(t) & 2.6 \\ y(t) & 3.3 \\ {}_k y(t), {}_k\hat{y}(t), {}_k\check{y}(t) & 3.4 \\ {}_k x(t), {}_k\hat{x}(t), {}_k\check{x}(t) & 3.5 \end{array}$$

We also use series called $z_{\mathbb{H}}(t), \overline{z}_{\mathbb{C}}(t)$ and so on; the subscript is often omitted and assumed to be clear from the context. The various ring and module structures of the K-theory spectra behave in such a way as to make this harmless. The formal derivative of $z(t)$ will sometimes be written $\dot{z}(t)$; dashes (e.g. $\overline{z}'(t)$) are used for other things.

CHAPTER 1

K-Theory Spectra

In this chapter we describe the periodic K-theory spectra and various maps between them. We also describe their homotopy rings and the action of the maps on them. Most of the facts stated in this chapter (in particular, the truth of Bott periodicity) will be assumed as background. Some remarks will be made in sections 6.1 to 7.2 about how they are proved.

1.1. Spectra

We consider the spectra KU,KT,KO and KSp representing unitary, self conjugate, orthogonal and symplectic K-theory respectively. The symplectic spectrum KSp is equivalent to $\Sigma^4 KO$ but it will be convenient to consider it in its own right.

We shall let $\mathbb{K}$ denote any of the division algebras $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. We write $K_{\mathbb{K}}$ for the associated K-theory and $O_{\mathbb{K}}$ for the associated linear isometry group. We also write $K_T = KT$. Thus, if X is a finite dimensional CW-complex, then $K^0_{\mathbb{K}} X$ is the Grothendieck group of the category of (left) $\mathbb{K}$-vector bundles over X of locally constant rank. The ring $KT^0 X$ can also be described as the Grothendieck group of a category. The objects are pairs (χ, f) where χ is a complex bundle over X and f is a "self-conjugacy" of χ, i.e. a conjugate linear automorphism. The maps from (χ_0, f_0) to (χ_1, f_1) are the complex linear maps $g\colon \chi_0 \longrightarrow \chi_1$ such that f_0 is homotopic to $g^{-1} f_1 g$ through self-conjugacies.

We define a partial multiplication on $\{\mathbb{R}, \mathbb{C}, \mathbb{H}, T\}$ by

$$\mathbb{R}\mathbb{K} = \mathbb{K} = \mathbb{K}\mathbb{R} \tag{1.1.1}$$

$$\mathbb{R}T = T = T\mathbb{R} \tag{1.1.2}$$

$$\mathbb{C}\mathbb{C} = \mathbb{C} \tag{1.1.3}$$

$$TT = T \tag{1.1.4}$$

$$\mathbb{H}\mathbb{H} = \mathbb{R} \tag{1.1.5}$$

There are natural pairings $K_{\mathbb{K}} \wedge K_{\mathbb{L}} \longrightarrow K_{\mathbb{K}\mathbb{L}}$ corresponding to the various kinds of tensor product of bundles. This makes KU , KT and KO into ring spectra, and all four spectra into KO-modules. Also, the various pairings involving KO and KSp fit together to make $KO \oplus KSp$ into a ring spectrum; it classifies complex bundles with a conjugate linear automorphism f such that $f^4 = 1$. We shall write $d_{\mathbb{K}} = \dim_{\mathbb{R}} \mathbb{K}$.

We can also interpret these spectra in terms of Atiyah's "Real K-Theory" (section 6.2). This makes it easier to construct and understand maps and homotopy elements in a unified way, but obscures the connection with the classical groups.

The homotopy groups are:

$$\begin{array}{llll}
KO_* = \mathbb{Z}[\alpha, \beta, \lambda^{\pm 1}]/(\alpha^3, 2\alpha, \alpha\beta, \beta^2 - 4\lambda) & |\alpha| = 1 & |\beta| = 4 & |\lambda| = 8 \\
KT_* = \mathbb{Z}[\alpha, \gamma, \mu^{\pm 1}]/(\alpha^2, 2\alpha, \alpha\gamma, \gamma^2) & |\alpha| = 1 & |\gamma| = 3 & |\mu| = 4 \\
KU_* = \mathbb{Z}[\nu^{\pm 1}] & |\nu| = 2 & & \\
KSp_* = KO_*\theta & |\theta| = 4 & &
\end{array} \tag{1.1.6}$$

The KO-bilinear pairing $KSp \wedge KSp \longrightarrow KO$ is given in homotopy by $\theta^2 = \lambda$.

The elements α,ν,θ and λ correspond to the reduced canonical line bundles over the projective lines $\mathbb{R}P^1 = S^1$, $\mathbb{C}P^1 = S^2$, $\mathbb{H}P^1 = S^4$ and $\mathbb{O}P^1 = S^8$. Here $\mathbb{H}$ and $\mathbb{O}$ denote the quaternions and octonions respectively.

The element β is just the underlying real bundle of θ, and μ is just θ with the self-conjugacy $u \mapsto ju$. To construct γ we take the trivial bundle

$$S^3 \times \mathbb{H} \to S^3 = Sp(1) \tag{1.1.7}$$

and give it the self-conjugacy

$$(a, z + wj) \mapsto (a, (\overline{z} + \overline{w}j)a) \qquad\qquad (z, w \in \mathbb{C}) \tag{1.1.8}$$

For more information about these generators and relations, and for indications of the proof of their completeness, see sections 6.1 to 7.2. I do not guarantee the correctness of signs.

1.2. Maps

We list below a number of maps between our spectra. The descriptions given are in fact descriptions of continuous functors on categories of bundles, which can be converted to stable maps by a suitable machine; we pass over such matters. We use ξ, χ, ζ and η for typical unitary, self conjugate, orthogonal and symplectic bundles respectively.

$$\begin{array}{llll}
c\colon & KU \longrightarrow KU & \xi \mapsto \xi & \mathbb{C}\text{-structure twisted by conjugation} \\
l_U\colon & KU \longrightarrow KSp & \xi \mapsto \mathbb{H} \otimes_{\mathbb{C}} \xi & \\
l_O\colon & KO \longrightarrow KT & \zeta \mapsto \mathbb{C} \otimes_{\mathbb{R}} \zeta & \text{with self-conjugacy } z \otimes x \mapsto \overline{z} \otimes x \\
m_U\colon & KU \longrightarrow KT & \xi \mapsto \xi \oplus c\xi & \text{with self-conjugacy } (x, y) \mapsto (y, x) \\
m_T\colon & KT \longrightarrow KSp & \chi \mapsto \mathbb{H} \otimes_{\mathbb{C}} \chi & \\
m_O\colon & KO \longrightarrow KU & \zeta \mapsto \mathbb{C} \otimes_{\mathbb{R}} \zeta & \\
n_O\colon & KO \longrightarrow KSp & \zeta \mapsto \mathbb{H} \otimes_{\mathbb{R}} \zeta & \\
f_U\colon & KU \longrightarrow KO & \xi \mapsto \xi & \text{considered as an } \mathbb{R}\text{-bundle} \\
f_T\colon & KT \longrightarrow KU & \chi \mapsto \chi & \text{with self-conjugacy forgotten} \\
f_{Sp}\colon & KSp \longrightarrow KT & \eta \mapsto \eta & \text{with self-conjugacy } x \mapsto jx \\
g_T\colon & KT \longrightarrow KO & \chi \mapsto \chi & \text{considered as an } \mathbb{R}\text{-bundle} \\
g_{Sp}\colon & KSp \longrightarrow KU & \eta \mapsto \eta & \text{considered as a } \mathbb{C}\text{-bundle} \\
h_{Sp}\colon & KSp \longrightarrow KO & \eta \mapsto \eta & \text{considered as an } \mathbb{R}\text{-bundle}
\end{array} \tag{1.2.1}$$

The scheme is that maps called f forget one piece of structure; maps called g or h are composites of two or three f's. Maps called l freely add one piece of structure. Maps called m are composites of an f and an l.

The maps c, l_O, f_T and m_O are ring maps. All maps are KO-module maps.

Given a ring spectrum E, an E-module spectrum F, and a homotopy element $x \in \pi_n E$, we have an obvious map ("multiplication by x"):

$$\Sigma^n F \xrightarrow{x \wedge 1} E \wedge F \longrightarrow F \tag{1.2.2}$$

We shall write either m_x or just x for this map.

1.3. Diagrams

In this section we exhibit various commutative and exact diagrams. For justification, see section 7.1. The maps described above fit into the following commutative diagram:

$$
\begin{tikzcd}
KU \arrow[r, "l_U"] \arrow[d, "f_U"'] \arrow[dr, "m_U"] \arrow[dd, bend right, "1+c"'] & KSp \arrow[d, "f_{Sp}"] \arrow[ddl, "g_{Sp}"'] \arrow[dddl, "h_{Sp}"] \arrow[dd, bend left, "2"] \\
KO \arrow[r, "l_O"] \arrow[d, "m_O"'] \arrow[dr, "n_O"] \arrow[dd, bend right, "2"'] & KT \arrow[d, "m_T"] \arrow[dl, "f_T"'] \arrow[ddl, "g_T"] \\
KU \arrow[r, "l_U"'] \arrow[d, "f_U"'] \arrow[dr, "m_U"] & KSp \arrow[d, "f_{Sp}"] \\
KO \arrow[r, "l_O"'] & KT
\end{tikzcd}
\tag{1.3.1}
$$

These maps also fit into a number of cofibrations. In the next two diagrams, $f\colon A \multimap\!\!\to B$ means $f\colon A \to \Sigma B$. The diagrams are flattened out octahedra, as in Verdier's octahedral axiom [**10**, section 1.1]. All triangles in which the arrows circulate are cofibre triangles. All other parts of the diagrams commute.

$$
\begin{tikzcd}
 & & KT \arrow[dl, "\delta_O"'] \arrow[ddrr, bend left, "f_T"] & & \\
 & \Sigma^2 KO \arrow[rr, "\alpha^2"] \arrow[dr, "\Sigma\alpha"'] & & KO \arrow[ul, "l_O"'] \arrow[dr, "m_O"] & \\
\Sigma KU \arrow[ur, "\Sigma f_U \nu^{-1}"] \arrow[uurr, bend left, "\delta_T"] & & \Sigma KO \arrow[ll, "\Sigma m_O"] \arrow[ur, "\alpha"'] & & KU \arrow[ll, "f_U\ \nu^{-1}"] \arrow[llll, bend left, "\nu^{-1}(1-c)"]
\end{tikzcd}
\tag{1.3.2}
$$

$$
\begin{tikzcd}
 & & KT \arrow[dl, "\delta_{Sp}"'] \arrow[ddrr, bend left, "f_T"] & & \\
 & \Sigma^2 KSp \arrow[rr, "\alpha^2"] \arrow[dr, "\Sigma\alpha"'] & & KSp \arrow[ul, "f_{Sp}"'] \arrow[dr, "g_{Sp}"] & \\
\Sigma KU \arrow[ur, "\Sigma l_U \nu^{-1}"] \arrow[uurr, bend left, "\delta_T"] & & \Sigma KSp \arrow[ll, "\Sigma g_{Sp}"] \arrow[ur, "\alpha"'] & & KU \arrow[ll, "l_U\ \nu^{-1}"] \arrow[llll, bend left, "\nu^{-1}(1-c)"]
\end{tikzcd}
\tag{1.3.3}
$$

There is an isomorphism of degree 4 from the first of these diagrams to the second, given by

$$\begin{array}{lllll} \theta\colon & \Sigma^4 KO & \to & KSp & \\ \mu\colon & \Sigma^4 KT & \to & KT & \\ \nu^2\colon & \Sigma^4 KU & \to & KU & \end{array} \tag{1.3.4}$$

Note also that these diagrams involve three new boundary maps

$$\begin{array}{llll} \delta_T\colon & \Sigma KU & \to & KT \\ \delta_O\colon & KT & \to & \Sigma^3 KO \\ \delta_{Sp}\colon & KT & \to & \Sigma^3 KSp \end{array} \tag{1.3.5}$$

These are most easily defined as boundary maps of exact sequences for various pairs in Atiyah's Real K-Theory [**2**]. They are again KO-linear, and δ_T is even KT-linear.

1.4. Action of Maps on Homotopy Groups

The next table shows the effect of these maps in homotopy. Information not given explicitly can be deduced from ring and module structures mentioned in section 1.2.

$$\begin{array}{llllll}
c\colon & KU \to KU & \nu\mapsto -\nu & & & \\
l_U\colon & KU \to KSp & 1\mapsto \lambda^{-1}\beta\theta & \nu\mapsto 0 & \nu^2\mapsto 2\theta & \nu^3\mapsto \alpha^2\theta \\
l_O\colon & KO \to KT & \alpha\mapsto\alpha & \beta\mapsto 2\mu & \lambda\mapsto\mu^2 & \\
m_U\colon & KU \to KT & 1\mapsto 2 & \nu\mapsto 0 & \nu^2\mapsto 2\mu & \nu^3\mapsto 0 \\
m_T\colon & KT \to KSp & 1\mapsto \lambda^{-1}\beta\theta & \mu\mapsto 2\theta & \alpha,\gamma,\mu\alpha,\mu\gamma\mapsto 0 & \\
m_O\colon & KO \to KU & \alpha\mapsto 0 & \beta\mapsto 2\nu^2 & \lambda\mapsto\nu^4 & \\
n_O\colon & KO \to KSp & 1\mapsto \lambda^{-1}\beta\theta & \alpha\mapsto 0 & \beta\mapsto 4\theta & \\
f_U\colon & KU \to KO & 1\mapsto 2 & \nu\mapsto\alpha^2 & \nu^2\mapsto\beta & \nu^3\mapsto 0 \\
f_T\colon & KT \to KU & \alpha\mapsto 0 & \gamma\mapsto 0 & \mu\mapsto\nu^2 & \\
f_{Sp}\colon & KSp \to KT & \theta\mapsto\mu & & & \\
g_T\colon & KT \to KO & 1\mapsto 2 & \mu\mapsto\beta & \alpha,\gamma,\mu\alpha,\mu\gamma\mapsto 0 & \\
g_{Sp}\colon & KSp \to KU & \theta\mapsto\nu^2 & & & \\
h_{Sp}\colon & KSp \to KO & \theta\mapsto\beta & & & \\
\delta_T\colon & \Sigma KU \to KT & 1\mapsto\alpha & \nu\mapsto\gamma & \nu^2\mapsto\mu\alpha & \nu^3\mapsto\mu\gamma \\
\delta_O\colon & KT \to \Sigma^3 KO & 1\mapsto 0 & \gamma\mapsto 2 & \mu\mapsto\alpha & \mu\gamma\mapsto\beta \\
\delta_{Sp}\colon & KT \to \Sigma^3 KSp & 1\mapsto \lambda^{-1}\alpha\theta & \gamma\mapsto\lambda^{-1}\beta\theta & \mu\mapsto 0 & \mu\gamma\mapsto 2\theta
\end{array} \tag{1.4.1}$$

I have converted the above table into a form usable by the symbolic mathematics program *Mathematica* and used this to check that the table is consistent with the commutativity and exactness properties of the diagrams discussed in section 1.3.

1.5. Spaces

In this section we recall the classical description of the spaces in the Ω-spectra for our various K-theories. For any spectrum E we have infinite loop spaces $\underline{E}_n$ classifying the associated cohomology theory:

$$E^n(X) = [X,\ \underline{E}_n] \tag{1.5.1}$$

This equation is supposed to refer to the unreduced E-cohomology of a space X, and homotopy classes of unbased maps $X \to \underline{E}_n$. We write $\underline{E}'_n$ for the base component in $\underline{E}_n$. If E is one of our K-theory spectra, then the spaces $\underline{E}_n$ are periodic: $\underline{E}_{n+d} = \underline{E}_n$ where d is 2, 4, 8 or 8 as E is KU, KT, KO or KSp.

Also $\underline{KSp}_n = \underline{KO}_{n+4}$. As part of the data of any spectrum E, there are specified homotopy equivalences

$$\Omega\, \underline{E}_n = \Omega\, \underline{E}'_n \simeq \underline{E}_{n-1} \tag{1.5.2}$$

The spaces are listed in the following diagram:

$$\begin{tikzcd}
 & & \mathbb{Z}\times BSp \arrow[dl] & & \\
 & Sp \arrow[dl] & & U/Sp \arrow[ul] & \\
Sp/U \arrow[dr] & & & & O/U \arrow[ul] \\
 & U/O \arrow[dr] & & O \arrow[ur, "\Omega"] & \\
 & & \mathbb{Z}\times BO \arrow[ur] & &
\end{tikzcd}$$

$$\begin{tikzcd}
\Omega T \arrow[d] & T \arrow[l] \\
\Omega^2 T \arrow[r] & \mathbb{Z}\times BT \arrow[u, "\Omega"']
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[d, bend right=60] \\
\mathbb{Z}\times BU \arrow[u, bend right=60, "\Omega"']
\end{tikzcd} \tag{1.5.3}$$

Thus,

$$\begin{array}{lllllll}
\underline{KO}_0 & = & \underline{KO}_8 & = & \underline{KSp}_4 & = & \mathbb{Z}\times BO \\
\underline{KO}_{-1} & = & \underline{KO}_7 & = & \underline{KSp}_3 & = & O \\
\underline{KO}_{-2} & = & \underline{KO}_6 & = & \underline{KSp}_2 & = & O/U
\end{array} \tag{1.5.4}$$

and so on. The diagram for KT defines our notation rather than giving any real information. The following table (from [**23**]) might be considered more illuminating:

$$BT = (BO \times BSp)/BU \tag{1.5.5}$$
$$T = (O \times Sp)/U \tag{1.5.6}$$
$$\Omega T = U/(U \otimes \mathbb{C}) \tag{1.5.7}$$
$$\Omega^2 T = U/(O \times Sp) \tag{1.5.8}$$

If E is KO, KU, or KSp then the equivalences $\Omega\, \underline{E}_n \simeq \underline{E}_{n-1}$ are given explicitly in [**8**], for example.

The various stable maps which we consider induce maps of the spaces in the relevant Ω-spectra. If one obtained a map $U \to Sp$ (for example) in this way, one would probably guess that it was the symplectification map. Such guesses are almost always correct, although verification can be tedious. We shall usually be able to avoid worrying about such things.

For most of our spaces, the first few connective covers can also be described classically. In the following table, each row consists of successive connective covers of the first entry. Thus, the second column gives the base components, the third gives the universal covers of the base components, and so on.

$$\begin{array}{lllllllll}
\mathbb{Z}\times BO & \longleftarrow & BO & \longleftarrow & BSO & \longleftarrow & BSpin & \longleftarrow & BSpin \\
U/O & \longleftarrow & U/O & \longleftarrow & SU/SO & \longleftarrow & SU/Spin & & \\
Sp/U & \longleftarrow & Sp/U & \longleftarrow & Sp/U & & & & \\
Sp & \longleftarrow & Sp & \longleftarrow & Sp & \longleftarrow & Sp & & \\
\mathbb{Z}\times BSp & \longleftarrow & BSp & \longleftarrow & BSp & \longleftarrow & BSp & \longleftarrow & BSp \\
U/Sp & \longleftarrow & U/Sp & \longleftarrow & SU/Sp & & & & \\
O/U & \longleftarrow & SO/U & \longleftarrow & SO/U & \longleftarrow & Spin/SU & & \\
O & \longleftarrow & SO & \longleftarrow & Spin & \longleftarrow & Spin & & \\
\mathbb{Z}\times BO & \longleftarrow & BO & \longleftarrow & BSO & \longleftarrow & BSpin & \longleftarrow & BSpin \\
 & & & & & & & & \\
\mathbb{Z}\times BU & \longleftarrow & BU & \longleftarrow & BU & \longleftarrow & BSU & \longleftarrow & BSU \\
U & \longleftarrow & U & \longleftarrow & SU & \longleftarrow & SU & & \\
\mathbb{Z}\times BU & \longleftarrow & BU & \longleftarrow & BU & \longleftarrow & BSU & \longleftarrow & BSU
\end{array} \tag{1.5.9}$$

CHAPTER 2

Mod 2 Hopf Rings for K-Theories

In this chapter we describe the structure of various Hopf rings related to K-theories. The general theory of Hopf algebras and Hopf rings is discussed in chapter 4, as is the Hopf ring associated to a ring Ω-spectrum. Proofs will be given in chapter 7.

2.1. The System of Mod 2 Hopf Rings

We write $\mathbb{F}$ for the field $\mathbb{Z}/(2)$ and $\mathrm{H}\mathbb{F}_*X$ for the homology of X with coefficients in $\mathbb{F}$.

The homology of the various projective spaces is given by:

$$\mathrm{H}\mathbb{F}_*\mathbb{K}P^\infty = \mathbb{F}\{z_{\mathbb{K},ld} \mid l \geq 0\} \qquad d = \dim_{\mathbb{R}} \mathbb{K} \tag{2.1.1}$$

There is a map $\mathbb{K}P^\infty \longrightarrow 1 \times BO_{\mathbb{K}} \subset \mathbb{Z} \times BO_{\mathbb{K}}$ which classifies the unreduced canonical line bundle. It induces an embedding in homology, and we will not distinguish notationally between $z_{\mathbb{K},k}$ and its image. Note that $z_{\mathbb{K},0}$ is just the basis element of $\mathrm{H}\mathbb{F}_0(\mathbb{Z} \times BO_{\mathbb{K}}) = \mathbb{F}[\pi_0(\mathbb{Z} \times BO_{\mathbb{K}})]$ corresponding to the component $1 \times BO_{\mathbb{K}}$ of $\mathbb{Z} \times BO_{\mathbb{K}}$. We will also write [1] for this element – compare section 4.3.

It will be convenient to define $z_{\mathbb{K},l} = 0$ if l is not divisible by d, and also to consider the formal power series

$$z_{\mathbb{K}}(t) = \sum_{l \geq 0} z_{\mathbb{K},dl} t^l \tag{2.1.2}$$

The scalar extension maps

$$\mathbb{R}P^\infty \xrightarrow{m_O} \mathbb{C}P^\infty \xrightarrow{l_U} \mathbb{H}P^\infty \tag{2.1.3}$$

induce

$$z_{\mathbb{R}}(t) \mapsto z_{\mathbb{C}}(t^2) \mapsto z_{\mathbb{H}}(t^4) \tag{2.1.4}$$

We also define $z_{T,k} = l_O(z_{\mathbb{R},k})$, and $\overline{z}(t) = z(t)/[1]$ for any of our series z.

Let E be one of the spectra KO, KU and KT, and take $d = 1, 2$ or 4 and $z = z_{\mathbb{R}}, z_{\mathbb{C}}$ or z_T as appropriate. Then the Hopf ring $\mathrm{H}\mathbb{F}_* \, \underline{E}_*$ contains a subcoalgebra

$$\begin{aligned} C_{**} &= \mathbb{F}\{z_{dk} \mid k \geq 0\} \oplus \mathbb{F}\{1_1, e\} && (2.1.5)\\ &\subseteq \mathrm{H}\mathbb{F}_* \, \underline{E}_0 \oplus \mathrm{H}\mathbb{F}_0 \, \underline{E}_1 \oplus \mathrm{H}\mathbb{F}_1 \, \underline{E}_1 && (2.1.6)\\ \psi z(t) &= z(t) \otimes z(t) && (2.1.7)\\ \psi e &= e \otimes 1 + 1 \otimes e && (2.1.8) \end{aligned}$$

Here e denotes the usual fundamental class in $\mathrm{H}\mathbb{F}_1 \, \underline{E}_1$, and 1_1 is the unit for the star product in $\mathrm{H}\mathbb{F}_0 \, \underline{E}_1$.

We also have a sub-Hopf ring

$$\mathbb{F}[E_*] = \mathrm{H}\mathbb{F}_0 \, \underline{E}_* \subseteq \mathrm{H}\mathbb{F}_* \, \underline{E}_* \tag{2.1.9}$$

It will transpire that in each case $\mathrm{H}\mathbb{F}_* \, \underline{E}_*$ is the Hopf ring over $\mathbb{F}[E_*]$ generated by the subcoalgebra C_{**} modulo a list of relations which will be given in the next three sections. By this I mean that we have the following universal property: the set of Hopf ring maps $\phi\colon \mathrm{H}\mathbb{F}_* \, \underline{KO}_* \to A_{**}$ bijects in the obvious way with the set of pairs (ϕ_0, ϕ_1) where ϕ_0 is a Hopf ring map $\mathbb{F}[E_*] \to A_{**}$ and ϕ_1 is a coalgebra map $C_{**} \to A_{**}$ and the images in the evident sense of the stated relations hold in A_{**}. See section 4.2 for more discussion of this.

For a start, we have the following relations:

PROPOSITION 2.1.1.

$$1_1 = [0_1] \tag{2.1.10}$$
$$z_0 = [1] \tag{2.1.11}$$
$$z(s) \circ z(t) = z(s+t) \tag{2.1.12}$$
$$e \circ [\alpha] = \overline{z}_{\mathbb{R},1} \tag{2.1.13}$$
$$e^{\circ 2} \circ [\nu] = \overline{z}_{\mathbb{C},2} \tag{2.1.14}$$
$$e^{\circ 4} \circ [\lambda] = [\beta] \circ \overline{z}_{\mathbb{R},4} \tag{2.1.15}$$
$$(e^{\circ n})^2 = e^{\circ n} \circ z_n \tag{2.1.16}$$

□

This is proved in section 7.3. Note that 1_1 is given as an element of C_{**} and $[0_1]$ as an element of $\mathbb{F}[E_*]$, so it is necessary to remark that they are the same. If we use our knowledge of various maps and their action in homotopy to move the above list around, we can generate all the relations in all the Hopf rings we consider.

REMARK 2.1.1. Many of the Hopf rings which have been computed are associated to complex oriented ring spectra [**21, 22, 26**]. In this case, a complex orientation gives rise to a map from the coalgebra

$$\mathrm{H}_*(\mathbb{C}P^\infty; R) = R\{b_{2k} \mid k \geq 0\} \tag{2.1.17}$$

to $\mathrm{H}_*(\underline{E}_2; R)$. It also gives rise [**20**] to a formal group law over E_*, which produces a power series relation involving circle products of the b's. Often, the even-space Hopf ring $\mathrm{H}_*(\underline{E}_{2*}; R)$ with suitable coefficients R turns out to be precisely the Hopf ring over $R[E_{2*}]$ generated by $\mathrm{H}_*(\mathbb{C}P^\infty; R)$ modulo these relations. To get the odd spaces, we need to throw in one more primitive generator (the fundamental class $e \in \mathrm{H}_1\,\underline{E}_1$) and one more relation ($e^{\circ 2} = b_2$). This is precisely the case with complex K-theory. If γ denotes the tautological line bundle over $\mathbb{C}P^\infty$, considered as an element of $K^0\mathbb{C}P^\infty$, then the usual complex orientation is $x = \nu^{-1}(\gamma - 1) \in \tilde{K}^2\mathbb{C}P^\infty$. The formal group law is then

$$x +_F y = x + y + \nu xy \tag{2.1.18}$$

giving the relation

$$b(s+t) = b(s) +_{[F]} b(t) \ := b(s)b(t)([\nu] \circ b(s) \circ b(t)) \tag{2.1.19}$$

In the mod 2 case, this is related to the notation of this thesis by

$$b(t) = [\nu^{-1}] \circ (z(t)/[1]) \tag{2.1.20}$$

and the relation is equivalent to $z(s) \circ z(t) = z(s+t)$. The integral case is very similar. In the orthogonal case, the sub-Hopf ring $\mathrm{H}\mathbb{F}_*\,\underline{KO}_{8*}$ can be described in much the same way as $\mathrm{H}\mathbb{F}_*\,\underline{KU}_{2*}$. In place of the formal group relations coming from $\mathbb{C}P^\infty$, we have analogous relations coming from $\mathbb{R}P^\infty$. To fill in the intermediate spaces, we cannot get away with just one more generator and one more relation, but a short, finite list will suffice. By contrast, in integral homology we need further infinite families of generators and relations. Similar remarks apply in the self-conjugate case.

2.2. The Mod 2 Hopf Ring for KO

In this section we write z for $z_{\mathbb{R}}$ and $\overline{z}$ for $z/[1]$.

PROPOSITION 2.2.1. The following relations hold in $H\mathbb{F}_* \underline{KO}_*$:

$$
\begin{aligned}
1_1 &= [0_1] && (2.2.1)\\
z_0 &= [1] && (2.2.2)\\
z(s) \circ z(t) &= z(s+t) && (2.2.3)\\
e \circ [\alpha] &= \overline{z}_1 = z_1/[1] && (2.2.4)\\
e^{\circ 2} \circ [\beta] &= [\alpha^2] \circ \overline{z}_2 && (2.2.5)\\
e^{\circ 4} \circ [\lambda] &= [\beta] \circ \overline{z}_4 && (2.2.6)\\
[\beta] \circ z_2 &= 0 && (2.2.7)\\
e \circ z_1 &= e^2 && (2.2.8)\\
e^{\circ 2} \circ z_2 &= (e^{\circ 2})^2 && (2.2.9)\\
(e^{\circ 3})^2 &= 0 && (2.2.10)
\end{aligned}
$$

□

This is proved in section 7.3.

THEOREM 2.2.2. The Hopf ring $H\mathbb{F}_* \underline{KO}_*$ is generated by $\mathbb{F}[KO_*]$ and the subcoalgebra C_{**} of (2.1.5) modulo the relations above. Space by space it has the following description:

$$
\begin{array}{lcll}
H\mathbb{F}_*(\mathbb{Z}\times BO) & = & P[[\lambda]\circ z_k][-\lambda] & \\
H\mathbb{F}_*(U/O) & = & P[e\circ[\lambda]\circ z_{2k}] & e\circ[\lambda]\circ z_{2k-1} = (e\circ[\lambda]\circ z_{k-1})^2\\
H\mathbb{F}_*(Sp/U) & = & P[e^{\circ 2}\circ[\lambda]\circ z_{4k}] & e^{\circ 2}\circ[\lambda]\circ z_{4k-2} = (e^{\circ 2}\circ[\lambda]\circ z_{2k-2})^2\\
H\mathbb{F}_*(Sp) & = & E[e^{\circ 3}\circ[\lambda]\circ z_{4k}] & \\
H\mathbb{F}_*(\mathbb{Z}\times BSp) & = & P[[\beta]\circ z_{4k}][-\beta] & e^{\circ 4}\circ[\lambda] = [\beta]\circ\overline{z}_4\\
 & & & [\beta]\circ z_{2k+1} = 0 = [\beta]\circ z_{4k+2}\\
H\mathbb{F}_*(U/Sp) & = & E[e\circ[\beta]\circ z_{4k}] & \\
H\mathbb{F}_*(O/U) & = & E[\overline{[\alpha^2]}\circ z_{2k}] & e^{\circ 2}\circ[\beta] = \overline{[\alpha^2]}\circ\overline{z}_2\\
 & & & \overline{[\alpha^2]}\circ z_{2k+1} = 0\\
H\mathbb{F}_*(O) & = & E[\overline{[\alpha]}\circ z_k] & e\circ\overline{[\alpha^2]} = \overline{[\alpha]}\circ\overline{z}_1\\
H\mathbb{F}_*(\mathbb{Z}\times BO) & = & P[z_k][-1] & e\circ\overline{[\alpha]} = \overline{z}_1
\end{array}
\qquad (2.2.11)
$$

□

Note that in $H\mathbb{F}_*(O)$ we use $\overline{[\alpha]} = [\alpha] - [0]$ rather than $[\alpha]$ simply so that all the generators square to zero. For $k > 0$, we have $\overline{[\alpha]} \circ z_k = [\alpha] \circ z_k$ anyway. There are two parts to the proof of the above theorem. First, we have to derive all the relations implicit in the table from the list in proposition 2.2.1. This will show that there is a map from the Hopf ring described in the table to $H\mathbb{F}_* \underline{KO}_*$. Next, we have to show that this map is iso. The first part is done in section 7.3, and the second in 7.6.

2.3. The Mod 2 Hopf Ring for KT

In this section we write z for z_T.

PROPOSITION 2.3.1. The following relations hold in $\mathrm{H}\mathbb{F}_* \underline{KT}_*$:

$$
\begin{aligned}
1_1 &= [0_1] && (2.3.1)\\
z_0 &= [1] && (2.3.2)\\
z(s) \circ z(t) &= z(s+t) && (2.3.3)\\
e \circ [\alpha] &= \overline{z}_1 && (2.3.4)\\
e^{\circ 2} \circ [\gamma] &= [\alpha] \circ \overline{z}_2 && (2.3.5)\\
e \circ z_1 &= e^2 && (2.3.6)\\
e^{\circ 2} \circ z_2 &= (e^{\circ 2})^2 && (2.3.7)\\
(e^{\circ 3})^2 &= 0 && (2.3.8)\\
[\gamma] \circ \overline{z}_2 &= (e \circ [\mu])^2 && (2.3.9)
\end{aligned}
$$

□

This is proved in section 7.3.

THEOREM 2.3.2. The Hopf ring $\mathrm{H}\mathbb{F}_* \underline{KT}_*$ is generated by $\mathbb{F}[KT_*]$ and the sub-coalgebra C_{**} of (2.1.5) modulo the relations above. Space by space it has the following description:

$$
\begin{aligned}
\mathrm{H}\mathbb{F}_*(\mathbb{Z}\times BT) &= P[[\mu] \circ z_{2k}][-\mu] \otimes E[[\mu] \circ z_{2k+1}]\\
\mathrm{H}\mathbb{F}_*(\Omega^2 T) &= P[e \circ [\mu] \circ z_{2k}, [\gamma] \circ z_{4k}][-\gamma]\\
&\quad e \circ [\mu] \circ z_{2k-1} = (e \circ [\mu] \circ z_{k-1})^2\\
&\quad [\gamma] \circ \overline{z}_2 = (e \circ [\mu])^2\\
&\quad [\gamma] \circ z_1 = 0\\
\mathrm{H}\mathbb{F}_*(\Omega T) &= P[e^{\circ 2} \circ [\mu] \circ z_{4k}] \otimes E[e \circ [\gamma] \circ z_{4k}]\\
&\quad e^{\circ 2} \circ z_{4k-2} = (e^{\circ 2} \circ z_{2k-2})^2 \qquad (2.3.10)\\
\mathrm{H}\mathbb{F}_*(T) &= E[\overline{[\alpha]} \circ z_{2k}, e^{\circ 3} \circ [\mu] \circ z_{4k}]\\
&\quad \overline{[\alpha]} \circ z_{2k+1} = 0\\
&\quad e^{\circ 2} \circ [\gamma] = [\alpha] \circ \overline{z}_2\\
\mathrm{H}\mathbb{F}_*(\mathbb{Z}\times BT) &= P[z_{2k}][-1] \otimes E[z_{2k+1}]\\
&\quad e \circ [\alpha] = \overline{z}_1
\end{aligned}
$$

□

The proof is again split between sections 7.3 and 7.6. Perhaps a word or two more is called for about how to express the elements $[\gamma] \circ z_{4k+2}$ in terms of the generators offered. If we take the circle product of the relation $[\gamma] \circ \overline{z}_2 = (e \circ [\mu])^2$ with $z(t)$ and rearrange a little, we obtain

$$\sum_{k\geq 0} [\gamma] \circ z_{4k+2} t^{4k+2} = \left([\mu] \circ z(t) \circ \frac{t^2 e^2}{1 + t^2 e^2} \right) \sum_{k \geq 0} [\gamma] \circ z_{4k} t^{4k} \qquad (2.3.11)$$

which provides the required information.

2.4. The Mod 2 Hopf Ring for KU

In this section we write z for $z_{\mathbb{C}}$.

PROPOSITION 2.4.1. The following relations hold in $H\mathbb{F}_* \underline{KU}_*$:

$$
\begin{aligned}
1_1 &= [0_1] && (2.4.1)\\
z_0 &= [1] && (2.4.2)\\
z(s) \circ z(t) &= z(s+t) && (2.4.3)\\
e^2 &= 0 && (2.4.4)\\
e^{\circ 2} \circ [\nu] &= \overline{z}_2 && (2.4.5)
\end{aligned}
$$

□

This is proved in section 7.3.

THEOREM 2.4.2. The Hopf ring $H\mathbb{F}_* \underline{KU}_*$ is generated by $\mathbb{F}[KU_*]$ and the subcoalgebra C_{**} of (2.1.5) modulo the relations above. Space by space it has the following description:

$$
\begin{aligned}
H\mathbb{F}_*(\mathbb{Z}\times BU) &= P[[\nu] \circ z_{2k}][-\nu] \\
H\mathbb{F}_*(U) &= E[e \circ [\nu] \circ z_{2k}] \\
H\mathbb{F}_*(\mathbb{Z}\times BU) &= P[z_{2k}][-1] \qquad e^{\circ 2} \circ [\nu] = \overline{z}_2
\end{aligned}
\tag{2.4.6}
$$

□

The proof is again split between sections 7.3 and 7.6.

2.5. The Mod 2 Hopf Ring for KSp

The title of this section is of course an abuse of language; $H\mathbb{F}_* \underline{KSp}_*$ is not a Hopf ring but rather a free Hopf module on one generator $[\theta]$ over $H\mathbb{F}_* \underline{KO}_*$. This is supposed to mean that the map

$$
H\mathbb{F}_* \underline{KO}_* \longrightarrow H\mathbb{F}_* \underline{KSp}_{*-4} \qquad x \mapsto x \circ [\theta] \tag{2.5.1}
$$

is iso. We are left with the task of understanding $z_{\mathbb{H}}$ in these terms, and of computing the tensor product $KSp \wedge KSp \to KO$.

$$
\begin{aligned}
z_{\mathbb{H}}(t) &= [\lambda^{-1}\beta\theta] \circ z(t) && (2.5.2)\\
[\theta] \circ [\theta] &= [\lambda] && (2.5.3)
\end{aligned}
$$

The first of these is proved in section 7.3. The second is of course equivalent to the analogous statement in the homotopy groups, which was recorded at the beginning of this chapter.

2.6. Primitives and Duality

In this section we look at the cohomology of the classifying spaces $BO_{\mathbb{K}}$ and the group of primitives in the dual homology ring $H\mathbb{F}_*(BO_{\mathbb{K}}) = P[\overline{z}_{dk} \mid k > 0]$. Proofs are given in section 7.4. Given a multiindex $\alpha = (\alpha_1, \alpha_2, \dots)$ with α_k a nonnegative integer which vanishes for almost all k, we define

$$
\overline{z}^{\alpha} = \prod_{k>0} \overline{z}_{dk}^{\alpha_k} \tag{2.6.1}
$$

We write e_k for the multiindex with a 1 in the k'th place and zeros elsewhere, so $\overline{z}^{e_k} = \overline{z}_{dk}$. The monomials $\overline{z}^{\alpha}$ for all possible α form a basis for $H\mathbb{F}_*(BO_{\mathbb{K}})$. There are thus elements $\overline{z}'_{dk}$ of the dual space $H\mathbb{F}^*(BO_{\mathbb{K}})$ uniquely characterised by

$$
\langle \overline{z}'_{dk}, \overline{z}^{\alpha} \rangle = \begin{cases} 1 & \text{if } \alpha = ke_d \\ 0 & \text{otherwise} \end{cases} \tag{2.6.2}
$$

We write $\overline{z}'(t) = \sum_k \overline{z}'_{dk} t^k$. The structure of the cohomology is as follows:

$$\begin{aligned} \mathrm{H}\mathbb{F}^*(BO_{\mathbb{K}}) &= P[\overline{z}'_{dk} \mid k > 0] && (2.6.3) \\ \psi\overline{z}'(t) &= \overline{z}'(t) \otimes \overline{z}'(t) && (2.6.4) \end{aligned}$$

We write p_α for the element of $\mathrm{H}\mathbb{F}_*BO$ dual to $(\overline{z}')^\alpha$ w.r.t. the monomial basis of $\mathrm{H}\mathbb{F}^*BO$, and collect these elements together into a power series in infinitely many variables:

$$\sum_\alpha p_\alpha t^\alpha \in \mathrm{H}\mathbb{F}_*BO[\![t_1, t_2 \dots]\!] \tag{2.6.5}$$

We find that

$$\begin{aligned} \psi p(\underline{t}) &= p(\underline{t}) \otimes p(\underline{t}) && (2.6.6) \\ \psi p_\alpha &= \sum_{\alpha=\beta+\gamma} p_\beta \otimes p_\gamma && (2.6.7) \\ p(s, 0, 0, \dots) &= \overline{z}(s) && (2.6.8) \end{aligned}$$

For various purposes it is important to understand the primitives in $\mathrm{H}\mathbb{F}_*X$, i.e. the elements x such that $\psi x = x \otimes 1 + 1 \otimes x$. See chapter 4 for general facts about primitives.

We start by defining

$$q_{\mathbb{K}}(t) = \sum_{l>0} q_{\mathbb{K},ld} t^{ld} = t\dot{z}_{\mathbb{K}}(t)/z_{\mathbb{K}}(t) = \mathrm{d}\log z_{\mathbb{K}}(t)/\mathrm{d}t \tag{2.6.9}$$

The series $q_{\mathbb{K}}$ and therefore the coefficients $q_{\mathbb{K},ld}$ are primitive. The first few terms of $q_{\mathbb{R}}(t)$ are as follows:

$$\begin{aligned} q(t) = {} & \overline{z}_1 t + \\ & \overline{z}_1^2 t^2 + \\ & (\overline{z}_3 + \overline{z}_1\overline{z}_2 + \overline{z}_1^3)t^3 + \\ & \overline{z}_1^4 t^4 + \\ & (\overline{z}_1^5 + \overline{z}_2\overline{z}_1^3 + \overline{z}_3\overline{z}_1^2 + \overline{z}_2^2\overline{z}_1 + \overline{z}_4\overline{z}_1 + \overline{z}_3\overline{z}_2 + \overline{z}_5)t^5 + \\ & (\overline{z}_1^6 + \overline{z}_2^2\overline{z}_1^2 + \overline{z}_3^2)t^6 + \\ & (\overline{z}_1^7 + \overline{z}_2\overline{z}_1^5 + \overline{z}_3\overline{z}_1^4 + \overline{z}_4\overline{z}_1^3 + \overline{z}_3\overline{z}_2\overline{z}_1^2 + \overline{z}_5\overline{z}_1^2 + \overline{z}_2^3\overline{z}_1 + \\ & \quad \overline{z}_3^2\overline{z}_1 + \overline{z}_6\overline{z}_1 + \overline{z}_3\overline{z}_2^2 + \overline{z}_5\overline{z}_2 + \overline{z}_4\overline{z}_3 + \overline{z}_7)t^7 + \\ & O(t^8) \end{aligned}$$

In fact $q_k = p_{e_k}$, and these are all the primitives in $\mathrm{H}\mathbb{F}_*BO$. We need some formulae for circle products:

$$\begin{aligned} \overline{z}_1 \circ z(t) &= q(t)/t && (2.6.10) \\ \overline{z}_1 \circ \overline{z}(t) &= q(t)/t + \overline{z}_1 && (2.6.11) \\ q(s) \circ z(t) &= \frac{s}{s+t} q(s+t) && (2.6.12) \\ q(s) \circ q(t) &= \frac{st}{(s+t)^2} q_{ev}(s+t) && (2.6.13) \end{aligned}$$

2.7. The Homology Suspension

The following table gives the iterated homology suspensions $e^{\circ n} \circ x$ of various elements x. See section 7.3 for justification.

$[\alpha]$	$[\alpha^2]$	$[\nu]$	$[\gamma]$	$[\beta]$	$[\mu]$	$[\lambda]$
$\overline{z}_1$	$[\alpha]\circ\overline{z}_1$	.	.	.	.	.
e^2	$\overline{z}_1^2$	$\overline{z}_2$	$[\alpha]\circ\overline{z}_2$	$[\alpha^2]\circ\overline{z}_2$	.	.
0	0	.	q_3	$[\alpha]\circ q_3$	.	.
		$(e^{\circ 2})^2$	e^4	$\overline{z}_1^4$	$\overline{z}_2^2$	$[\beta]\circ\overline{z}_4$
		0	0	0	0	.
						$[\alpha^2]\circ(\overline{z}_6+\overline{z}_4\overline{z}_2)$
						$[\alpha]\circ q_7$
						$\overline{z}_1^8$
						0

(2.7.1)

The dot under $[\gamma]$ for example, simply indicates that we have no useful description of $e\circ[\gamma]$ other than as $e\circ[\gamma]$. The fact that in all cases we eventually hit zero is a reflection of the fact that the stable integral homology of each of our spectra is rational so that the stable homology mod 2 vanishes.

2.8. Steenrod and Kudo-Araki Operations

In this section we discuss the action of the Steenrod and Kudo-Araki[1] operations on our Hopf rings. Proofs are given in section 7.5.

The right action of the total Steenrod operation $Sq(s) = \sum_{k\geq 0} s^k Sq^k$ is as follows:

$$z(t)Sq(s) = z(s^d t^2 + t) \tag{2.8.1}$$
$$e\,Sq(s) = e \tag{2.8.2}$$
$$[a]Sq(s) = [a] \tag{2.8.3}$$
$$(xy)Sq(s) = (x\,Sq(s))(y\,Sq(s)) \tag{2.8.4}$$
$$(x\circ y)Sq(s) = (x\,Sq(s))\circ(y\,Sq(s)) \tag{2.8.5}$$

Here z can be $z_{\mathbb{R}}$, $z_{\mathbb{C}}$, $z_{\mathbb{H}}$ or z_T and d is 1, 2, 4 or 1 accordingly. It is amusing to verify that this is consistent with the Bullett-Macdonald formulation [**7**] of the Adem relations:

$$Sq(s^2+st)Sq(t^2) = Sq(t^2+ts)Sq(s^2) \tag{2.8.6}$$

Equivalently, we can give the coaction of the dual Steenrod algebra. We first recall the description of this Hopf algebra:

$$\mathcal{A}_* = P[\xi_k \mid k>0] \qquad \xi_0 = 1 \tag{2.8.7}$$
$$\xi(t) = \sum_{k\geq 0}\xi_k t^{2^k} \tag{2.8.8}$$
$$\psi\xi(t) = (1\otimes\xi)(\xi(t)\otimes 1) \tag{2.8.9}$$

The pairing between $\mathcal{A}^*$ and $\mathcal{A}_*$ is

$$\langle Sq(s), f(\xi(t))\rangle = f(t+t^2 s) \qquad f(u)\in\mathbb{F}_2[\![u]\!] \tag{2.8.10}$$

[1] We deviate here from the usual practice of calling these Dyer-Lashof operations, and adding a footnote to explain that they are really due to Kudo and Araki.

The coaction $\alpha\colon H\mathbb{F}_*X \to \mathcal{A}_* \otimes H\mathbb{F}_*X$ is as follows:

$$\alpha(z(t^d)) = (1 \otimes z)(\xi(t)^d \otimes 1) \tag{2.8.11}$$
$$\alpha(e) = 1 \otimes e \tag{2.8.12}$$
$$\alpha([a]) = 1 \otimes [a] \tag{2.8.13}$$
$$\alpha(xy) = \alpha(x)\alpha(y) \tag{2.8.14}$$
$$\alpha(x \circ y) = \alpha(x) \circ \alpha(y) \tag{2.8.15}$$

For the last equation we must interpret $(\zeta \otimes x)\circ(\eta \otimes y)$ as $\zeta\eta \otimes (x \circ y)$. In cohomology, we have

$$Sq(st)\overline{z}'(s + t^2 s) = \overline{z}'(ts + t^2 s)\overline{z}'(s + ts) \tag{2.8.16}$$

The total Kudo-Araki operation $Q(s) = \sum_{k\geq 0} s^k Q^k$ acts as follows:

$$Q(s)z(s^d(t+t^2)) = z(s^d(1+t))z(s^d t) \tag{2.8.17}$$
$$Q(s)z(s^d t) = ([1]z(s^d)) \circ z(s^d \sum_k t^{2^k}) \tag{2.8.18}$$
$$Q^n[1] = [1]z_n \tag{2.8.19}$$
$$Q(s)(e \circ x) = e \circ Q(s)x \tag{2.8.20}$$
$$Q(s)(xy) = Q(s)x\, Q(s)y \tag{2.8.21}$$
$$Q(s)([a] \circ x) = [a] \circ Q(s)x \tag{2.8.22}$$

(The first of these equations is a translation into formal power series language of a result of Priddy [**19**]. The second equation is equivalent to the first, and the third is a consequence)

2.9. Bocksteins

We next record the homology of $H\mathbb{F}_* \underline{E}_*$ with respect to the action of the Bockstein, which we write as $H(H\mathbb{F}_* \underline{E}_*, \beta)$. The basic data are:

$$\beta z_{2k+2} = z_{2k+1} \tag{2.9.1}$$
$$\beta z_{2k+1} = \beta[x] = \beta e = 0 \tag{2.9.2}$$
$$\beta(xy) = \beta(x)y + x\beta(y) \tag{2.9.3}$$
$$\beta(x \circ y) = \beta(x) \circ y + x \circ \beta(y) \tag{2.9.4}$$

Proposition 2.9.1. In most of the spaces under consideration, the Bockstein vanishes so the β-homology is just $H\mathbb{F}_*X$. The exceptions are as follows:

$$\begin{aligned}
H(H\mathbb{F}_*(O),\beta) &= E[\overline{[\alpha]}] \otimes E[[\alpha] \circ q_{4k+3}] \\
H(H\mathbb{F}_*(\mathbb{Z}\times BO),\beta) &= P[z_{2k}^2 \mid k \geq 0][-1] \\
H(H\mathbb{F}_*(U/O),\beta) &= E[e] \otimes E[h_{4k+1} \mid k > 0] \\
& \quad h_{4k+1} = e \circ z_{4k} + (e \circ z_{2k})(e \circ z_{2k-1}) \\
H(H\mathbb{F}_*(\mathbb{Z}\times BT),\beta) &= P[z_{2k}^2][-1] \otimes E[q_{4k+3}] \\
H(H\mathbb{F}_*(\Omega^2 T),\beta) &= E[e] \otimes E[h_{4k+5}] \otimes P[[\mu^{-1}\gamma] \circ z_{4k}][-\mu^{-1}\gamma]
\end{aligned} \tag{2.9.5}$$

Moreover $[\mu^{-1}\gamma] \circ z_{4k+2}$ is a β-boundary in $H\mathbb{F}_*(\Omega^2 T)$. □

This is proved in section 7.7.

CHAPTER 3

Torsion Free Hopf Rings for K-Theories

In this chapter we develop a little theory of torsion free Hopf rings, and state what happens in the K-theory case. Proofs are in sections 7.8 to 7.10.

3.1. Torsion Quotients

In this section we discuss the torsion quotient of an Abelian group, and give a Künneth formula for torsion free homology. This material would be simplified if we assumed that all connected components of the spaces considered had homology finitely generated in each dimension, and this would cover all cases used elsewhere in this thesis. However, the results are true without such hypotheses so it would be a shame not to prove them that way.

Given an Abelian group A, we write

$$tA = \{a \in A \mid \exists n > 0 \; na = 0\} = \text{ torsion subgroup of } A \tag{3.1.1}$$

$$fA = A/tA = \text{ torsion quotient of } A \tag{3.1.2}$$

Note that fA is torsion free. We also write $\overline{\mathrm{H}}_*(X) = f\mathrm{H}_*(X;\mathbb{Z})$ and $A*B = \mathrm{Tor}(A,B)$. Recall that a short exact sequence

$$0 \to A \rightarrowtail B \twoheadrightarrow C \to 0 \tag{3.1.3}$$

gives rise to a six term sequence

$$0 \to A*D \rightarrowtail B*D \to C*D \to A \otimes D \to B \otimes D \twoheadrightarrow C \otimes D \to 0 \tag{3.1.4}$$

Recall also that $A*B = 0$ if A or B is torsion free. (In other words, torsion free groups are flat).

Lemma 3.1.1.
(1) $A*B$ is always a torsion group.
(2) $f(A \otimes B) = fA \otimes fB$
(3) $\overline{\mathrm{H}}_*(X \times Y) = \overline{\mathrm{H}}_*(X) \otimes \overline{\mathrm{H}}_*(Y)$

Proof. First, choose a free resolution $R \rightarrowtail F \twoheadrightarrow B$ for B. Taking the tensor product first by $\mathbb{Q}$ and then by A, we find that

$$0 \to A \otimes R \otimes \mathbb{Q} \to A \otimes F \otimes \mathbb{Q} \twoheadrightarrow A \otimes B \otimes \mathbb{Q} \to 0 \tag{3.1.5}$$

is exact. On the other hand, if we take the tensor product first by A and then by $\mathbb{Q}$, we find that the following is exact:

$$0 \to (A*B) \otimes \mathbb{Q} \to A \otimes R \otimes \mathbb{Q} \rightarrowtail A \otimes F \otimes \mathbb{Q} \twoheadrightarrow A \otimes B \otimes \mathbb{Q} \to 0$$

We conclude that $(A*B) \otimes \mathbb{Q} = 0$ and thus that $A*B$ is a torsion group.

Next, we observe that the sequence $tA \rightarrowtail A \twoheadrightarrow fA$ remains short exact when tensored by any group D (as fA is flat). This gives us a 3 by 3 exact diagram:

$$\begin{tikzcd}
tA \otimes tB \arrow[r, tail] \arrow[d, tail] & tA \otimes B \arrow[r, two heads] \arrow[d, tail] & tA \otimes fB \arrow[d, tail] \\
A \otimes tB \arrow[r, tail] \arrow[d, two heads] & A \otimes B \arrow[r, two heads] \arrow[d, two heads] & A \otimes fB \arrow[d, two heads] \\
fA \otimes tB \arrow[r, tail] & fA \otimes B \arrow[r, two heads] & fA \otimes fB
\end{tikzcd} \tag{3.1.6}$$

By chasing in this diagram we find that $A \otimes B \to fA \otimes fB$ is epi with kernel the torsion group $tA \otimes B + A \otimes tB$. Moreover, $fA \otimes fB$ is torsion free. To see this, note that fA is flat (so $fA \otimes (-)$ preserves monos) and $n.1_{fB}$ is mono (for $n \neq 0$) so $n.1_{fA\otimes fB}$ is mono. It follows easily that $f(A \otimes B) = fA \otimes fB$.

Finally, suppose we have spaces X and Y. For brevity, set $A = \mathrm{H}_*(X)$, $B = \mathrm{H}_*(Y)$ and $C = \mathrm{H}_*(X \times Y)$. The Künneth theorem gives a short exact sequence $A \otimes B \rightarrowtail C \twoheadrightarrow A{*}B$ which splits unnaturally. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & K \arrow[d, tail] \\
t(A \otimes B) \arrow[r, tail] \arrow[d, tail] & A \otimes B \arrow[r, two heads] \arrow[d, tail] & f(A \otimes B) \arrow[d, two heads] \\
tC \arrow[r, tail] \arrow[d, two heads] & C \arrow[r, two heads] \arrow[d, two heads] \arrow[u, bend right] & fC \arrow[d] \\
L \arrow[r, two heads] & A{*}B \arrow[r] \arrow[u, tail, bend right] & 0
\end{tikzcd} \tag{3.1.7}$$

The groups K and L are defined as the kernel and cokernel of the evident maps. The middle two rows are short exact by definition, and the middle column is split exact by the Künneth theorem. It follows easily that $t(A \otimes B) \to tC$ is mono. The torsion group $A{*}B$ can only map trivially to fC. Using this and the splitting $C \simeq (A \otimes B) \oplus (A{*}B)$ we find that $f(A \otimes B) \to fC$ is epi. The snake lemma now gives a short exact sequence $K \rightarrowtail L \twoheadrightarrow A{*}B$. As K is a subgroup of the quotient L of tC, it must be a torsion group. However, K is also a subgroup of the torsion free group $f(A \otimes B)$, so it must vanish. Thus $f(A \otimes B) \to fC$ is iso. $\square$

We refer to $\overline{\mathrm{H}}_*(X)$ as the torsion free homology of X. By the Künneth formula just proven, the groups $\overline{\mathrm{H}}_*(\,\underline{E}_*)$ form a Hopf ring for any ring spectrum E. In the rest of this chapter, we shall write $=$ for equations in H_* or $\mathrm{H}\mathbb{F}_*$, and $\equiv$ for equations in $\overline{\mathrm{H}}_*$ or $\mathrm{H}_*(\mathrm{H}\mathbb{F}_*, \beta)$.

LEMMA 3.1.2. Let $\phi\colon A \to B$ be a map of graded Abelian groups such that $A[\frac{1}{2}] \to B[\frac{1}{2}]$ and $A\otimes\mathbb{F} \to B\otimes\mathbb{F}$ are iso. If either of the following conditions holds then ϕ is iso:

(1) A and B are torsion free
(2) A is finitely generated in each dimension and B is torsion free.

PROOF. Let U, V and W denote the kernel, image and cokernel of ϕ. As $\mathbb{Z}[\frac{1}{2}]$ is flat, we then have exact sequences:

$$0 \to U \rightarrowtail A \to B \twoheadrightarrow W \to 0 \tag{3.1.8}$$

$$0 \to U[\tfrac{1}{2}] \rightarrowtail A[\tfrac{1}{2}] \xrightarrow{\simeq} B[\tfrac{1}{2}] \twoheadrightarrow W[\tfrac{1}{2}] \to 0 \tag{3.1.9}$$

This shows that $U[\frac{1}{2}] = 0 = W[\frac{1}{2}]$.

As $A \to V$ is epi, so is $A\otimes\mathbb{F} \to V\otimes\mathbb{F}$. On the other hand, we are given that the composite $A\otimes\mathbb{F} \to V\otimes\mathbb{F} \to B\otimes\mathbb{F}$ is iso. It follows first that $A\otimes\mathbb{F} \to V\otimes\mathbb{F}$ is mono, and then that both $A\otimes\mathbb{F} \to V\otimes\mathbb{F}$ and $V\otimes\mathbb{F} \to B\otimes\mathbb{F}$ are iso. We next consider the exact sequence

$$B{*}\mathbb{F} = 0 \to W{*}\mathbb{F} \to V\otimes\mathbb{F} \xrightarrow{\simeq} B\otimes\mathbb{F} \to W\otimes\mathbb{F} \to 0 \tag{3.1.10}$$

This shows that $W{*}\mathbb{F} = 0 = W\otimes\mathbb{F}$. On the other hand, $W{*}\mathbb{F}$ and $W\otimes\mathbb{F}$ are the kernel and cokernel of $W \xrightarrow{2} W$, so we conclude that this map is iso and thus that W is a $\mathbb{Z}[\frac{1}{2}]$-module. This means that $W = W[\frac{1}{2}] = 0$ and so $V = B$. This leaves us with an exact sequence

$$0 \to U{*}\mathbb{F} \to A{*}\mathbb{F} \to B{*}\mathbb{F} = 0 \to U\otimes\mathbb{F} \to A\otimes\mathbb{F} \xrightarrow{\simeq} B\otimes\mathbb{F} \to 0$$

which shows that $U\otimes\mathbb{F} = 0$. If A is torsion free then $A{*}\mathbb{F} = 0$ so $U{*}\mathbb{F} = 0$ and U vanishes for the same reason as W does. On the other hand, if A is finitely generated in each dimension then the same is true of $U \leq A$ and it follows from $U\otimes\mathbb{F} = 0 = U[\frac{1}{2}]$ and the structure theory that $U = 0$. Either way, we conclude that ϕ is iso. □

3.2. Divided Squares and Square Roots

All the homology rings which we have encountered so far have been polynomial or exterior. Indeed, the structure theory of bicommutative Hopf algebras over a field assures us that little more complication is possible. However, more subtle phenomena can occur integrally – we investigate some of them in this section.

Let A_* be an augmented graded commutative k-algebra. We shall mainly be interested in the cases $k = \mathbb{Z}$ or $\mathbb{F}$, but it could be any ring. We let $I = \tilde{A}$ denote the augmentation ideal.

DEFINITION 3.2.1. A divided square operator on A is a map $\gamma\colon I_* \to I_{2*}$ satisfying

(1) $\forall a \in I\ \ 2\gamma(a) = a^2$
(2) $\forall a \in I, b \in A\ \ \gamma(ab) = (-1)^{1+\epsilon}\gamma(a)b^2$
(3) $\forall a, b \in I\ \ \gamma(a+b) = \gamma(a) + \gamma(b) + \epsilon ab$

where $\epsilon = 0$ if $|a|$ and $|b|$ are both odd, and $\epsilon = 1$ otherwise.

In the cases of interest, all odd dimensional elements are annihilated by 2, so the signs can be ignored. If A_* is torsion free then γ depends only on the algebra structure. In any case, if A_* is generated by a graded subset X_* then γ is determined by its action on X_*. Note that the Frobenius endomorphism of $A\otimes\mathbb{F}$ (which sends x to x^2) annihilates $I\otimes\mathbb{F}$. It follows that $\gamma(ab) = 0$ for $a, b \in I\otimes\mathbb{F}$, but γ need not annihilate a sum of such terms.

Given an integer n write $\alpha(n)$ for the number of ones in the binary expansion and $\nu(n)$ for the 2-adic valuation so that $\nu(n!) = n - \alpha(n)$. We can write $n = \sum_{k\in S} 2^k$ where $|S| = \alpha(n)$ and $\min S = \nu(n)$. We write

$$a^{(n)} = \prod_{k\in S} \gamma^k(a) \tag{3.2.1}$$

so that $2^{n-\alpha(n)}a^{(n)} = a^n$. If k is 2-local, we may define

$$a^{[n]} = \frac{2^{n-\alpha(n)}}{n!}a^{(n)} \tag{3.2.2}$$

and this gives a divided power structure on A_* in the usual sense.

Let B_* be an augmented graded commutative k-algebra. By the usual methods of universal algebra we can construct a B_*-algebra A_* with divided squares, with the property that maps $A_* \to C_*$ of divided-square algebras biject naturally with k-algebra maps $B_* \to C_*$. We refer to A_* as the divided-square envelope of B_*. Consider the k-algebra

$$D[x_i \mid i \in I] = P[x_{i,j}|i \in I, j \geq 0]/(x_{i,j}^2 - 2x_{i,j+1}) \tag{3.2.3}$$

This has a unique divided-square structure in which $\gamma(x_{i,j}) = x_{i,j+1}$. One checks easily that $D[x_i]$ is the divided-square envelope of $P[x_i]$. If k has characteristic 2 then $D[x_i] = E[x_{i,j}]$. If k is torsion free then $P[x_i] \subset D[x_i] \subset P[x_i/2]$.

We need to understand the arithmetic properties of the power series

$$\sqrt{1+x^2} = \sum_k c_k x^{2k} = \sum_k \frac{1}{2}\left(\frac{-1}{2}\right)\dots\left(\frac{3-2k}{2}\right)\frac{x^{2k}}{k!} \tag{3.2.4}$$

First note that $\nu(c_k) = \alpha(k) - 2k = \alpha(2k) - 2k < 0$. Also, by squaring and arguing inductively we see that $c_k \in \mathbb{Z}[\frac{1}{2}]$. We conclude that $d_k = 2^{2k-\alpha(k)}c_k \in 1 + 2\mathbb{Z}$. Thus, if $x^{(2k)}$ is defined and $y = \sum_k d_k x^{(2k)}$ converges in a suitable sense, then it satisfies $y^2 = 1 + x^2$.

Take $k = \mathbb{Z}$ and consider the ring $A_* = P[x_{2k} \mid k \geq 0]/(x(t)x(-t) - 1)$. Here $x(t) = \sum_{k\geq 0} x_{2k}t^k$ and the ideal of relations is supposed to be generated by the coefficients of the series $x(t)x(-t) - 1$.

Lemma 3.2.1. $A_* = \mathbb{Z}[x_0]/(x_0^2 - 1) \otimes D[x_{4k+2} \mid k \geq 0]$

Proof. We first define

$$\hat{x}(t) = \sum_k x_{4k+2}t^{2k+1} = (x(t) - x(-t))/2 \tag{3.2.5}$$

$$\check{x}(t) = \sum_k x_{4k}t^{2k} = (x(t) + x(-t))/2 \tag{3.2.6}$$

so $\check{x}(t)^2 - \hat{x}(t)^2 = 1$. By analogy, we define three series over the ring $D_* = \mathbb{Z}[y_0]/(y_0^2 - 1) \otimes D[\hat{y}_{4k+2}]$:

$$\hat{y}(t) = \sum_k \hat{y}_{4k+2}t^{2k+1} \tag{3.2.7}$$

$$\check{y}(t) = y_0\sqrt{1 + \hat{y}(t)^2} \tag{3.2.8}$$

$$y(t) = \hat{y}(t) + \check{y}(t) \tag{3.2.9}$$

The remarks before the lemma ensure that $\check{y}(t)$ is indeed a series over D_*. We find that $\hat{y}(-t) = -\hat{y}(t)$ so $\check{y}(-t) = \check{y}(t)$ so (as D_* is torsion free) $\check{y}(t) = \sum_k \check{y}_{4k}t^{2k}$ say. We also find that $y(t)y(-t) = 1$ so we deduce a map $f \colon A_* \to D_*$ sending $x(t)$ to $y(t)$. After inverting 2 it is not hard to see that the equation $\check{x}(t)^2 - \hat{x}(t)^2 = 1$ can be solved uniquely to give $\breve{x}(t) = x_0\sqrt{1+\hat{x}(t)^2}$ and thus that our map becomes iso. On the other hand, one sees that

$$\begin{aligned} \mathbb{F}\otimes A_* &= \mathbb{F}[x_0]/(x_0^2-1)\otimes E[x_{2k+2}] && (3.2.10)\\ \mathbb{F}\otimes D_* &= \mathbb{F}[y_0]/(y_0^2-1)\otimes E[y_{4k+2}^{(2^l)}] && (3.2.11)\end{aligned}$$

In $\mathbb{F}\otimes A_*$, all squares of reduced elements vanish. It follows that all squares of reduced elements in A_* are divisible by 2. We can use this to choose a map $g\colon D_*\to A_*$ with $fg = 1_D$. Indeed, we may set $g(\hat{y}_{4k+2}) = x_{4k+2}$ and recursively choose $g\Big(\hat{y}_{4k+2}^{(2^l)}\Big)$ such that

$$2g\Big(\hat{y}_{4k+2}^{(2^l)}\Big) = g\Big(\hat{y}_{4k+2}^{(2^{l-1})}\Big)^2 \tag{3.2.12}$$

Using the fact that D_* is torsion free, we see that $fg = 1_D$ as required. It follows that f is epi mod 2 and by comparing Poincaré series, we see that it is iso mod 2. It is also iso after inverting 2, the source is finitely generated in each dimension and the target is torsion free. It follows by lemma 3.1.2 that f itself is iso. □

This shows that A_* has a divided square structure, which is unique as there is no torsion. By expanding the equation $\breve{x}(t)^2 - \hat{x}(t)^2 = 1$ we find that

$$\begin{aligned} \gamma(x_0-1) &= 1-x_0 && (3.2.13)\\ \gamma(x_{2m}) &= -\sum_{\substack{k<l\\ k+l=2m}} (-1)^{l+m}x_{2k}x_{2l} && (3.2.14)\end{aligned}$$

3.3. The Torsion Free Hopf Ring for KU

We omit most proofs for this section, as they are very similar to the mod 2 case. There are some remarks in section 7.8, however. In this case there is no torsion in the integral homology, so $\overline{\mathrm{H}}$ coincides with H.

PROPOSITION 3.3.1.

$$\begin{aligned} \mathrm{H}_*\mathbb{C}P^\infty &= \mathbb{Z}\{y_{2k}\mid k\geq 0\} && (3.3.1)\\ y(t) &= \sum_{k\geq 0} y_{2k}t^k && (3.3.2)\\ \psi y(t) &= y(t)\otimes y(t) && (3.3.3)\\ y(s)\circ y(t) &= y(s+t) && (3.3.4)\\ y_0 &= [1] && (3.3.5)\\ e^{\circ 2}\circ[\nu] &= \overline{y}_2 = y_2/[1] && (3.3.6)\\ e^{\circ 2}\circ y(t) &= [\nu^{-1}]\circ \mathrm{dlog}\, y(t)/\mathrm{d}t && (3.3.7)\\ c[\nu] &= [-\nu] && (3.3.8)\\ cy(t) &= y(-t) && (3.3.9)\\ \rho y(t) &= z_{\mathbb{C}}(t) && (3.3.10)\end{aligned}$$

□

Here c denotes the complex conjugation map and ρ the mod 2 reduction $\mathrm{H}\to\mathrm{H}\mathbb{F}$. As in the mod 2 case we are considering the homology of $\mathbb{C}P^\infty$ as being embedded in that of $\mathbb{Z}\times BU$ via the map which classifies the unreduced canonical line bundle.

THEOREM 3.3.2. The integral Hopf ring for KU is as follows:

$$
\begin{array}{lcll}
H_*(\mathbb{Z}\times BU) & = & P[[\nu]\circ y_{2k}][-\nu] & \\
H_*(U) & = & E[e\circ[\nu]\circ y_{2k}] & \\
H_*(\mathbb{Z}\times BU) & = & P[y_{2k}][-1] & y_0 = [1]
\end{array}
\tag{3.3.11}
$$

□

3.4. The Torsion Free Hopf Ring for KO

The case of KO is rather more complicated. Proofs are given in section 7.9. We write ${}_ky_l$ for $f_U([\nu^k]\circ y_l) \in H_l\,\underline{KO}_{-2k}$. Note that KO-linearity gives $[\lambda]\circ{}_ky(t) = {}_{(k+4)}y(t)$. Recall that we write $\equiv$ for equations in H_* mod torsion or in $H\mathbb{F}_*$ mod the image of the Bockstein β.

$$
\begin{array}{rcll}
{}_ky(-t) & = & {}_ky(t) & k \text{ even} \\
{}_ky(-t) & = & {}_ky(t)^{-1} & k \text{ odd} \\
{}_ky_{4l+2} & \equiv & 0 & k \text{ even} \\
{}_ky_{4l} & \in & P[{}_ky_{4l+2}/2][{}_ky_0^{\pm 1}] & k \text{ odd}
\end{array}
\tag{3.4.1}
$$

When k is odd, it is convenient to make the following definitions (as in section 3.2 above):

$$
\begin{array}{rcccl}
{}_k\breve{y}(t) & = & \sum_l {}_ky_{4l}t^{2l} & \equiv & ({}_ky(t) + {}_ky(-t))/2 \\
{}_k\hat{y}(t) & = & \sum_l {}_ky_{4l+2}t^{2l+1} & \equiv & ({}_ky(t) - {}_ky(-t))/2
\end{array}
\tag{3.4.2}
$$

We find that

$$ {}_ky(\pm t) = {}_k\breve{y}(t) \pm {}_k\hat{y}(t) \tag{3.4.3} $$

$$ {}_k\breve{y}(t)^2 - {}_k\hat{y}(t)^2 = 1 \tag{3.4.4} $$

$$ {}_k\breve{y}(t) \equiv {}_k\breve{y}_0\sqrt{1 + {}_k\hat{y}(t)^2} \tag{3.4.5} $$

The square root in the second equation above is given by the usual power series. Note that ${}_{(4m+1)}\breve{y}_0 = [\lambda^m\alpha^2]$ and ${}_{(4m+3)}\breve{y}_0 = [0] = 1$. Below we shall give various structure formulae in terms of the series ${}_ky(t)$. They can if necessary be converted into formulae involving only the generator series ${}_k\hat{y}(t)$ by means of the above equations.

The complexification map is given by:

$$ m_O({}_ky(t)) = [\nu^k]\circ(y(t)y(-t)^\epsilon) \qquad \epsilon = (-1)^k \tag{3.4.6} $$

$$ m_O({}_k\hat{y}(t)) = \frac{[\nu^k]}{2}\circ\frac{y(t)^2 - y(-t)^2}{y(t)y(-t)} \tag{3.4.7} $$

The series ${}_ky(t)$ are grouplike, i.e.

$$ \psi({}_ky(t)) = {}_ky(t)\otimes{}_ky(t) \tag{3.4.8} $$

We can deduce the action of ψ on the other series when k is odd:

$$ \psi({}_k\hat{y}(t)) = {}_k\hat{y}(t)\otimes{}_k\breve{y}(t) + {}_k\breve{y}(t)\otimes{}_k\hat{y}(t) \tag{3.4.9} $$

$$ \psi({}_k\breve{y}(t)) = {}_k\hat{y}(t)\otimes{}_k\hat{y}(t) + {}_k\breve{y}(t)\otimes{}_k\breve{y}(t) \tag{3.4.10} $$

The double suspension acts as follows:

$$ e^{\circ 2}\circ{}_{(k+1)}y(t) = \mathrm{dlog}({}_ky(t))/\mathrm{d}t \tag{3.4.11} $$

$$ e^{\circ 2}\circ{}_{(k+1)}\breve{y}(t) \equiv 0 \tag{3.4.12} $$

We also have:

$$\begin{aligned}
e\circ[\alpha] &\equiv 0 && (3.4.13)\\
e\circ[\alpha^2] &\equiv 0 && (3.4.14)\\
[\alpha]\circ {}_ky(t) &= 1 && (3.4.15)\\
[\beta]\circ {}_ky(t) &= {}_{(k+2)}y(t)^2 && (3.4.16)\\
{}_ky(s)\circ {}_ly(t) &= {}_{(k+l)}y(t+s)\ {}_{(k+l)}y(t-s)^{\epsilon} \qquad \epsilon=(-1)^l && (3.4.17)
\end{aligned}$$

From this we can readily compute circle products of the other series we have mentioned, using standard Hopf ring properties. For example

$${}_1\hat{y}(s)\circ {}_1\hat{y}(t) \equiv \frac{1}{2}\left(\frac{{}_2y(s+t)}{{}_2y(s-t)} - \frac{{}_2y(s-t)}{{}_2y(s+t)}\right) \qquad (3.4.18)$$

The mod 2 reduction $\rho\colon \mathrm{H}_* \to \mathrm{H}\mathbb{F}_*$ satisfies

$$\begin{aligned}
\rho({}_0y(t^2)) &= z(t)^2 && (3.4.19)\\
\rho(e\circ {}_0y_{4k}/2) &= h_{4k+1} && (3.4.20)\\
&= e\circ z_{4k} + (e\circ z_{2k})(e\circ z_{2k-1}) && (3.4.21)\\
\rho({}_1y(t^2)) &= [\alpha^2]\circ z(t) && (3.4.22)\\
\rho(e\circ {}_1y(t^2)) &= [\alpha]\circ q(t) && (3.4.23)\\
\rho({}_2y(t^2)) &= [\beta]\circ z(t) && (3.4.24)\\
\rho({}_3\hat{y}(t^2)/2) &= \sum_k e^{\circ 2}\circ[\lambda]\circ z_{4k}t^{4k+2} && (3.4.25)\\
&= [\lambda]\circ(t^2e^{\circ 2} + (t^2e^{\circ 2})^2)\circ z(t) && (3.4.26)
\end{aligned}$$

More precisely, for each $k\geq 0$ there is a unique element $a\in \mathrm{H}_{4k+1}(U/O)$ such that $2a = e\circ {}_0y_{4k}$ and $\rho(a) = h_{4k+1}$; we call this element $e\circ {}_0y_{4k}/2$. Of course, the image of a in $\overline{\mathrm{H}}_{4k+1}(U/O)$ is already fixed by the first of the above two criteria. On the other hand, ${}_3\hat{y}_{4k+2}/2$ lies in $\mathrm{H}_{4k+2}(Sp/U)$ which is torsion free, so it is uniquely determined by the requirement that it gives ${}_3\hat{y}_{4k+2}$ on multiplication by 2.

THEOREM 3.4.1. The torsion free Hopf ring for KO is as follows:

$$\begin{aligned}
\overline{\mathrm{H}}_*(\mathbb{Z}\times BO) &= P[{}_4y_{4k}][-\lambda] && {}_4y_0=[2\lambda]\\
\overline{\mathrm{H}}_*(U/O) &= E[e\circ {}_4y_{4k}/2] &&\\
\mathrm{H}_*(Sp/U) &= P[{}_3y_{4k+2}/2] &&\\
\mathrm{H}_*(Sp) &= E[e\circ {}_3y_{4k+2}/2] &&\\
\mathrm{H}_*(\mathbb{Z}\times BSp) &= P[{}_2y_{4k}][-\beta] && {}_2y_0=[\beta]\\
\mathrm{H}_*(U/Sp) &= E[e\circ {}_2y_{4k}] &&\\
\mathrm{H}_*(O/U) &= D[{}_1y_{4k+2}]\otimes\mathbb{Z}[\mathbb{F}\alpha^2] &&\\
\overline{\mathrm{H}}_*(O) &= E[e\circ {}_1y_{4k+2}]\otimes\mathbb{Z}[\mathbb{F}\alpha] &&\\
\overline{\mathrm{H}}_*(\mathbb{Z}\times BO) &= P[{}_0y_{4k}][-1] && {}_0y_0=[2]
\end{aligned} \qquad (3.4.27)$$

□

For five of the eight spaces, there is no torsion in the integral homology so again $\overline{\mathrm{H}}_* = \mathrm{H}_*$. The complexification map $m_O\colon \overline{\mathrm{H}}_*\, \underline{KO}_* \longrightarrow \mathrm{H}_*\, \underline{KU}_*$ is injective except on O and O/U. The space O has two components SO and αSO , and $*$-multiplication by the homotopy element α is an involution which exchanges them. As $m_O(\alpha)=0$, the composites $SO \xrightarrow{m_O} U$ and $SO \xrightarrow{\alpha} \alpha SO \xrightarrow{m_O} U$ are equal. The homology of each component is mapped injectively, but $m_O[\alpha]=[0]=1$ so $m_O([\alpha]x) = m_O(x)$. The case of O/U is similar, with α^2 replacing α.

3.5. The Torsion Free Hopf Ring for KT

Proofs for the self conjugate case are in section 7.10. We consider the series ${}_kx(t) = \delta_T([\nu^k] \circ y(t))$. We have

$$\begin{array}{rcll} \psi({}_kx(t)) & = & {}_kx(t) \otimes {}_kx(t) & \\ [\mu] \circ {}_kx(t) & = & {}_{(k+2)}x(t) & \\ e^{\circ 2} \circ {}_{(k+1)}x(t) & = & \mathrm{dlog}({}_kx(t))/\mathrm{d}t & \\ {}_1x_0 & = & [\gamma] & \\ {}_kx(-t) & = & {}_kx(t) & k \text{ odd} \\ 2\ {}_kx_{4l+2} & = & 0 & k \text{ odd} \\ {}_kx(-t) & = & {}_kx(t)^{-1} & k \text{ even} \\ {}_kx_{4l} & \in & P[{}_kx_{4l+2}/2][{}_kx_0^{\pm 1}] & k \text{ even} \end{array} \tag{3.5.1}$$

We also denote $l_O({}_ky(t)) = m_U([\nu^k] \circ y(t))$ simply by ${}_ky(t)$. We find that

$$[\mu] \circ {}_ky(t) = {}_{(k+2)}y(t) \tag{3.5.2}$$

$$[\gamma] \circ {}_ky(t) = {}_{(k+1)}x(t)^2 \tag{3.5.3}$$

$$e \circ [\gamma] \circ {}_ky(t) = 2e \circ {}_{(k+1)}x(t) \tag{3.5.4}$$

$${}_ky(s) \circ {}_ly(t) = {}_{(k+l)}y(t+s)\ {}_{(k+l)}y(t-s)^{\epsilon} \qquad \epsilon = (-1)^l \tag{3.5.5}$$

$${}_kx(s) \circ {}_ly(t) = {}_{(k+l)}x(s+t)\ {}_{(k+l)}x(s-t)^{\epsilon} \qquad \epsilon = (-1)^l \tag{3.5.6}$$

$${}_kx(s) \circ {}_lx(t) = 1 \tag{3.5.7}$$

The reduction mod 2 is as follows:

$$\rho({}_0y(t^2)) = z(t)^2 \tag{3.5.8}$$

$$\rho(e \circ {}_0y_{4k}/2) = h_{4k+1} \tag{3.5.9}$$

$$\rho({}_1y_{4k+2}/2) = e^{\circ 2} \circ [\mu] \circ z_{4k} \tag{3.5.10}$$

$$\rho(e \circ {}_1y_{4k+2}/2) = e^{\circ 3} \circ [\mu] \circ z_{4k} \tag{3.5.11}$$

$$\rho({}_0x(t^2)) = [\alpha] \circ z(t) \tag{3.5.12}$$

$$\rho(e \circ {}_0x(t^2)) = q(t) \tag{3.5.13}$$

$$\rho({}_1x(t^2)) = [\gamma] \circ z(t) \tag{3.5.14}$$

THEOREM 3.5.1. The torsion free Hopf ring for KT is as follows:

$$\begin{array}{rcl} \overline{\mathrm{H}}_*(\mathbb{Z}\times BT) & = & P[{}_2y_{4k}][-\mu] \otimes E[e \circ {}_2x_{4k+2}] \\ \overline{\mathrm{H}}_*(\Omega^2 T) & = & E[e \circ {}_2y_{4k}/2] \otimes P[{}_1x_{4k}][-\gamma] \\ \mathrm{H}_*(\Omega T) & = & P[{}_1y_{4k+2}/2] \otimes E[e \circ {}_1x_{4k}] \\ \mathrm{H}_*(T) & = & E[e \circ {}_1y_{4k+2}/2] \otimes D[{}_0x_{4k+2}] \otimes \mathbb{Z}[\mathbb{F}\alpha] \\ \overline{\mathrm{H}}_*(\mathbb{Z}\times BT) & = & P[{}_0y_{4k}][-\mu] \otimes E[e \circ {}_0x_{4k+2}] \end{array} \tag{3.5.15}$$

□

CHAPTER 4

Hopf Algebras and Hopf Rings

4.1. Hopf Algebras

In this section we recall a little theory of Hopf algebras, mainly as background to the discussion of Hopf rings in the next section. We shall only consider biassociative bicommutative Hopf algebras. The material in this section comes mainly from [**17**].

Let k_* be a graded commutative ring (commutative will always mean in the graded sense). In the rest of this thesis, k_* is either $\mathbb{F} = \mathbb{Z}/(2)$ or $\mathbb{Z}$, concentrated in degree zero, or a ring of formal power series over one of these.

Let $\mathcal{M}_*$ denote the category of graded (left) modules over k_*. Note that a graded left module is a graded right module via $mr = (-1)^{|r||m|}rm$. We can define a tensor product on $\mathcal{M}_*$ by

$$(M_* \otimes_{k_*} N_*)_n = \bigoplus_{n=s+t} M_s \otimes_{\mathbb{Z}} N_t/(mr \otimes n - m \otimes rn) \tag{4.1.1}$$

and a twist map

$$\tau \colon M \otimes N \longrightarrow N \otimes M \qquad\qquad m \otimes n \mapsto (-1)^{|m||n|} n \otimes n \tag{4.1.2}$$

This makes $\mathcal{M}_*$ into a symmetric monoidal category. Let $\mathcal{C}_*$ denote the category of commutative comonoid objects in $\mathcal{M}_*$. Such a beast is an object C of $\mathcal{M}_*$ equipped with maps

$$k \xleftarrow{\epsilon} C \xrightarrow{\psi} C \otimes C \tag{4.1.3}$$

such that the usual diagram commutes:

$$\begin{tikzcd}
C \otimes C \arrow[d, "1 \otimes \epsilon"'] & C \otimes C \arrow[l, "\tau"'] \arrow[r, "\psi \otimes 1"] & C \otimes C \otimes C \\
C & C \arrow[l, "1"] \arrow[r, "\psi"'] \arrow[u, "\psi"'] \arrow[ul, "\psi"] & C \otimes C \arrow[u, "1 \otimes \psi"']
\end{tikzcd} \tag{4.1.4}$$

Given objects C and D of $\mathcal{C}_*$, we can take $\epsilon_C \otimes \epsilon_D$ as an augmentation and the following composite as a coproduct on $C \otimes D$:

$$C \otimes D \xrightarrow{\psi_C \otimes \psi_D} C \otimes C \otimes D \otimes D \xrightarrow{1 \otimes \tau \otimes 1} C \otimes D \otimes C \otimes D \tag{4.1.5}$$

This makes $C \otimes D$ into an object of $\mathcal{C}_*$. The maps

$$C \xleftarrow{1 \otimes \epsilon_D} C \otimes D \xrightarrow{\epsilon_C \otimes 1} D \tag{4.1.6}$$

are $\mathcal{C}_*$-morphisms and present $C \otimes D$ as the categorical product of C and D in $\mathcal{C}_*$. This means that given $\mathcal{C}_*$-maps $f \colon B \to C$ and $g \colon B \to D$ there is a unique $\mathcal{C}_*$-map $h \colon B \to C \otimes D$ such that $(1 \otimes \epsilon_D)h = f$ and $(\epsilon_C \otimes 1)h = g$.

By a C-comodule we mean an object M of $\mathcal{M}_*$ equipped with a coaction map $\alpha\colon M \to C \otimes M$ making the usual counit and coassociativity diagrams commute. We shall not make this explicit here.

Let $\mathcal{T}_k$ denote the category of topological spaces such that $\mathrm{H}_*(X; k_*)$ is flat over k_*. This is a symmetric monoidal category under the Cartesian product, and $\mathrm{H}_*(-; k_*)$ is a symmetric monoidal functor $\mathcal{T}_k \to \mathcal{C}_*$.

By a Hopf algebra, we shall mean a commutative group object in $\mathcal{C}_*$. (The same name is often used elsewhere for weaker concepts). A Hopf algebra A thus has

- a multiplication map $\sigma\colon A \otimes A \to A$
- a unit map $\eta\colon A \to k$
- an inversion map ("antipode") $\chi\colon A \to A$

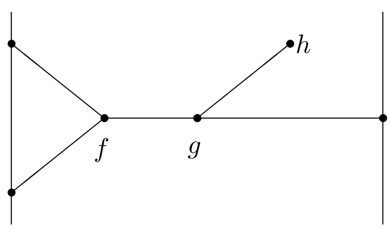
The multiplication is commutative and associative, with unit η. This is equivalent to the commutativity of a diagram similar to (4.1.4), but with the arrows reversed. It is supposed that σ and η are coalgebra maps, or equivalently that ψ and ϵ are algebra maps. This comes down to the commutativity of the following diagram:

$$\begin{tikzcd}
k & A \otimes A \arrow[l, "\epsilon \otimes \epsilon"'] \arrow[r, "\psi \otimes \psi"] \arrow[dl, "\sigma"] & A \otimes A \otimes A \otimes A \arrow[dd, "1 \otimes \tau \otimes 1"] \\
A \arrow[u, "\epsilon"] \arrow[dr, "\psi"] & & \\
k \arrow[u, "\eta"] \arrow[uu, bend left=60, "1"] \arrow[r, "\eta \otimes \eta"'] & A \otimes A & A \otimes A \otimes A \otimes A \arrow[l, "\sigma \otimes \sigma"]
\end{tikzcd} \tag{4.1.7}$$

Finally, the following diagram characterises the antipode:

$$\begin{tikzcd}
A \arrow[r, "\psi"] \arrow[d, "\epsilon"'] & A \otimes A \arrow[r, "\chi \otimes 1"] & A \otimes A \arrow[d, "\sigma"] \\
k \arrow[rr, "\eta"'] & & A
\end{tikzcd} \tag{4.1.8}$$

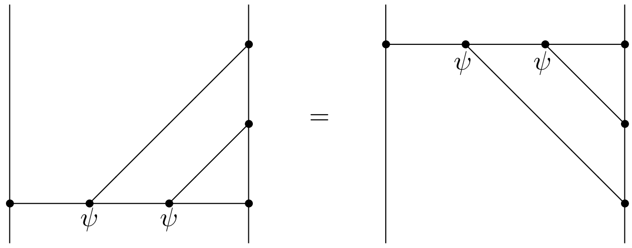
To relieve the tedium of yet another exposition of this material, we can convert the commutative diagrams above into Penrose diagrams *à la* Joyal [**12**]. These diagrams are most easily explained by an example. If $f\colon A \otimes B \to C$ and $g\colon C \to D \otimes E$ and $h\colon D \to k$ then the picture:



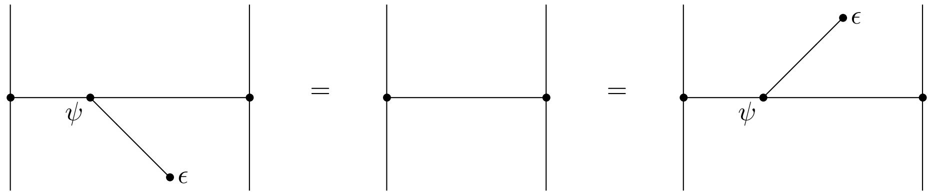
represents the map

$$A \otimes B \xrightarrow{f} C \xrightarrow{g} D \otimes E \xrightarrow{h \otimes 1} E$$

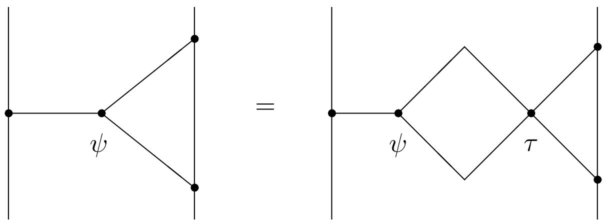
Coassociativity of ψ:



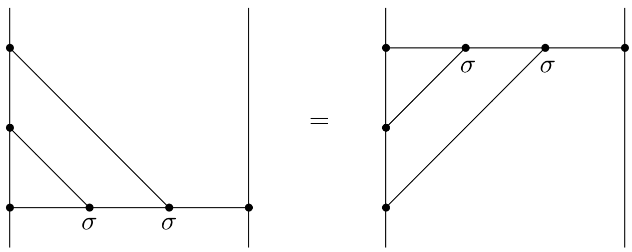
Counitary properties of ϵ:



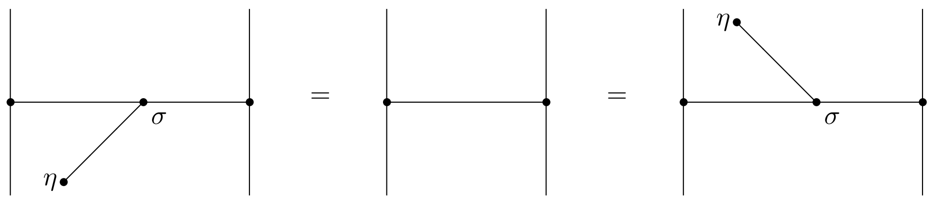
Cocommutativity of ψ:



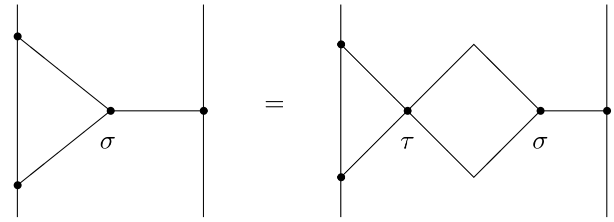
Associativity of σ:



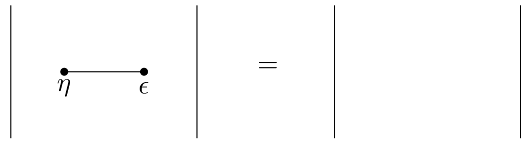
Unitary properties of η:



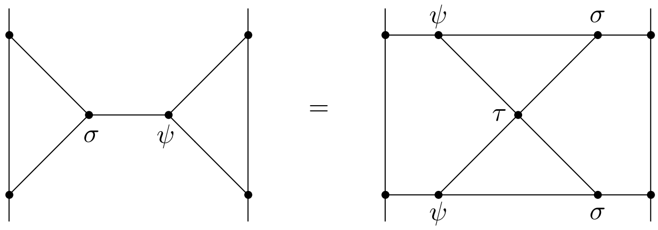
Commutativity of σ:



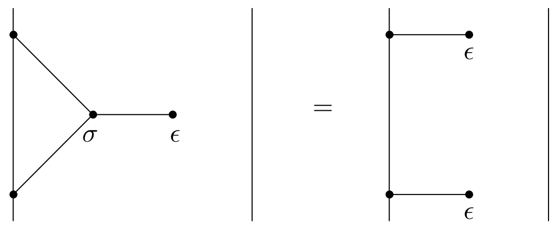
Interaction of η and ϵ:



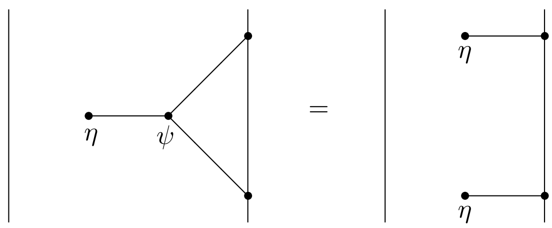
Interaction of ψ and σ:



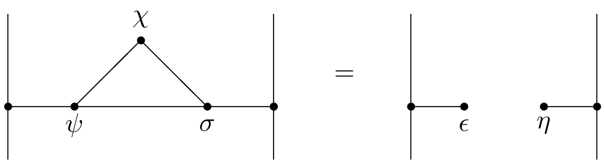
Interaction of σ and ϵ:



Interaction of ψ and η:



Inverse property of χ:



If G is an H-space (which we shall take to mean: an Abelian group object in the pointed homotopy category) and $\mathrm{H}_*(G;k_*)$ is flat over k_*, then $\mathrm{H}_*(G;k_*)$ becomes a Hopf algebra in an evident way. The product and coproduct are induced by the multiplication $G \times G \to G$ and the diagonal $G \to G \times G$ respectively. If G is a discrete Abelian group (which we write additively), then this is just the group ring $k_*[G]$, which is a free k_*-module on one generator $[g]$ for each element g of G. The structure maps are:

$$\begin{aligned} \eta(1) &= [0] && (4.1.9)\\ \epsilon[g] &= 1 && (4.1.10)\\ [g][h] &= [g+h] && (4.1.11)\\ \psi[g] &= [g]\otimes[g] && (4.1.12)\\ \chi[g] &= [-g] && (4.1.13) \end{aligned}$$

Even if G is not discrete, $k_*[\pi_0 G] = \mathrm{H}_0(G; k_*)$ is a sub-Hopf algebra of $\mathrm{H}_*(G; k_*)$.

Given a Hopf algebra A_*, we define various other algebraic gadgets:

$$\begin{aligned} \tilde{A}_* &= \ker(\epsilon\colon A_* \to k_*) && (4.1.14)\\ \mathcal{Q}_*A &= \tilde{A}/\tilde{A}^2 && (4.1.15)\\ \mathcal{P}_*A &= \{a \in A \mid \psi a = a\otimes 1 + 1\otimes a\} && (4.1.16)\\ \mathcal{G}A &= \{a \in A \mid \psi a = a \otimes a\} && (4.1.17) \end{aligned}$$

These are called the augmentation ideal, the indecomposable quotient, the submodule of primitives and the group of grouplike elements, respectively. The reason for the term grouplike comes from the example above, of course. Grouplike elements necessarily lie in dimension zero. If $A_k = 0$ for $k < 0$ then grouplike elements are rather thin on the ground. However, we shall often (implicitly) consider algebras like $\mathrm{H}_*(G; \mathbb{F}[\![t]\!])$ where $\dim t < 0$, in which they are abundant. (Strictly speaking, we should say something at this point about completed tensor products, but this can safely be glossed over.) On the other hand, there is a natural Hopf algebra map $k[\mathcal{G}A] \to A$, which is mono if k is a field.

We say that an element is indecomposable if it lies in $\tilde{A} \setminus \tilde{A}^2$ (and so maps nontrivially to $\mathcal{Q}_*A$).

Note that if k is an $\mathbb{F}$-algebra then the Frobenius map $F\colon a \mapsto a^2$ is a Hopf algebra morphism, except that it doubles degrees. In particular, it induces a map $\mathcal{P}_*A \to \mathcal{P}_{2*}A$.

If A and B are Hopf algebras over k, then we can define a Hopf algebra structure on $A \otimes_k B$ in a natural way. The maps

$$A \xrightarrow{1\otimes\eta} A\otimes B \xleftarrow{\eta\otimes 1} B \tag{4.1.18}$$

$$A \xleftarrow{1\otimes\epsilon} A\otimes B \xrightarrow{\epsilon\otimes 1} B \tag{4.1.19}$$

are maps of Hopf algebras, and this presents $A\otimes B$ as the biproduct (=simultaneous product and coproduct) of A and B. This makes the category $\mathcal{H}$ of Hopf algebras over k into an additive category. The sum of two maps $f, g\colon A \to B$ is the composite

$$A \xrightarrow{\psi} A\otimes A \xrightarrow{f\otimes g} B\otimes B \xrightarrow{\sigma} B \tag{4.1.20}$$

and the negative of f is just $\chi \circ f = f \circ \chi$. The functors above behave as follows:

$$\begin{aligned} \mathcal{Q}(A\otimes B) &\simeq \mathcal{Q}A \oplus \mathcal{Q}B && (4.1.21)\\ \mathcal{P}(A\otimes B) &\simeq \mathcal{P}A \oplus \mathcal{P}B && (4.1.22)\\ \mathcal{G}(A\otimes B) &\simeq \mathcal{G}A \times \mathcal{G}B && (4.1.23) \end{aligned}$$

Suppose we are given a diagram in $\mathcal{H}$:

$$A \xleftarrow{f} B \xrightarrow{g} C \tag{4.1.24}$$

The tensor product $A \otimes_B C$ inherits a Hopf algebra structure. It is the pushout of the diagram. In particular, the cokernel in $\mathcal{H}$ of a map $B \to A$ is

$$A//B = A \otimes_B k = A/(B\tilde{A}) \tag{4.1.25}$$

Note that $A \otimes_B C$ is the coequaliser of the two maps:

$$A \otimes B \otimes C \xrightarrow{1\otimes f\otimes 1} A \otimes A \otimes C \xrightarrow{\sigma\otimes 1} A \otimes C \tag{4.1.26}$$

and

$$A \otimes B \otimes C \xrightarrow{1\otimes g\otimes 1} A \otimes C \otimes C \xrightarrow{1\otimes\sigma} A \otimes C \tag{4.1.27}$$

Dually, suppose we have a diagram:

$$A \xrightarrow{f} B \xleftarrow{g} C \tag{4.1.28}$$

We define the cotensor product by$A\Box_B C$ as the equaliser of

$$A \otimes C \xrightarrow{\psi\otimes 1} A \otimes A \otimes C \xrightarrow{1\otimes f\otimes 1} A \otimes B \otimes C \tag{4.1.29}$$

and

$$A \otimes C \xrightarrow{1\otimes\psi} A \otimes C \otimes C \xrightarrow{1\otimes g\otimes 1} A \otimes B \otimes C \tag{4.1.30}$$

This is an k-algebra, and even a Hopf algebra provided k is a field. If so, it is the $\mathcal{H}$-pullback of f and g. In particular, the Hopf algebra kernel of $g\colon B \to C$ is

$$B\Box_C k = \{b \in B \mid (1 \otimes g)(\psi b) = b \otimes 1\} \tag{4.1.31}$$

Note that this contains all primitives $b \in \mathcal{P}B$ such that $g(b) = 0$.

To define $M \otimes_A N$, we only need M and N to be A-modules. The tensor product is right exact on the category of such modules, and the derived functors are written $\mathrm{Tor}^A_{**}(M,N)$. Dually, if k is a field then $M\Box_A N$ is a left exact functor on A-comodules with derived functors $\mathrm{Cotor}^A_{**}(M,N)$. For more details, and the case when k is not a field, see [**20**, Appendix A].

Given a short exact sequence $G \longrightarrow H \longrightarrow K$ of groups, a useful technique is to choose a transversal to G in H, or equivalently to choose maps of sets $G \longleftarrow H \longleftarrow K$ which split the sequence. This gives rise to an isomorphism $G \times K \to H$ of G-sets, if not of groups. The following theorem provides a nice Hopf-algebraic analogue. We shall assume for simplicity that k is a field, concentrated in dimension 0. A Hopf algebra A is said to be connected if $A_n = 0$ for $n < 0$ and $\eta\colon k \to A_0$ is iso.

Theorem 4.1.1 (Milnor-Moore). *Suppose that $A \rightarrowtail B \twoheadrightarrow C$ is a sequence of connected Hopf algebras, with $C = B//A$. (Note that $A \otimes C$ is an A-module and a C-comodule). Then there is a bijective map $f\colon A \otimes C \longrightarrow B$ which is both A-linear and C-colinear, and which makes the following diagram commute:*

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, tail] & B \arrow[d, two heads] \\
A \otimes C \arrow[ur, "f"] \arrow[r, two heads] & C
\end{tikzcd} \tag{4.1.32}$$

For the proof, see [**17**, 4.7]. It follows from the conclusion of the theorem that $A = B\Box_C k$. This is a major step towards the proof that the category of connected Hopf algebras is Abelian.

Suppose again that A is connected. Taking into account $(\epsilon \otimes 1) \circ \psi = 1 = (1 \otimes \epsilon) \circ \psi$, we see that any element of minimal degree in $\tilde{A} \setminus 0$ is primitive and indecomposable. Similarly, if $f \colon A \to B$ is a map of connected Hopf algebras, then the kernel is a subcoalgebra of $\tilde{A}$ and a nonzero element of minimal dimension is forced to be primitive. Thus, if $\mathcal{P}f \colon \mathcal{P}A \to \mathcal{P}B$ is mono, then f itself is mono. By a similar argument, if $\mathcal{Q}f$ is epi then f is.

4.2. Algebraic Theory of Hopf Rings

In this section we review the general theory of Hopf Rings. A Hopf ring is a graded ring object in the category $\mathcal{C}_*$ of coalgebras defined in the last section. To be more explicit, it is a sequence $\{A_{*t}\}_{t\in\mathbb{Z}}$ of objects of $\mathcal{M}_*$, equipped with structure maps as follows:

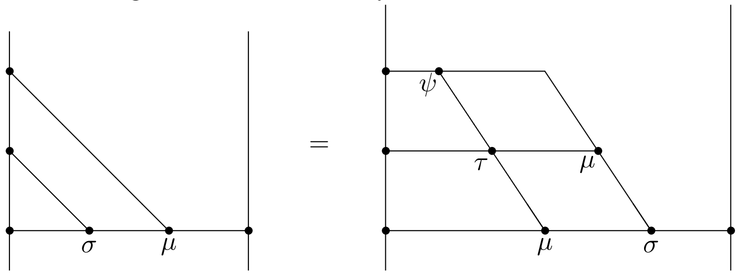
$$\begin{tikzcd}
A_{*t} \otimes_{k_*} A_{*t} \arrow[d, shift right, "\sigma"'] & A_{*t} \otimes_{k_*} A_{*s} \arrow[d, "\mu"] & \\
A_{*t} \arrow[u, shift right, "\psi"'] \arrow[d, shift right, "\epsilon"'] \arrow[loop left, "\chi"] & A_{*,t+s} & A_{*0} \\
k_* \arrow[u, shift right, "\eta"'] & & k_* \arrow[u, "\theta"']
\end{tikzcd} \tag{4.2.1}$$

The maps ψ and ϵ are of course just the $\mathcal{C}$-structure maps of A_{*t}. The maps η, σ and χ are the zero, addition and negation of the underlying graded Abelian group object of the graded ring object A. They make A_{*t} into a Hopf algebra. The multiplication and multiplicative identity of the ring object are the maps μ and θ. In addition to the Hopf algebra diagrams for(A, ψ, σ), we require that

- μ defines a commutative and associative multiplication, with unit θ.
- μ and θ are coalgebra maps, so there is a commutative diagram analogous to 4.1.7.
- The following diagram commutes ("distributivity"):

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\psi \otimes 1"'] \arrow[d, "1 \otimes \sigma"'] & A \otimes A \otimes A \otimes A \arrow[r, "1 \otimes \tau \otimes 1"'] & A \otimes A \otimes A \otimes A \arrow[r, "\mu \otimes \mu"'] & A \otimes A \arrow[d, "\sigma"] \\
A \otimes A \arrow[rrr, "\mu"'] & & & A
\end{tikzcd}$$

The Penrose diagram for distributivity is:



(4.2.2)

We write $a * b$ or just ab for $\sigma(a \otimes b)$ and $a \circ b$ for $\mu(a \otimes b)$. We refer to σ and μ as the star and circle products. Also, we write

$$\psi(a) = \sum a' \otimes a'' \tag{4.2.3}$$

In this notation, the characterisation of χ is

$$\sum a' \chi(a'') = \eta\epsilon(a) \tag{4.2.4}$$

and the distributivity law says

$$a \circ (b\, c) = \sum \pm (a' \circ b)(a'' \circ c) \tag{4.2.5}$$

where the sign is the usual one for exchanging a'' and b. Note in particular that

$$\begin{array}{rrcll} a \in \mathcal{G}A \Rightarrow & a \circ (b\, c) & = & (a \circ b)(a \circ c) & \\ a \in \mathcal{P}A \Rightarrow & a \circ (b\, c) & = & (a \circ b)\epsilon(c) \pm \epsilon(b)(a \circ c) & \\ \Rightarrow & a \circ b & = & 0 & \text{if } b \in \tilde{A}^2 \end{array} \tag{4.2.6}$$

We have used the fact that $1 \circ a = \eta\epsilon(a)$; to prove this, construct the diagram corresponding to the fact that $0x = 0$ in a ring.

A basic example of a Hopf ring is the "ring-ring" $A_{**} = k_*[\pi_*]$ of a graded ring π_*. In this case, A_{*t} is the group ring $k_*[\pi_t]$ of the additive group π_t. The structure formulae are:

$$\eta_t(1) = [0_t] \tag{4.2.7}$$
$$\epsilon[a] = 1 \tag{4.2.8}$$
$$[a][b] = [a+b] \tag{4.2.9}$$
$$[a] \circ [b] = [ab] \tag{4.2.10}$$
$$\psi[a] = [a] \otimes [a] \tag{4.2.11}$$
$$\chi[a] = [-a] \tag{4.2.12}$$

For a general Hopf ring A, we write $[0_t]$ or just $[0]$ for the identity element $\eta_t(1)$ in A_{0t}. We also define

$$[1] = \theta(1) \in A_{00} \tag{4.2.13}$$
$$[n] = [1]^n \qquad (n \geq 0) \tag{4.2.14}$$
$$[-n] = \chi[n] \tag{4.2.15}$$

One checks easily that these satisfy the equations given above for $k[\pi]$, so we have a Hopf ring map $k[\mathbb{Z}] \to A$. If A was a ring-ring in the first place, then these elements $[n]$ are what you think they are. We also have

$$[-1] \circ a = \chi(a) \tag{4.2.16}$$
$$[0] \circ a = \eta\epsilon(a) \tag{4.2.17}$$
$$[1] \circ a = a \tag{4.2.18}$$
$$[2] \circ a = \sum a'a'' \tag{4.2.19}$$

EXAMPLE.
Elsewhere in this thesis we have a Hopf ring over $\mathbb{F}$ containing elements z_k with

$$\begin{aligned}
\psi z_k &= \sum_{k=l+m} z_l \otimes z_m && (4.2.20)\\
z_k \circ z_l &= (k,l)z_{k+l} = (k+l)!/k!l!\ z_{k+l} && (4.2.21)\\
z_0 &= [1] && (4.2.22)\\
\epsilon z_k &= \delta_{k0} && (4.2.23)
\end{aligned}$$

We write $\overline{z}_k = z_k/z_0 = [-1]z_k$. Note that $\overline{z}_0 = [0] = 1$. Let us compute $\overline{z}_1 \circ \overline{z}_2$. First observe that

$$\begin{aligned}
\psi\overline{z}_1 &= \psi([-1])\psi(z_1)\\
&= ([-1]\otimes[-1])([1]\otimes z_1 + z_1\otimes[1])\\
&= [0]\otimes\overline{z}_1 + \overline{z}_1\otimes[0]
\end{aligned}$$

so $\overline{z}_1$ is primitive. Using this, the distributivity law, and the fact that $[0]\circ z_2 = \epsilon z_2 = 0$, we find

$$\begin{aligned}
\overline{z}_1\circ\overline{z}_2 &= \overline{z}_1\circ([-1]z_2)\\
&= ([0]\circ[-1])(\overline{z}_1\circ z_2) + (\overline{z}_1\circ[-1])([0]\circ z_2)\\
&= \overline{z}_1\circ z_2
\end{aligned}$$

Next, we use the equations $[-1]\circ a = \chi a$ and $\sum a'\chi a'' = \epsilon a$ to find $[-1]\circ z_k$ for $k = 1, 2$.

$$\chi[1] = [-1]\circ[1] = [-1] \tag{4.2.24}$$

$$z_1\chi[1] + [1]\chi z_1 = \epsilon z_1 = 0 \Longrightarrow [-1]\circ z_1 = [-1]z_1/[1] = [-2]z_1 \tag{4.2.25}$$

$$z_2\chi[1] + z_1\chi z_1 + [1]\chi z_2 = 0 \Longrightarrow [-1]\circ z_2 = [-2]z_2 + [-3]z_1^2 \tag{4.2.26}$$

The binomial coefficient $(1,2) = 3$ is odd, so $z_1\circ z_2 = z_3$. However, $(1,1) = 2$ so $z_1\circ z_1 = 0$. Thus

$$\begin{aligned}
\overline{z}_1\circ z_2 &= ([-1]z_1)\circ z_2\\
&= ([-1]\circ[1])(z_1\circ z_2) + ([-1]\circ z_1)(z_1\circ z_1) + ([-1]\circ z_2)(z_1\circ[1])\\
&= [-1]z_3 + 0 + ([-2]z_2z_1 + [-3]z_1^3)\\
&= \overline{z}_3 + \overline{z}_2\overline{z}_1 + \overline{z}_1^3 \qquad \square
\end{aligned}$$

We need a few comments about another way to construct Hopf rings. Suppose we start with a Hopf ring A and a sequence $\{C_{*t}\}_{t\in\mathbb{Z}}$ of coalgebras (all defined over k). We look for a universal example of a Hopf ring $B = A[C]$ equipped with a Hopf ring map $A \to B$ and a coalgebra map $C \to B$. We can construct such a thing by the Adjoint Functor Theorem [**13**] or by the methods of universal algebra. It is horribly large; we have to take the free graded-commutative algebra on C to serve as the module generated by the circle products of elements of C, take the tensor product with A and finally take the free graded-commutative algebra on *that* to produce star products. More generally, one can look for a universal solution modulo a given list of relations. Many known examples of Hopf rings that arise in topology (as explained in the next section) have a simple description in these terms (see [**21**], for example). In the cases described elsewhere in this thesis, the answers are even of a tolerable size.

4.3. The Hopf Ring Associated to an Ω Spectrum

Let E be a ring spectrum. We write $\underline{E}_t$ for the t'th space of the associated Ω-spectrum, and $\underline{E}'_t$ for the base component in $\underline{E}_t$. Thus, if X is a space we have

$$E^n(X) = [X,\ \underline{E}_n] \tag{4.3.1}$$

(this means unreduced cohomology and unbased maps). Also, we are given a specified homotopy equivalence

$$\Omega\,\underline{E}_{n+1} = \Omega\,\underline{E}'_{n+1} \longrightarrow \underline{E}_n = \underline{\Sigma E}_{n-1} \tag{4.3.2}$$

One can show that

$$\underline{E}_n = \pi_{-n}E \times \underline{E}'_n \tag{4.3.3}$$

Suppose that for all n, $\mathrm{H}_*(\underline{E}_n; k)$ is flat over k. (We shall suppress the coefficients for the rest of this section.) Then $\{\underline{E}_t\}_{t\in\mathbb{Z}}$ is a graded ring object in the category $\mathcal{T}_k$. In the case of K-theory (of whatever brand)

- The addition $\underline{E}_t \times \underline{E}_t \to \underline{E}_t$ classifies Whitney sum of bundles. It agrees with the loop sum under the identification $\underline{E}_t = \Omega\,\underline{E}_{t+1}$.
- The multiplication $\underline{E}_t \times \underline{E}_l \to \underline{E}_{t+l}$ classifies the tensor product of bundles.
- The unit $1 \to \underline{E}_0$ classifies the one dimensional trivial bundle

It follows that $\mathrm{H}_*\,\underline{E}_*$ is a graded ring object in $\mathcal{C}$, that is, a Hopf ring.

We know that $\pi_0\,\underline{E}_n = \pi_{-n}E$ and that $\mathrm{H}_0\,\underline{E}_n = k[\pi_0\,\underline{E}_n]$. We conclude that the sub-Hopf ring $\mathrm{H}_0\,\underline{E}_*$ is just the ring-ring $k[\pi_{-*}E]$. We use the notation $[a]$ for the basis element of $\mathrm{H}_0\,\underline{E}_n$ corresponding to $a \in \pi_{-n}E$. Note that this extends the notation $[n]$ for $n \in \mathbb{Z}$ defined earlier.

Given $a \in \pi_nE$, recall that we have a stable map m_a:

$$\Sigma^n E \xrightarrow{a\wedge 1} E \wedge E \xrightarrow{\mu} E \tag{4.3.4}$$

This gives an infinite loop map $\underline{E}_m \to \underline{E}_{m-n}$. In homology, this sends x to $[a]\circ x$. It follows that this is a Hopf algebra morphism, commuting with Steenrod and Kudo-Araki operations.

The isomorphism

$$\mathrm{H}_1(\underline{E}'_1;\mathbb{Z}) \simeq \pi_1\,\underline{E}'_1 \simeq \tilde{E}^1S^1 \simeq \tilde{E}^0S^0 \ni 1 \tag{4.3.5}$$

gives a canonical element e of $\mathrm{H}_1\,\underline{E}_1$, called the fundamental class or the suspension class. The reason for the second name is the following fact:

$$e \circ x = s_*(x - \eta\epsilon(x)) \tag{4.3.6}$$

where s_* denotes the homology suspension. To prove this, one first notes that $e\circ[0] = \eta\epsilon(e) = 0$. Next, recall that the circle product $\mu\colon\ \underline{E}_t \times \underline{E}_l \to \underline{E}_{t+l}$ comes from a stable map $E\wedge E \to E$. Together with a little general nonsense, this yields the diagram

$$\begin{tikzcd}
\underline{E}_1 \wedge \underline{E}_n \arrow[dr] & \Sigma\,\underline{E}_n \arrow[l, "e\wedge 1"'] \arrow[r] \arrow[d] & \Sigma\Omega\,\underline{E}_{n+1} \arrow[d] \arrow[dl, "\mathrm{eval}"] \\
\underline{E}_1 \times \underline{E}_n \arrow[u] \arrow[r, "\mu"'] & \underline{E}_{n+1} & \underline{E}'_{n+1} \arrow[l, hook']
\end{tikzcd} \tag{4.3.7}$$

Recall that s_* is by definition the composite

$$\tilde{\mathrm{H}}_{s-1}(\Omega\ \underline{E}_{n+1}) \simeq \tilde{\mathrm{H}}_s(\Sigma\Omega\ \underline{E}_{n+1}) \xrightarrow{\mathrm{eval}_*} \tilde{\mathrm{H}}_s\ \underline{E}_{n+1} \tag{4.3.8}$$

Using this, it is easy to verify the claim.

Note also that e is carried by a based map $S^1 \to \underline{E}'_1$, so it is necessarily primitive.

CHAPTER 5

Spectral Sequences

In this chapter, we assemble the arsenal of spectral sequences which we will need to prove the claims made in chapters 2 and 3. We shall mostly use mod 2 coefficients. Much of the material can be found in [**16**].

5.1. The Bockstein Spectral Sequence

In this section we examine the problem of recovering the integral homology from the mod 2 homology. This material seems to be well known, but I do not know a good reference. The spectral sequence of the same name in [**6**] (for example) is not quite the same as ours. In the notation defined below, it converges to $(\mathrm{H}_*(C)/\text{torsion}) \otimes \mathbb{F}$, whereas ours converges to $\mathrm{H}_*(C)$.

Let C_* be a chain complex of free Abelian groups, with differential $d\colon C_k \to C_{k-1}$. Our spectral sequence is that associated to the filtration of C_* by the subcomplexes $\{2^s C_*\}$, but to understand it fully we need to look a little closer. Define

$$
\begin{array}{rcll}
Z^r_{st} & = & \{c \in 2^s C_{t-s} \mid dc \in 2^{s+r} C_{t-s-1}\} & (s \geq 0) \\
Z^r_{st} & = & 0 & (s < 0) \\
E^r_{st} & = & Z^r_{st}/(Z^{r-1}_{s+1,t+1} + dZ^{r-1}_{s-r+1,t-r+2}) &
\end{array}
\tag{5.1.1}
$$

One finds that the boundary map in C induces a differential

$$d_r\colon E^r_{s,t} \longrightarrow E^r_{s+r,t+r-1} \tag{5.1.2}$$

whose homology is naturally identified with E^{r+1}_{**}. In other words, we have a spectral sequence. Consider the polynomial ring $P[\tau]$ where τ has bidegree $(1,1)$. It acts on our spectral sequence by $\tau[c] = [2c]$. Multiplication by τ gives an isomorphism $E^r_{st} \longrightarrow E^r_{s+1,t+1}$ for $s \geq r-1$ and an epimorphism for $s \geq 0$. If $s < 0$ then $E^r_{st} = 0$. Thus E^r_{**} is generated over $P[\tau]$ by E^r_{0*}.

For $s \geq 0$ and $r \geq 1$ we have

$$\overline{Z}^r_{st} = Z^r_{st}/B^1_{st} \quad = \quad \mathrm{im}[\mathrm{H}_{t-s}C/2^r \to \mathrm{H}_{t-s}C/2]\tau^s \tag{5.1.3}$$

$$\overline{B}^r_{st} = B^r_{st}/B^1_{st} \quad = \quad \mathrm{im}[\mathrm{ann}(2^{\min(s,r-1)}, \mathrm{H}_{t-s}C) \to \mathrm{H}_{t-s}C/2]\tau^s \tag{5.1.4}$$

Thus an element $\tau^s x$ of E^r_{st} corresponds to an element u of C_{t-s} with $du = 2^r w$ say. The differential is given by $d_r(\tau^s x) = \tau^{r+s}[w] \in E^r_{s+r,t+r-1}$. Note also that given s and t, the groups B^r_{st} are constant for $r > s$.

Suppose that for each k there is an integer n_k such that 2^{n_k} annihilates the 2-torsion subgroup T_k of $\mathrm{H}_k C$. This holds if $\mathrm{H}_* C$ is finitely generated in each degree, for example. We have a morphism of short exact sequences:

$$\begin{tikzcd} C \arrow[r, tail, "2^r"'] \arrow[d, "2^{r-1}"'] & C \arrow[r, two heads] \arrow[d, "1"'] & C/2^r \arrow[d] \\ C \arrow[r, tail, "2"'] & C \arrow[r, two heads] & C/2 \end{tikzcd} \tag{5.1.5}$$

Consider the resulting map of Künneth sequences when $r > n_{t-s-1}$:

$$\begin{tikzcd} (\mathrm{H}_{t-s}C)/2^r \arrow[r, tail] \arrow[d] & \mathrm{H}_{t-s}(C/2^r) \arrow[r, two heads] \arrow[d] \arrow[dl] & (\mathrm{H}_{t-s-1}C) * \mathbb{Z}/2^r \arrow[d, "2^{r-1} = 0"] \\ (\mathrm{H}_{t-s}C)/2 \arrow[r, tail] & \mathrm{H}_{t-s}(C/2) \arrow[r, two heads] & (\mathrm{H}_{t-s-1}C) * \mathbb{Z}/2 \end{tikzcd} \tag{5.1.6}$$

According to the ancient tradition, we write $A{*}B$ for $\mathrm{Tor}(A, B)$. The diagonal map exists because the right hand vertical vanishes and the rows are short exact. This implies that

$$\overline{Z}^r_{st} = \overline{Z}^\infty_{st} = \mathrm{im}[\mathrm{H}_{t-s}C \to \mathrm{H}_{t-s}(C/2)]\tau^s \tag{5.1.7}$$

We conclude that

$$E^1_{st} = \tau^s \mathrm{H}_{t-s}(C/2) \qquad\qquad d_1 x = \beta(x)\tau \tag{5.1.8}$$

$$E^2_{**} = \ker[\beta\colon \mathrm{H}\mathbb{F}_*X \to \mathrm{H}\mathbb{F}_{*-1}X] \otimes P[\tau] \;/\; \mathrm{im}[\beta\colon \mathrm{H}\mathbb{F}_{*+1}X \to \mathrm{H}\mathbb{F}_*X] \otimes \tau P[\tau] \tag{5.1.9}$$

$$\begin{aligned} E^\infty_{s,t} &\simeq \mathrm{H}_{t-s}C/(2\mathrm{H}_{t-s}C + \mathrm{ann}(2^s, \mathrm{H}_{t-s}C)) \\ &\simeq 2^s\mathrm{H}_{t-s}C/2^{s+1}\mathrm{H}_{t-s}C \end{aligned}$$

This is the bigraded group associated to a filtration of $\mathrm{H}_*C/(\text{odd torsion})$.

Let $\tilde\beta$ denote the Bockstein $\mathrm{H}\mathbb{F}_* \to \mathrm{H}_{*-1}$ and ρ the reduction $\mathrm{H} \to \mathrm{H}\mathbb{F}$. Then $\rho\tilde\beta = \beta$ and $\ker[\tilde\beta] = \mathrm{image}[\rho]$. It follows that everything in $\mathrm{image}[\beta] \subseteq E^2_{0*}$ lifts to H_*C and is thus a permanent cycle. If the composite

$$\mathrm{H}_*C \to \mathrm{image}[\beta] \twoheadrightarrow \mathrm{H}_*(\mathrm{H}\mathbb{F}_*(C), \beta) \tag{5.1.10}$$

is epi, then it follows in turn that everything in E^2_{0*} is a permanent cycle. Using $P[\tau]$-linearity, we see that the whole sequence collapses. We can then read off from the E^∞-page that the 2-torsion subgroup is annihilated by 2 and that

$$\overline{\mathrm{H}}_*(C) \otimes \mathbb{F} \to \mathrm{H}_*(\mathrm{H}\mathbb{F}_*(C), \beta) \tag{5.1.11}$$

is iso, where $\overline{\mathrm{H}}_*(C) = \mathrm{H}_*(C)/\text{torsion}$.

On the other hand, suppose $\beta = 0$. It is elementary that there is then no even torsion. For if not, we can find x such that $0 \neq [x] \in \mathrm{H}\mathbb{F}_*C$ but $2[x] = 0$, so $2x = dy$ say. In that case, y is a cycle mod 2 with $\beta[y] = [x] \neq 0$ contrary to hypothesis.

Combining the above remarks with lemma 3.1.2, we obtain:

THEOREM 5.1.1. Let X be a space such that $\mathrm{H}_*(Y)$ is finitely generated in each dimension for each component Y of X. Suppose also that $\mathrm{H}[\frac{1}{2}]_*(X)$ is torsion free.

(1) If $\beta\colon \mathrm{H}\mathbb{F}_*(X) \to \mathrm{H}\mathbb{F}_{*-1}(X)$ vanishes then $\mathrm{H}_*(X)$ is torsion free and the Bockstein spectral sequence collapses.
(2) If $\mathrm{H}_*(X) \to \mathrm{H}_*(\mathrm{H}\mathbb{F}_*(X), \beta)$ is epi then the Bockstein spectral sequence collapses and $\overline{\mathrm{H}}_*(X) \otimes \mathbb{F} \to \mathrm{H}_*(\mathrm{H}\mathbb{F}_*(X), \beta)$ is iso.
(3) Suppose $\phi\colon A_* \to \mathrm{H}_*(X)$, where A_* is torsion free. If $A[\frac{1}{2}]_* \to \mathrm{H}[\frac{1}{2}]_*(X)$ and $A \otimes \mathbb{F} \to \mathrm{H}(\mathrm{H}\mathbb{F}_*(X), \beta)$ are iso, then ϕ is iso.

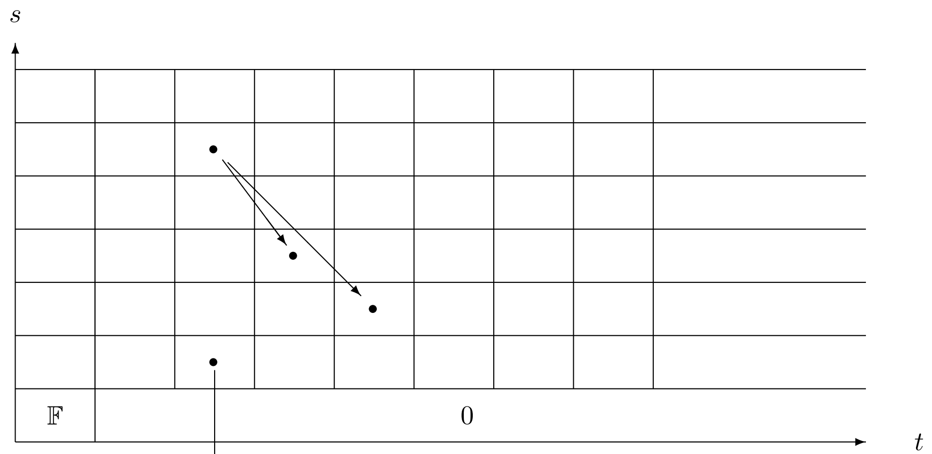
$\square$

5.2. The Rothenberg-Steenrod Spectral Sequence

Our next spectral sequence computes the homology of the classifying space of an infinite loop space. It is often called the bar spectral sequence. See [**26**] or [**22**] for more information and references.

$$E^2_{s,t} = \mathrm{Tor}^{H\mathbb{F}_* \underline{E}_k}_{s,t}(\mathbb{F}, \mathbb{F}) \Longrightarrow \mathrm{H}\mathbb{F}_{t+s}\, \underline{E}'_{k+1} \tag{5.2.1}$$

$$d_r \colon E^r_{s,t} \longrightarrow E^r_{s-r,t+r-1} \tag{5.2.2}$$



$$\begin{tikzcd}
E^2_{1,t} = \mathrm{Tor}^{H\mathbb{F}_* \underline{E}_k}_{1,t} \arrow[rr, two heads] & & E^\infty_{1,t} \arrow[d, tail] \\
\tilde{\mathrm{H}}\mathbb{F}_t\, \underline{E}_k \arrow[u, two heads, "\sigma"] \arrow[rr, "e \circ (-)"] & & \mathrm{H}\mathbb{F}_{t+1}\, \underline{E}'_{k+1}
\end{tikzcd} \tag{5.2.3}$$

The E^2 term can be computed using the bar resolution. Let A_* be a Hopf algebra (e.g. $A_* = \mathrm{H}\mathbb{F}_*\, \underline{E}_k$) and set

$$\Omega_{s,t} = (\tilde{A} \otimes \dots \tilde{A})_t \qquad (s \text{ factors }) \tag{5.2.4}$$

A typical element of $\Omega_{s,t}$ will be written as $\langle a_1 | \dots | a_s \rangle$ where $a_i \in \tilde{A}_{t_i}$ and $\sum_i t_i = t$. This modifies the usual notation slightly to avoid confusion with $[x] \in \mathrm{H}_0\, \underline{E}_k$ as defined previously. A boundary mapping is given by:

$$d\langle a_1|\dots|a_s\rangle = \sum_{i=1}^{s-1}\langle a_1|\dots|a_i a_{i+1}|\dots|a_s\rangle \tag{5.2.5}$$

There is also a product on Ω, making it a DGA:

$$\langle a_1|\dots|a_r\rangle\langle a_{r+1}|\dots|a_{r+s}\rangle = \sum\{\text{shuffles of } \langle a_1|\dots a_{r+s}\rangle\} \tag{5.2.6}$$

The homology of Ω agrees with our E^2 term $\mathrm{Tor}_{**}^{H_*\underline{E}_k} = \mathrm{Tor}_{**}^{H_*\underline{E}_k}(\mathbb{F},\mathbb{F})$ as an algebra.

Suppose $a \in \tilde{\mathrm{H}}_t\,\underline{E}_k$. Then $\sigma(a) = \langle a\rangle$ is a cycle in $\Omega_{1,|a|}$ and in fact a permanent cycle in the spectral sequence, representing $e \circ a$ which lies in the bottom filtration of $\mathrm{H}_{t+1}\,\underline{E}'_{k+1}$. Suppose further that $a^2 = 0$. Then $B_k(a) = \langle a|\dots a\rangle$($k$ factors) is again a cycle in Ω. It is immediate from the definitions that

$$B_k(a)B_l(a) = \frac{(k+l)!}{k!l!}B_{k+l}(a) \tag{5.2.7}$$

so, defining $B(s)(a) = \sum_k B_k(a)s^k$ we obtain

$$B(s)(a)B(t)(a) = B(s+t)(a) \tag{5.2.8}$$

As we are working over a field, the evident external product

$$\mathrm{Tor}_{**}^{A} \otimes \mathrm{Tor}_{**}^{B} \longrightarrow \mathrm{Tor}_{**}^{A\otimes B} \tag{5.2.9}$$

is an isomorphism. The standard examples of Tor algebras are:

$$\mathrm{Tor}_{**}^{P[x]} = E[\sigma(x)] \tag{5.2.10}$$

$$\mathrm{Tor}_{**}^{E[x]} = D[\sigma(x)] = \mathbb{F}\{B_k(x)|k \geq 0\} = \bigotimes_{l\geq 0} E[B_{2^l}(x)] \tag{5.2.11}$$

There is a circle product pairing

$$\Omega_{k,s,t} \otimes \mathrm{H}\mathbb{F}_r\,\underline{E}_l \longrightarrow \Omega_{k+l,s,t+r} \tag{5.2.12}$$

given by

$$\langle a_1|\dots|a_s\rangle \circ b = \sum\langle a_1 \circ b_{(1)}|\dots|a_s \circ b_{(s)}\rangle \tag{5.2.13}$$

where

$$\psi^{(s)}b = \sum b_{(1)} \otimes \dots b_{(s)} \tag{5.2.14}$$

is the iterated coproduct. This structure is compatible with the differentials:

$$d^r(\langle a_1|\dots|a_s\rangle \circ b) = (d^r\langle a_1|\dots|a_s\rangle) \circ b \tag{5.2.15}$$

It also converges to the usual circle product at E^∞. Note that if b is grouplike (i.e. $\psi b = b \otimes b$) then

$$\langle a_1|\dots|a_s\rangle \circ b = \langle a_1 \circ b|\dots|a_s \circ b\rangle \tag{5.2.16}$$

Provided that $\mathrm{H}_*\,\underline{E}_k$ is torsion free, there is an analogous spectral sequence for integral homology. One has to put in some signs, of course. We omit the details here.

5.3. The Eilenberg-Moore Spectral Sequence

We next consider the Eilenberg-Moore spectral sequence. The special case in which we are most interested is in a sense dual to the Rothenberg-Steenrod sequence above. It runs in the opposite direction ($\underline{E}_k \Rightarrow \underline{E}_{k-1}$ rather than $\underline{E}_{k-1} \Rightarrow \underline{E}_k$) and it uses the coalgebra structure via the Cotor functor rather than the algebra structure and the Tor functor. See [**16**] or [**9**] for the construction of this spectral sequence and its generalisations, and [**20**] for information about the Cotor functor.

We start with a pullback square:

$$\begin{tikzcd} X \arrow[r, "j_1"] \arrow[d, "j_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_1 \arrow[d, "p_1"] \\ Y_0 \arrow[r, "p_0"'] & Z \end{tikzcd} \tag{5.3.1}$$

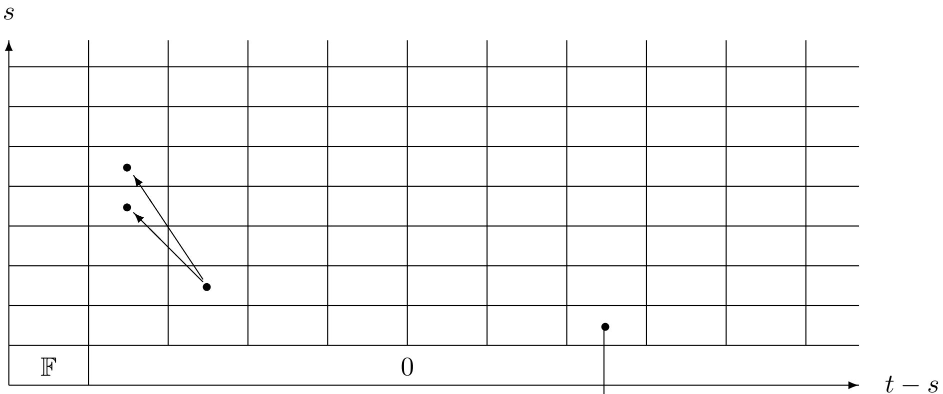
Suppose that at least one of p_0 and p_1 is a fibration. There is then a spectral sequence:

$$E^2_{s,t} = \mathrm{Cotor}^{H\mathbb{F}_* Z}_{s,t}(\mathrm{H}\mathbb{F}_* Y_0, \mathrm{H}\mathbb{F}_* Y_1) \Longrightarrow \mathrm{H}\mathbb{F}_{t-s} X \tag{5.3.2}$$

$$d_r \colon E^r_{s,t} \longrightarrow E^r_{s+r,t+r-1} \tag{5.3.3}$$

The most interesting case for us is when $Z = \underline{E}'_k$ and both Y_0 and Y_1 are the path space $\mathrm{P}\,\underline{E}'_k$ so that $X = \Omega\,\underline{E}'_k = \underline{E}_{k-1}$. If we abbreviate $\mathrm{Cotor}^{A_*}_{**}(\mathbb{F}, \mathbb{F})$ as just Cotor^{A}_{**} then we have

$$E^2_{s,t} = \mathrm{Cotor}^{H\mathbb{F}_* \underline{E}'_k}_{s,t} \Longrightarrow \mathrm{H}\mathbb{F}_{t-s}\,\underline{E}_{k-1} \tag{5.3.4}$$



$$\begin{tikzcd} E^\infty_{1,t} \arrow[r, tail] & E^2_{1,t} = \mathrm{Prim}\mathrm{H}\mathbb{F}_t\,\underline{E}'_k \arrow[d, tail] \\ \mathrm{H}\mathbb{F}_{t-1}\,\underline{E}_{k-1} \arrow[u, two heads] \arrow[r, "e \circ (-)"'] & \mathrm{H}\mathbb{F}_t\,\underline{E}'_k \end{tikzcd}$$

This is a spectral sequence of differential bigraded Hopf algebras. The coproduct on $E^{\infty}_{*,*}$ corresponds to that on $\mathrm{H}_* \, \underline{E}_{k-1}$, but the products need not correspond. The Hopf algebra $\mathrm{Cotor}^{H\mathbb{F}_* \underline{E}'_k}_{**}$ depends only on the coalgebra structure of $\mathrm{H}\mathbb{F}_* \, \underline{E}'_k$. It can be computed using the cobar resolution. Let A_* be a connected Hopf algebra over $\mathbb{F}$ (e.g. $A_* = \mathrm{H}\mathbb{F}_* \, \underline{E}'_k$) and set

$$\Upsilon_{s,t} = (\tilde{A} \otimes \dots \tilde{A})_t \qquad (s \text{ factors }) \tag{5.3.5}$$

$$d\langle a_1 | \dots | a_s \rangle = \sum_{i=1}^{s} \sum \langle a_1 | \dots a_{i-1} | a'_i | a''_i | \dots a_s \rangle \tag{5.3.6}$$

$$\langle a_1 | \dots a_r \rangle \langle a_{r+1} | \dots a_{r+s} \rangle = \langle a_1 | \dots a_{r+s} \rangle \tag{5.3.7}$$

This makes Υ_{**} into a homotopy commutative DGA, whose homology is $\mathrm{Cotor}^{A_*}_{**}$.

There is a Künneth theorem:

$$\mathrm{Cotor}^{A_* \otimes B_*}_{**} = \mathrm{Cotor}^{A_*}_{**} \otimes \mathrm{Cotor}^{B_*}_{**} \tag{5.3.8}$$

In the rest of this section $d = \deg(x)$. The basic examples of coalgebras are:

$$\begin{array}{rclll} E[x] & = & \mathbb{F}\{1, x\} & \deg x = d & \psi x = x \otimes 1 + 1 \otimes x \\ D[x] & = & \mathbb{F}\{x^{[k]} \mid k \geq 0\} & \deg x^{[k]} = kd & \psi x^{[n]} = \sum_{n=i+j} x^{[i]} \otimes x^{[j]} \\ P[x] & = & \mathbb{F}\{x^k \mid k \geq 0\} & \deg x^k = kd & \psi x^n = \sum_{n=i+j} (i,j) x^i \otimes x^j \end{array} \tag{5.3.9}$$

The corresponding Cotor algebras are:

$$\begin{array}{rcll} \mathrm{Cotor}^{E[x]} & = & P[a] & a = \langle x \rangle \in \mathrm{Cotor}_{1,d} \\ \mathrm{Cotor}^{D[x]} & = & E[b] & b = \langle x \rangle \in \mathrm{Cotor}_{1,d} \\ \mathrm{Cotor}^{P[x]} & = & P[a_i \mid i \geq 0] & a_i = \langle x^{2^i} \rangle \in \mathrm{Cotor}_{1,2^i d} \end{array} \tag{5.3.10}$$

5.4. The Serre Spectral Sequence

In this section we discuss the Serre spectral sequence, with particular reference to the case of an infinite-loop fibration. Some proofs are given at the end of the section. The rest of the material is standard.

Let $U \to V \to W$ be a cofibration of spectra. We then have a fibration of spaces:

$$\underline{U}_n \longrightarrow \underline{V}_n \longrightarrow \underline{W}_n \tag{5.4.1}$$

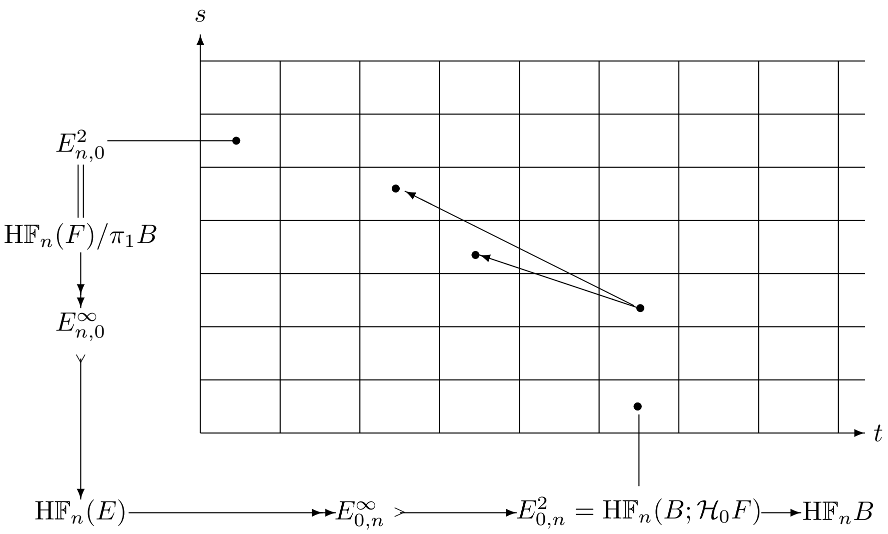
Let $\underline{W}''_n$ denote the image of the right hand map. As this map is a fibration, $\underline{W}''_n$ is the union of certain components of $\underline{W}_n$. Let us write

$$F = \underline{U}_n \tag{5.4.2}$$
$$F' = \underline{U}'_n \tag{5.4.3}$$
$$E = \underline{V}_n \tag{5.4.4}$$
$$B = \underline{W}''_n \tag{5.4.5}$$

so $F \to E \to B$ is still a fibration. Using simplicial methods [**15, 5**], we may assume that it is also a sequence of homomorphisms of topological groups (only Abelian up to homotopy, of course). We have a bundle of Abelian groups (= local coefficient system) $\mathcal{H}_0 F = \coprod_{b \in B} \mathbb{F}[\pi_0 E_b]$ over B. In the present context, the Serre spectral sequence is a spectral sequence of Hopf algebras, and it takes the form:

$$E^2_{s,t} = \mathrm{H}\mathbb{F}_s(B; \mathcal{H}_0 F) \otimes \mathrm{H}\mathbb{F}_t(F') \Longrightarrow \mathrm{H}\mathbb{F}_{t+s}(E) \tag{5.4.6}$$

$$d_r \colon E^r_{s,t} \longrightarrow E^r_{s-r,t+r-1} \tag{5.4.7}$$



If the boundary map $\pi_1 B \to \pi_0 F$ vanishes, then the local coefficient system is trivial and the E^2 page is just $\mathrm{H}\mathbb{F}_* B \otimes \mathrm{H}\mathbb{F}_* F$. However, in the worst cases we need two more spectral sequences to compute the initial term. Firstly, we need to calculate the homology of the universal cover $\tilde{B}$ of B. We write $\mathrm{H}\mathbb{F}_*(\pi_1 B)$ for the group homology, i.e. the homology of the classifying space. We use the Eilenberg-Moore spectral sequence of the following square:

$$\begin{tikzcd}
\tilde{B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E\pi_1 B \arrow[d] \\
B \arrow[r] & B\pi_1 B
\end{tikzcd} \tag{5.4.8}$$

Here $B\pi_1 B$ is the classifying space of the group $\pi_1 B$, and $E\pi_1 B$ is the (contractible) total space of the universal principal $\pi_1 B$-bundle.

$$E^2_{s,t} = \mathrm{Cotor}^{H\mathbb{F}_*(\pi_1 B)}_{s,t}(\mathbb{F}, \mathrm{H}\mathbb{F}_* B) \Longrightarrow \mathrm{H}\mathbb{F}_{t-s}\tilde{B} \tag{5.4.9}$$

$$d_r \colon E^r_{s,t} \longrightarrow E^r_{s+r,t+r-1} \tag{5.4.10}$$

We can then use a Künneth spectral sequence to calculate the homology with local coefficients.

$$E^2_{s,t} = \mathrm{Tor}^{\mathbb{F}[\pi_1 B]}_s(\mathrm{H}\mathbb{F}_t\tilde{B}, \mathbb{F}[\pi_0 F]) \Longrightarrow \mathrm{H}\mathbb{F}_{t+s}(B;\mathcal{H}_0 F) \tag{5.4.11}$$

$$d^r \colon E^r_{s,t} \longrightarrow E^r_{s-r,t+r-1} \tag{5.4.12}$$

This is explained in more detail at the end of the section.

Consider the two edge maps:

$$\begin{aligned}
\mathrm{H}\mathbb{F}_t(E) &\longrightarrow \mathrm{H}\mathbb{F}_t(B;\mathcal{H}_0 F) && (5.4.13)\\
\mathrm{H}\mathbb{F}_t(F)/\pi_1 B &\longrightarrow \mathrm{H}\mathbb{F}_t(E) && (5.4.14)
\end{aligned}$$

If the first is epi, then all differentials must vanish on E^r_{0*}. They also vanish on E^r_{*0}, for dimensional reasons. Using the algebra structure and induction on r, we find that the sequence collapses. This implies that the second edge map is mono. By looking at the dual spectral sequence, we find similarly that if the second edge map is mono then the sequences collapses and the first is epi. In this case, the maps

$$\mathrm{H}\mathbb{F}_*(F)/\pi_1 B \longrightarrow \mathrm{H}\mathbb{F}_*(E)\square_{H\mathbb{F}_*(B;\mathcal{H}_0F)}\mathbb{F} \tag{5.4.15}$$
$$\mathrm{H}\mathbb{F}_*(E)\otimes_{H\mathbb{F}_*F}\mathbb{F} \longrightarrow \mathrm{H}\mathbb{F}_*(B;\mathcal{H}_0F) \tag{5.4.16}$$

are iso.

We shall need a few facts about the transgression. For simplicity, we shall only consider the case in which the local coefficients are simple. The transgression (written τ) is then the following additive relation from $\mathrm{H}\mathbb{F}_*(B)$ to $\mathrm{H}\mathbb{F}_*(F)$:

$$\mathrm{H}\mathbb{F}_k(B) \twoheadrightarrow \mathrm{H}\mathbb{F}_k(B,*) \leftarrow \mathrm{H}\mathbb{F}_k(E,F) \xrightarrow{\partial} \mathrm{H}\mathbb{F}_{k-1}(F) \tag{5.4.17}$$

It agrees [**16**, section 6.1] with the additive relation

$$\mathrm{H}\mathbb{F}_k(B) \hookleftarrow E^k_{k,0} \xrightarrow{d_k} E^k_{0,k-1} \twoheadleftarrow \mathrm{H}\mathbb{F}_{k-1}(F) \tag{5.4.18}$$

Elements in the domain of τ are described as transgressive. An element $x \in \mathrm{H}\mathbb{F}_k(B)$ is transgressive if and only if it is a cycle for each of the differentials $d_2, \dots, d_{k-1}$. If so, $d_k(x)$ lies in the quotient $E^k_{0,k-1}$ of $\mathrm{H}\mathbb{F}_{k-1}(F)$; in particular, it is a coset in $\mathrm{H}\mathbb{F}_{k-1}(F)$. This coset is just $\tau(x)$. This construction is of course functorial. The most important application of this is as follows:

Lemma 5.4.1. If $s_*\colon \tilde{\mathrm{H}\mathbb{F}}_{k-1}(\Omega B) \to \mathrm{H}\mathbb{F}_k(B)$ is the homology suspension, and $\partial\colon \Omega B \to F$ is the usual connecting map, then $s_*(x)$ is transgressive with $\tau s_*(x) = \partial_*(x)$.

Proof. This follows by considering the following morphism of fibre sequences:

$$\begin{tikzcd} \Omega B \arrow[r] \arrow[d] & F \arrow[d] \\ PB \arrow[r] \arrow[d] & E \arrow[d] \\ B \arrow[r] & B \end{tikzcd} \tag{5.4.19}$$

One also needs to bear in mind the homotopy equivalence $(C\Omega B, \Omega B) \to (PB, \Omega B)$ which sends $(t,\omega) \in C\Omega B$ to the path $s \mapsto \omega(st)$. $\square$

A few of the statements above need some justification. Firstly, the Serre spectral sequence is usually discussed under the assumption that the base is connected. Provided that we restrict the base so that $E \to B$ is surjective (as we did) and then take into account the transitive action of the topological group E, we can ignore this hypothesis.

Secondly, let ω be a loop in B. To understand how it acts on F, we are required to find a homotopy $h_t\colon F \to E$ for $0 \le t \le 1$, such that h_0 is the inclusion and the composite

$$F \xrightarrow{h_t} E \longrightarrow B \tag{5.4.20}$$

is the constant map with value $\omega(t)$; the action of ω is given by $h_1 : F \to F$. To do this, we choose a path $\tilde{\omega} : I \to E$ lifting ω. Note that the image of ω under the boundary map $\partial : \pi_1 B \to \pi_0 F$ is just the component of $\tilde{\omega}(1)$. As we assume that our spaces are topological groups and our maps are homomorphisms, we can take $h_t(x) = x\tilde{\omega}(t)$. Thus, ω acts via multiplication by $\partial(\omega)$. If $a \in F$, then multiplication by a is a homeomorphism $F' \to F'a$ and $F'a$ is the component of a in F. Using this one sees that $\mathrm{H}\mathbb{F}_*(F) = \mathbb{F}[\pi_0 F] \otimes \mathrm{H}\mathbb{F}_*(F')$ as $\pi_1 B$-modules, where $\pi_1 B$ acts on the first factor via ∂ and trivially on the second factor.

Essentially by definition,

$$\mathrm{H}\mathbb{F}_*(B; \mathcal{H}_* F) = \mathrm{H}_*(C_*\tilde{B} \otimes_{\pi_1 B} \mathrm{H}\mathbb{F}_* F) = \mathrm{H}_*(C_*\tilde{B} \otimes_{\pi_1 B} \mathbb{F}[\pi_0 F]) \otimes \mathrm{H}\mathbb{F}_*(F') \tag{5.4.21}$$

Let $Q_* \twoheadrightarrow \mathbb{F}[\pi_0 F]$ be a free resolution over $\mathbb{F}[\pi_1 B]$. Consider the two spectral sequences associated to the double complex $C_*\tilde{B} \otimes_{\pi_1 B} Q_*$, noting that both factors are free. If we take homology w.r.t. the right hand factor first, then the spectral sequence degenerates to $\mathrm{H}_*(C_*\tilde{B} \otimes_{\pi_1 B} \mathbb{F}[\pi_0 F])$, so the other spectral sequence converges to this also. The E^2 page of the other sequence is just $\mathrm{Tor}_s^{\mathbb{F}[\pi_1 B]}(\mathrm{H}\mathbb{F}_t \tilde{B}, \mathbb{F}[\pi_0 F])$, as above. This is the Künneth-type spectral sequence mentioned previously.

5.5. Examples of Serre Spectral Sequences

We have mentioned a large number of stable cofibrations, some of which gives rise to as many as 24 different unstable fibrations. Although unnecessary for our computations, it is nonetheless interesting to analyse how the associated Serre sequences behave. Life being short, we examine only four cases, which appear to cover most of the observed phenomena. Recall (from 1.3.2) the cofibre sequence

$$\Sigma KO \xrightarrow{\alpha} KO \xrightarrow{m_O} KU \xrightarrow{f_U \nu^{-1}} \Sigma^2 KO \tag{5.5.1}$$

From this we extract our fibrations:

$$\mathbb{Z} \times BSp \xrightarrow{m_O} \mathbb{Z} \times BU \xrightarrow{f_U \nu^{-1}} O/U \tag{5.5.2}$$

$$\mathbb{Z} \times BU \xrightarrow{f_U \nu^{-1}} O/U \xrightarrow{\alpha} U/Sp \tag{5.5.3}$$

$$O/U \xrightarrow{\alpha} U/Sp \xrightarrow{m_O} U \tag{5.5.4}$$

$$U/Sp \xrightarrow{m_O} U \xrightarrow{f_U \nu^{-1}} SO \tag{5.5.5}$$

Analysis of (5.5.2).

$$\mathbb{Z} \times BSp \xrightarrow{m_O} \mathbb{Z} \times BU \xrightarrow{f_U \nu^{-1}} O/U \tag{5.5.6}$$

In homology, we have

$$P[[\beta] \circ z_{4k}][-\beta] \longrightarrow P[[\nu^2] \circ z_{2k}][-\nu^2] \longrightarrow E[\overline{[\alpha^2]} \circ z_{2k}] \tag{5.5.7}$$

$$[\beta] \circ z_{4k} \mapsto [\nu^2] \circ [2] \circ z_{4k} = [\nu^2] \circ z_{2k}^2 \tag{5.5.8}$$

$$[\nu^2] \circ z_{2k} \mapsto [\alpha^2] \circ z_{2k} \tag{5.5.9}$$

It is clear that the first map is mono and the second is epi. The local coefficients are thus trivial and the spectral sequence collapses. □

Analysis of (5.5.3).

$$\mathbb{Z}\times BU \xrightarrow{f_U \nu^{-1}} O/U \xrightarrow{\alpha} U/Sp \tag{5.5.10}$$

$$P[[\nu^2]\circ z_{2k}][-\nu^2] \longrightarrow E[\overline{[\alpha^2]}\circ z_{2k}] \longrightarrow E[e\circ[\beta]\circ z_{4k}] \tag{5.5.11}$$

$$[\nu^2]\circ z_{2k} \mapsto [\alpha^2]\circ z_{2k} \mapsto \delta_{k0} \tag{5.5.12}$$

In this case the coefficients are not simple. To analyse them, we need to compute the Cotor groups of $H\mathbb{F}_*(B)$ as a comodule over $H\mathbb{F}_*(B\mathbb{Z}) = H\mathbb{F}_*(S^1) = E[a]$. This comodule structure arises from a map $B \to B\mathbb{Z}$ whose fibre is the universal cover of B. As $e\circ[\beta]$ is the Hurewicz image of a generator of $\pi_1 B$, it must map to a. Other generators map to zero. The coaction is induced by the map

$$B \xrightarrow{\Delta} B\times B \to S^1 \times B \tag{5.5.13}$$

As the generators $e\circ[\beta]\circ z_{4k}$ are primitive, it follows that

$$H\mathbb{F}_*(B) \simeq E[a]\otimes E[e\circ[\beta]\circ z_{4k+4}] \tag{5.5.14}$$

is an extended comodule. Thus

$$\mathrm{Cotor}^{E[a]}_{**}(\mathbb{F}, H\mathbb{F}_*(B)) = \mathrm{Cotor}^{E[a]}_{0*}(\mathbb{F}, H\mathbb{F}_*(B)) = E[e\circ[\beta]\circ z_{4k+4}] \tag{5.5.15}$$

It follows that the Eilenberg-Moore sequence collapses to give

$$H\mathbb{F}_*(\tilde{B}) = E[e\circ[\beta]\circ z_{4k+4}] \tag{5.5.16}$$

We next have to use the Künneth sequence

$$\mathrm{Tor}^{\mathbb{F}[\mathbb{Z}\beta]}_*(E[e\circ[\beta]\circ z_{4k+4}], \mathbb{F}[\mathbb{Z}\nu^2]) \Longrightarrow H\mathbb{F}_*(B;\mathcal{H}_0 F) \tag{5.5.17}$$

The connecting map $\partial\colon \mathbb{Z}\times BSp = \Omega B \to F = O/U$ is m_O. In particular, $\partial(\beta) = 2\nu^2$. It follows that $\mathbb{F}[\mathbb{Z}\nu^2]$ is a free module on two generators (viz. $[0]$ and $[\nu^2]$) over $\mathbb{F}[\mathbb{Z}\beta]$. On the other hand, $\mathbb{Z}\beta$ acts trivially on $E[e\circ[\beta]\circ z_{4k+4}]$ (because this maps injectively to $H\mathbb{F}_*(B)$). We conclude that the Tor group lies on the 0-line, and that the Serre E^2 is

$$E[e\circ[\beta]\circ z_{4k+4}]\otimes \mathbb{F}[\mathbb{F}\nu^2]\otimes P[[\nu^2]\circ z_{2k+2}] \tag{5.5.18}$$

Using the usual relation between suspension and transgression, (lemma 5.4.1) we find that

$$\begin{aligned} d_r(e\circ[\beta]\circ z_{4k+4}) &= 0 \qquad (r < 4k+5) && (5.5.19)\\ d_{4k+5}(e\circ[\beta]\circ z_{4k+4}) &= [\nu^2]\circ z_{2k+2}^2 && (5.5.20)\end{aligned}$$

The other generators are on the vertical axis, and therefore are permanent cycles. We can now prove by induction that

$$E^{4k+5}_{**} = E[e\circ[\beta]\circ z_{4l+4} \mid l\ge k]\otimes \mathbb{F}[\mathbb{F}\nu^2]\otimes E[[\nu^2]\circ z_{2l+2} \mid l<k]\otimes P[[\nu^2]\circ z_{2l+2} \mid l\ge k] \tag{5.5.21}$$

so

$$E^{\infty}_{**} = \mathbb{F}[\mathbb{F}\nu^2]\otimes E[[\nu^2]\circ z_{2l+2}] \tag{5.5.22}$$

This is concentrated on the vertical axis, so there are no extensions. The answer agrees with what we already know as soon as we identify the image of $[\nu^2]$ under the fibre inclusion $f_U\nu^{-1}\colon \mathbb{Z}\times BU \to O/U$ as $[\alpha^2]$. □

Analysis of (5.5.4).

$$O/U \xrightarrow{\alpha} U/Sp \xrightarrow{m_O} U \tag{5.5.23}$$

$$E[\overline{[\alpha^2]} \circ z_{2k}] \xrightarrow{\eta\epsilon} E[e \circ [\beta] \circ z_{4k}] \xrightarrow{\eta\epsilon} E[e \circ [\nu^2] \circ z_{2k}] \tag{5.5.24}$$

Again, the coefficients are not simple. The Eilenberg-Moore sequence giving the homology of the universal cover behaves much as in the previous case. We find that

$$\mathrm{H}\mathbb{F}_*(\tilde{B}) = E[e \circ [\nu^2] \circ z_{2k+2}] \tag{5.5.25}$$

with trivial action of $\pi_1(B) = \mathbb{Z}\nu^2$. On the other hand, $\mathbb{F}[\mathbb{Z}\nu^2]$ acts on $\mathrm{H}\mathbb{F}_0(F) = \mathbb{F}[\mathbb{F}\alpha^2]$ via the epimorphism $\mathbb{Z}\nu^2 \twoheadrightarrow \mathbb{F}\alpha^2$ sending ν^2 to α^2. Using the obvious minimal resolution

$$0 \to \mathbb{F}[\mathbb{Z}\nu^2] \xrightarrow{[\nu^2]-[0]} \mathbb{F}[\mathbb{Z}\nu^2] \longrightarrow \mathbb{F}[\mathbb{F}\alpha^2] \to 0 \tag{5.5.26}$$

We find that the Künneth E^2 is

$$\mathrm{Tor}_{**}^{\mathbb{F}[\mathbb{Z}\nu^2]}(E[e \circ [\nu^2] \circ z_{2k+2}], \mathbb{F}[\mathbb{F}\alpha^2]) = E[a] \otimes E[e \circ [\nu^2] \circ z_{2k+2}] \tag{5.5.27}$$

where a has bidegree $(1,0)$ and the second factor is $E^2_{0,*}$. As the differential d_r has bidegree $(-r, r-1)$, only d_2 can be nontrivial. A nonzero element of smallest possible degree in the image of d_2 must be primitive and cannot lie on the horizontal axis; a is the only candidate. There is nothing in bidegree $(0,2)$ to support a d_2 hitting a, so the sequence collapses. To see what a represents, consider the following morphism of fibrations:

$$\begin{tikzcd}
S^0 \arrow[r] \arrow[d] & S^1 \arrow[r, "2"'] \arrow[d, "\beta"'] & S^1 \arrow[d, "\nu^2"] \\
O/U \arrow[r] & U/Sp \arrow[r] & U
\end{tikzcd} \tag{5.5.28}$$

A naturality argument shows that $a = e \circ [\beta]$. We find that the Serre E^2 is as follows, with the first two factors on the horizontal axis and the third on the vertical one:

$$E^2_{**} = E[e \circ [\beta]] \otimes E[e \circ [\nu^2] \circ z_{2k+2}] \otimes E[[\alpha^2] \circ z_{2k+2}] \tag{5.5.29}$$

By lemma 5.4.1, there are transgressive differentials

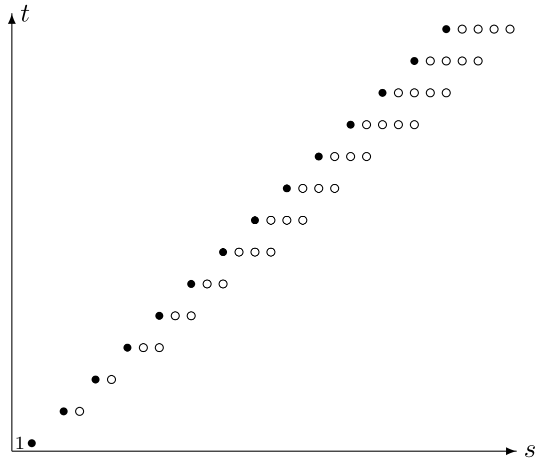
$$d_{2k+1}(e \circ [\nu^2] \circ z_{2k+2}) = [\alpha^2] \circ z_{2k+2} \tag{5.5.30}$$

This implies

$$\begin{aligned} E^{2k+3}_{**} \quad = \quad & E[e \circ [\beta]] \otimes E[e \circ [\nu^2] \circ z_{2l+2} \mid l \geq k] \otimes \\ & E[e \circ [\nu^2] \circ z_{2l+2} \otimes [\alpha^2] \circ z_{2l+2} \mid l < k] \otimes E[[\alpha^2] \circ z_{2l+2} \mid l \geq k] \end{aligned} \tag{5.5.31}$$

$$E^\infty_{**} = E[e \circ [\beta]] \otimes E[e \circ [\nu^2] \circ z_{2l+2} \otimes [\alpha^2] \circ z_{2l+2}] \tag{5.5.32}$$

The E^∞ term looks like this:



(5.5.33)

The solid circles represent the generators $e \circ [\beta]$ or $e \circ [\nu^2] \circ z_{2l+2} \otimes [\alpha^2] \circ z_{2l+2}$. They lie on the line $s = t + 1$. All decomposables (the open circles) lie strictly below this line and only 1 lies above it. By considering various bidegrees, we see that this forces the generators to be primitive in E^∞, which we recall is the associated graded Hopf algebra corresponding to a filtration of $H\mathbb{F}_*(U/Sp)$ by Hopf ideals. On the other hand, we see from the diagram that the generators lie in the bottom filtration, so they correspond to primitives in $H\mathbb{F}_*(U/Sp)$ itself. Using our previous description of $H\mathbb{F}_*(U/Sp)$, this implies that $e \circ [\nu^2] \circ z_{2l+2} \otimes [\alpha^2] \circ z_{2l+2}$ maps to $e \circ [\beta] \circ z_{4k+2}$. It would be interesting to have a more equational proof of this, perhaps involving the Kudo transgression theorem [**16**] or Kudo-Araki operations in the spectral sequence. □

Analysis of (5.5.5).
This is much the most subtle and interesting of our examples.

$$U/Sp \xrightarrow{m_O} U \xrightarrow{f_U \nu^{-1}} SO \tag{5.5.34}$$

$$E[e \circ [\beta] \circ z_{4k}] \xrightarrow{\eta\epsilon} E[e \circ [\nu^2] \circ z_{2k}] \longrightarrow E[[\alpha] \circ z_{k+1}] \tag{5.5.35}$$

$$(f_U \nu^{-1})_*(e \circ [\nu^2] \circ z_{2k}) = [\alpha] \circ q_{2k+1} \tag{5.5.36}$$

The coefficients are simple as the fibre is connected. To prove the last equation, recall that $f_U(\nu) = \alpha^2$ (1.4.1) and that $e \circ [\alpha^2] = (e \circ [\alpha]) \circ [\alpha] = \overline{z}_1 \circ [\alpha]$ (2.1.13) and that $\overline{z}_1 \circ \overline{z}_k = q_{k+1}$ (2.6.10). As the connecting map is $m_\alpha \colon O/U \to U/Sp$, this implies a transgressive differential

$$d_{2k+3}([\alpha] \circ q_{2k+3}) = (m_\alpha)_*([\alpha^2] \circ z_{2k+2}) = [\alpha^3] \circ z_{2k+2} = 0 \tag{5.5.37}$$

showing that $[\alpha] \circ q_{2k+3}$ is a permanent cycle. For dimensional reasons, $[\alpha] \circ q_1$ is also.

We shall need to use different generators to understand this spectral sequence. For brevity, we write

$$e_{4k+1} = e \circ [\beta] \circ z_{4k} \tag{5.5.38}$$
$$A_* = H\mathbb{F}_*(U/Sp) = E[e_{4k+1}] \tag{5.5.39}$$
$$u_{k+1} = [\alpha] \circ z_{k+1} \tag{5.5.40}$$
$$B_* = H\mathbb{F}_*(SO) = E[u_{k+1}] \tag{5.5.41}$$

We also set $u_0 = 1$ and $u(t) = \sum_k u_k t^k$. It is easy to show by induction that there are unique elements $w_k \in B_k$ with $w_0 = 0$ and

$$u(t) = \prod_k (1 + w_k t^k) \tag{5.5.42}$$

Moreover, $w_k \equiv u_k$ mod decomposables, so $B_* = E[w_{k+1}]$. The perceptive reader will realise that these generators are the circle products of $[\alpha]$ with the Witt generators [**4**] of $H\mathbb{F}_*(BO)$. We shall need some notation involving binary expansions. Recall that $0 \in \mathbb{N}$ according to our conventions.

$$\begin{aligned}
S(k) &\quad = \quad \text{the unique } S \subset \mathbb{N} \text{ such that } k = \sum_{i \in S} 2^i && (5.5.43)\\
2^l \in k &\quad \Leftrightarrow \quad l \in S(k) && (5.5.44)\\
n \perp m &\quad \Leftrightarrow \quad S(n) \cap S(m) = \emptyset && (5.5.45)\\
l = n \sqcup m &\quad \Leftrightarrow \quad n \perp m \text{ and } n + m = l && (5.5.46)
\end{aligned}$$

Note that mod 2 we have

$$(s+t)^k = \prod_{2^i \in k} (s^{2^i} + t^{2^i}) = \sum_{k = l \sqcup m} s^l t^m \tag{5.5.47}$$

For k odd, we write

$$\begin{aligned}
B(k)_* &= E[w_{2^l k} \mid l \geq 0] && (5.5.48)\\
{}_k w(t) &= \prod_l (1 + w_{2^l k} t^{2^l}) && (5.5.49)\\
&= \sum_{m \geq 0} {}_k w_{mk} t^m && (5.5.50)
\end{aligned}$$

so

$$\begin{aligned}
B_* &= \bigotimes_{k \text{ odd}} B(k)_* && (5.5.51)\\
w(t) &= \prod_{k \text{ odd}} {}_k w(t) && (5.5.52)\\
{}_k w_{mk} &= \prod_{2^l \in m} w_{2^l k} && (5.5.53)\\
B(k)_* &= \mathbb{F}\{{}_k w_{mk} \mid m \geq 0\} && (5.5.54)
\end{aligned}$$

Lemma 5.5.1. *For k odd:* (1) ${}_k w(s+t) = {}_k w(s) {}_k w(t)$
(2) $\psi({}_k w(t)) = {}_k w(t) \otimes {}_k w(t)$

Proof. First note that

$${}_k w(s+t) = \sum_m {}_k w_{mk} (s+t)^m = \sum_{p \perp q} {}_k w_{(p+q)k} s^p t^q \tag{5.5.55}$$

On the other hand, using (5.5.53) and the fact that all squares of positive dimensional elements in B_* vanish, we see that

$${}_k w_{pk} \; {}_k w_{qk} = \begin{cases} {}_k w_{(p+q)k} & \text{if } p \perp q \\ 0 & \text{otherwise} \end{cases} \tag{5.5.56}$$

It follows directly that ${}_k w(s+t) = {}_k w(s) {}_k w(t)$. Consider the series ${}_k v(t) = {}_k w(t) \otimes {}_k w(t)$. Clearly, this also satisfies ${}_k v(s+t) = {}_k v(s) {}_k v(t)$. Running the argument above backwards, we conclude that

$${}_k v_{pk} \; {}_k v_{qk} = \begin{cases} {}_k v_{(p+q)k} & \text{if } p \perp q \\ 0 & \text{otherwise} \end{cases} \tag{5.5.57}$$

where ${}_kv(t) = \sum_m {}_kv_{mk}t^m$. It follows that the map

$$\phi\colon B(k)_* = \mathbb{F}\{{}_kw_{mk} \mid m \geq 0\} \longrightarrow B(k)_* \otimes B(k)_* \tag{5.5.58}$$

sending ${}_kw_{mk}$ to ${}_kv_{mk}$ is a ring homomorphism. We have

$$\prod_l (1 + \phi(w_{2^l k})t^{2^l}) = \phi({}_kw(t)) = {}_kw(t) \otimes {}_kw(t) \tag{5.5.59}$$

$$\begin{aligned}
\prod_m (1 + \phi(w_m)t^m) &= \prod_{k \text{ odd}} \phi({}_kw(t^k)) && (5.5.60)\\
&= \prod_{k \text{ odd}} {}_kw(t^k) \otimes {}_kw(t^k) && (5.5.61)\\
&= u(t) \otimes u(t) && (5.5.62)\\
&= \psi(u(t)) && (5.5.63)\\
&= \prod_m (1 + \psi(w_m)t^m) && (5.5.64)
\end{aligned}$$

It follows by induction on m that $\psi(w_m) = \phi(w_m)$. □

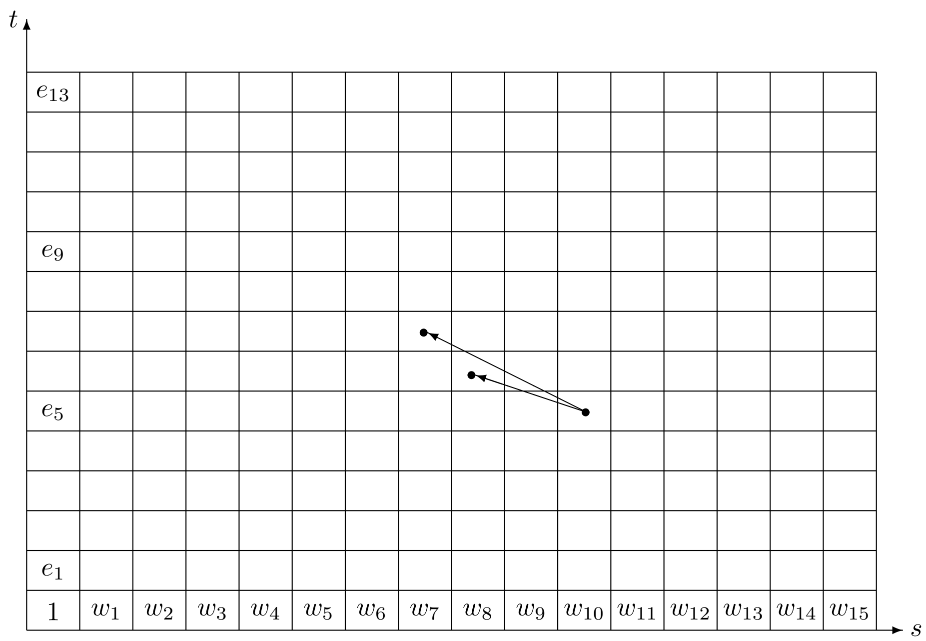
We note also that

$$\begin{aligned}
[\alpha] \circ q(t) &= t\mathrm{dlog} \prod_m (1 + w_m t^m)/\mathrm{d}t && (5.5.65)\\
&= \sum_m m w_m t^m/(1 + w_m t^m) && (5.5.66)\\
&= \sum_{m \text{ odd}} w_m t^m && (5.5.67)
\end{aligned}$$

so

$$w_{2k+1} = [\alpha] \circ q_{2k+1} \tag{5.5.68}$$

Consider the layout of the generators on the E^2 page:



(5.5.69)

For dimensional reasons, we must have $d_2 e_{4k+1} = 0$ and $d_2 u(t) = t^2 e_1 f(t)$ for some series f. Moreover, we know that $m_O(e_1) = 0$, so e_1 cannot survive. It follows that $\epsilon f(t) = f(0) = 1$. As d_2 is a Hopf algebra differential, we have

$$\begin{aligned} t^2\psi(f(t))(e_1 \otimes 1 + 1 \otimes e_1) &= \psi d_2 u(t) && (5.5.70)\\ &= d_2 \psi u(t) && (5.5.71)\\ &= d_2(u(t) \otimes u(t)) && (5.5.72)\\ &= t^2 f(t) e_1 \otimes u(t) + t^2 u(t) \otimes f(t) e_1 && (5.5.73)\end{aligned}$$

By projecting into the summand

$$(B \otimes B).(1 \otimes e_1) \subset (B \otimes B)\{1 \otimes 1, 1 \otimes e_1, e_1 \otimes 1, e_1 \otimes e_1\} \tag{5.5.74}$$

we find that $\psi(f(t)) = u(t) \otimes f(t)$. Applying $1 \otimes \epsilon_B$, we obtain $f(t) = u(t)$, so

$$d_2 u(t) = t^2 u(t) e_1 \tag{5.5.75}$$

In fact, essentially the same analysis applies to each factor ${}_k w$. We find that

$$d_2({}_k w(t)) = \begin{cases} t^2{}_1 w(t) & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases} \tag{5.5.76}$$

It follows that d_2 respects the decomposition

$$E^2_{**} = \bigotimes_{k \geq 0} B(2k+1) \otimes E[e_{4k+1}] \tag{5.5.77}$$

and acts nontrivially only on the first factor. Using

$$\begin{aligned} B(1)_* &= E[w_{2^l} \mid l \geq 0] && (5.5.78)\\ &= \mathbb{F}\{{}_1 w_m \mid m \geq 0\} && (5.5.79)\\ d_2({}_1 w_1) &= 0 && (5.5.80)\\ d_2({}_1 w_{m+2}) &= {}_1 w_m e_1 && (5.5.81)\end{aligned}$$

we find that

$$\begin{aligned} \mathrm{H}(B(1) \otimes E[e_1], d_2) &= E[w_1] && (5.5.82)\\ E^3_{**} &= E[w_1] \otimes \bigotimes_{k>0} (B(2k+1) \otimes E[e_{4k+1}]) && (5.5.83)\end{aligned}$$

The above chain of reasoning extends inductively to give

$$\begin{aligned} E^{4k+2} &= E[w_{2l+1} \mid l < k] \otimes \bigotimes_{l \geq k} (B(2l+1) \otimes E[e_{4l+1}]) && (5.5.84)\\ d_{4k+2}({}_{2l+1} w(t)) &= \begin{cases} t^{4k+2}{}_{2k+1} w(t) e_{4k+1} & \text{if } l = k \\ 0 & \text{if } l > k) \end{cases} && (5.5.85)\\ E^\infty &= E[w_{2l+1} \mid l \geq 0] && (5.5.86)\end{aligned}$$

This is as expected, as

$$w_{2l+1} = [\alpha] \circ q_{2l+1} = (f_U \nu^{-1})_*(e \circ [\nu^2] \circ z_{2k}) \tag{5.5.87}$$

□

CHAPTER 6

K-Theoretic Machinery

In this chapter we recall without detailed proof the basic ideas of Clifford modules and of Atiyah's Real K-theory. The Clifford theory comes from [**3**] and [**11**], and the Real K-theory from [**2**].

6.1. Clifford Modules

In this section we recall some of the theory of Clifford modules and their relation to the coefficient rings of various K-theories. We construct a ring A_* and a homomorphism $A_* \to KO_*$. A similar (but much simpler) procedure is possible with complex Clifford modules and KU-theory; we omit this. All this comes mainly from [**3**]. In the next section, we recall some results which can be used to prove that this map is an isomorphism.

Let V be a finite dimensional real inner product space. The associated Clifford algebra CV is the quotient of the (noncommutative) tensor algebra $TV = \bigoplus_{k\geq 0} V^{\otimes k}$ by the two sided ideal generated by elements $v \otimes v + \langle v, v\rangle$ where $v \in V$. The obvious map $j\colon V \to CV$ is injective; we usually suppress it from the notation. Thus the set of algebra maps $\phi\colon CV \to A$ bijects with the set of linear maps $\theta\colon V \to A$ such that $\theta(v)^2 = -\langle v, v\rangle$, via $\phi \mapsto \phi \circ j$.

The algebra CV can be graded over $\mathbb{Z}/(2)$ in a unique way such that $V \subseteq C^1V$. It is not commutative even in the graded sense. A nonzero vector $v \in V$ is a unit in CV with inverse $-v/\langle v, v\rangle$.

By a Clifford module for V , we mean a $\mathbb{Z}/(2)$-graded module $M = M^0 \oplus M^1$ over CV. Note that multiplication by a vector $v \in V$ exchanges M^0 and M^1.

Define

$$\begin{aligned} S_V &= \{v \in V \mid \|v\| = 1\} && (6.1.1)\\ B_V &= \{v \in V \mid \|v\| \leq 1\} && (6.1.2)\\ S^V &= V \cup \{\infty\} && (6.1.3)\\ B'_V &= \{v \in V \mid \|v\| \geq 1\} \cup \{\infty\} && (6.1.4)\end{aligned}$$

so that

$$\begin{aligned} B_V \cap B'_V &= S_V && (6.1.5)\\ B_V \cup B'_V &= S^V && (6.1.6)\end{aligned}$$

Let M be a Clifford module for $V \neq 0$. We can associate to M an element α_M of the reduced orthogonal K-theory $\tilde{KO}(S^V)$ as follows. Suppose $\dim M^0 = m$. Then, as multiplication by a nonzero vector gives an isomorphism between M^0 and M^1 , we have $\dim M^1 = m$ also. Write

$$GM = \{ \text{ ungraded vector subspaces } N < M^0 \oplus M^1 \mid \dim N = m\} \qquad (6.1.7)$$

This Grassmannian manifold thus has a tautological m-plane bundle γ with

$$E(\gamma) = \{(m, N) \mid m \in N \in GM\} \qquad (6.1.8)$$

We define a map $\eta\colon S^V \to GM$ by

$$\eta(v) = (1+v)M^0 = (\|v\|^{-1} + \hat{v})M^0 \tag{6.1.9}$$

$$\eta(\infty) = M^1 \tag{6.1.10}$$

Here $\hat{v} = v/\|v\|$. The continuity of η near ∞ follows from the second expression for $\eta(v)$ and the fact that $\hat{v}M^0 = M^1$ for all $v \neq 0$. We also write η for the corresponding bundle , i.e. the pullback of γ by η.

Over $S^V \setminus \{\infty\} = V$, we can use the projection

$$\pi_0\colon (1+v)M^0 \simeq M^0 \tag{6.1.11}$$

to trivialise η. Similarly, we can trivialise over $S^V \setminus \{0\}$ by

$$\pi_1\colon (\|v\|^{-1} + \hat{v})M^0 \simeq M^1 \tag{6.1.12}$$

The resulting clutching function over S_V is

$$v \mapsto (M^0 \xrightarrow{v} M^1) \tag{6.1.13}$$

We recall that the absolute orthogonal K-theory $KO(X)$ is the ring of formal differences of isomorphism classes of real bundles over X. The reduced group $\tilde{KO}(X)$ is the ideal of elements of formal dimension zero. The relative group $KO(X,Y)$ is defined in terms of symbols $d(\zeta,\phi,\eta)$ where ζ and η are bundles over X , and $\phi\colon \zeta \to \eta$ is a bundle map which is iso over Y.

We can define elements

$$\begin{aligned} \eta - M^1 &\in \tilde{KO}(S^V) && (6.1.14)\\ d(\eta,\pi_1,M^1) &\in KO(S^V, B'_V) && (6.1.15)\\ d(M^0,\sigma,M^1) &\in KO(B_V,S_V) && (6.1.16) \end{aligned}$$

Here M^0 refers to the trivial bundle with fibre M^0, and $\sigma(v,m) = (v,vm)$. There are canonical isomorphisms between the three groups involved, under which the given elements correspond to each other. We write α_M for any of these elements. If M has a complex or symplectic structure which is compatible with the grading and action of $C(V)$, then α_M lifts canonically to KU or KSp.

If we are given an orientation on V, or if we know that $2\alpha_M = 0$, then we can consider α_M as an element of $\tilde{KO}^0 S^n = \pi_n(\mathbb{Z}\times BO)$.

It is clear that $\alpha_{M\oplus N} = \alpha_M + \alpha_N$. We thus have a homomorphism from the Grothendieck group of Clifford modules over $C(V)$ to $\tilde{KO}^0 S^V$.

Suppose that the action of $C(V)$ on M extends to an action of $C(W)$ for some strictly larger vector space $W > V$. Then $\alpha_M \in \tilde{KO}^0 S^V$ is the restriction of the analogous element of $\tilde{KO}^0 S^W$. As the inclusion $S^V \to S^W$ is nullhomotopic, this implies that $\alpha_M = 0$.

Suppose we have two inner product spaces U and V. We can give $C(U)\otimes C(V)$ the obvious grading over $\mathbb{Z}/(2)$, and make it into an algebra via the multiplication

$$(a_0 \otimes b_0)(a_1 \otimes b_1) = (-)^{|b_0||a_1|} a_0a_1 \otimes b_0b_1 \tag{6.1.17}$$

Using the universal property of $C(U\oplus V)$ we see that there is a unique algebra map $C(U\oplus V) \to C(U)\otimes C(V)$ with

$$(u,v) \mapsto u\otimes 1 + 1 \otimes v \tag{6.1.18}$$

This map is actually an isomorphism, as one can see by induction on $\dim V$ for example. Using it we can make the graded tensor product of two Clifford modules M over $C(U)$ and N over $C(V)$ into a Clifford module $M\otimes N$ over $C(U\oplus V)$. The resulting element

$$\alpha_{M\otimes N} \in \tilde{KO}^0 S^{U\oplus V} = \tilde{KO}^0 (S^U \wedge S^V) \tag{6.1.19}$$

is just the usual external product $\alpha_M \alpha_N$ as one sees most easily from (6.1.16) above.

Let C_n denote the Clifford algebra of $\mathbb{R}^n = \mathbb{R}\{e_0, \dots e_{n-1}\}$ with the usual inner product, and M_n the Grothendieck group of Clifford modules over it. These groups form a graded ring M_*. We have scalar restriction homomorphisms $r_n \colon M_{n+1} \to M_n$; let A_n denote the cokernel. As $r(xy) = xr(y) = r(x)y$, we have an induced ring structure on A_*. There is thus a ring map $A_* \to KO_*$ sending $[M]$ to α_M.

We can identify many of the algebras C_n and their degree zero subalgebras C_n^0 in more familiar terms:

n	C_n	generators	C_n^0	generators
0	$\mathbb{R}$		$\mathbb{R}$	
1	$\mathbb{C}$	i	$\mathbb{R}$	
2	$\mathbb{H}$	j, k	$\mathbb{C}$	$i = jk$
3	$\mathbb{H}^2$	$(i, -i), (j, j), (k, k)$	$\mathbb{H}$	$(k, -k), (-j, j)$
4	$M_2\mathbb{H}$	$\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}, \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$	$\mathbb{H}^2$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -k \\ -k & 0 \end{pmatrix}, \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}$
5	$M_4\mathbb{C}$		$M_2\mathbb{H}$	
6	$M_8\mathbb{R}$		$M_4\mathbb{C}$	
7	$M_8\mathbb{R}^2$		$M_8\mathbb{R}$	
8	$M_{16}\mathbb{R}$		$M_8\mathbb{R}^2$	

We have used the (ungraded) isomorphism $C_n \to C_{n+1}^0$ sending e_k to $e_0 e_{k+1}$. The entry for $n = 2$ above (for example) means that there is an isomorphism $C_2 \to \mathbb{H}$ sending e_0 to j and e_1 to k. This induces an isomorphism $C_2^0 \to \mathbb{C}$ sending $e_0 e_1$ to i. We omit this information for $n > 4$, as it would take up too much space. I have *Mathematica* code which implements all this.

Using this, we can compute the ring M_* and the map r. The groups M_n are all free Abelian, and the fourth column below gives bases. Each basis element corresponds to an irreducible graded C_n-module. The element ζ is the one dimensional module over $C_0 = \mathbb{R}$ concentrated in degree 1. It follows that $\zeta.[M] = M^*$, where M^* is M with the degrees shifted by one. If $n > 0$, then $a_n = \dim_{\mathbb{R}} M^0$ is independent of which irreducible M we consider.

n	C_n	a_n	M_n	r
0	$\mathbb{R}$		$\zeta, 1 = \zeta^2$	$r\alpha = 1 + \zeta$
1	$\mathbb{C}$	1	$\alpha = \zeta\alpha$	$r\alpha^2 = 2\alpha$
2	$\mathbb{H}$	2	α^2	$r\alpha^3 = 2\alpha^2$
3	$\mathbb{H}^2$	4	α^3	$r\beta = \alpha^3$
4	$M_2\mathbb{H}$	4	$\beta, \zeta\beta$	$\alpha^4 = r^4\lambda = \beta + \zeta\beta$
5	$M_4\mathbb{C}$	8	$r^3\lambda$	$\alpha\beta = 2r^3\lambda$
6	$M_8\mathbb{R}$	8	$r^2\lambda$	$\alpha r^3\lambda = 2r^2\lambda$
7	$M_8\mathbb{R}^2$	8	$r\lambda = \zeta r\lambda$	$\alpha r^2\lambda = 2r\lambda$
8	$M_{16}\mathbb{R}$	8	$\lambda, \zeta\lambda$	$\alpha r\lambda = \lambda + \zeta\lambda$

(6.1.20)

This table can be extended so as to be essentially 8-periodic. In fact:

$$C_{n+8} \simeq M_{16}(C_n) \tag{6.1.21}$$
$$a_{n+8} = 16a_n \tag{6.1.22}$$
$$M_{n+8} \simeq M_n \quad \text{via multiplication by } \lambda \tag{6.1.23}$$

The Clifford module β over $C_4 = M_2\mathbb{H}$ is just $\mathbb{H}^2$ with the obvious left action. The grading is more subtle. The following table gives a homogeneous basis:

$$\begin{array}{lllll} \text{degree } 0: & \binom{i}{1} & \binom{-1}{i} & \binom{k}{j} & \binom{-j}{k} \\ \text{degree } 1: & \binom{-i}{1} & \binom{1}{i} & \binom{-k}{j} & \binom{j}{k} \end{array} \tag{6.1.24}$$

Note that the *right* action of $\mathbb{H}$ is homogeneous. Similarly, λ is just $\mathbb{R}^{16}$, with the obvious left action and a suitable grading which we shall not make precise. To complete the description of the multiplication in M_*, we have $\beta^2 = 4\lambda$. It follows that A_* has the same structure (in positive dimensions) as that given for KO_* in section 1.1. Moreover, the elements called α, β and λ here go over to the corresponding ones in KO_*.

6.2. Atiyah's Real K-Theory

In this section we recall the basic ideas of [**2**], and indicate how they help us to prove the statements in chapter 1.

Let the cyclic group $C_2 = \{1, \tau\}$ act on $\mathbb{C}$ by conjugation. We shall call a space with C_2 action a real space, and write $\overline{x}$ for τx. By a real vector bundle over a real space X, we mean a complex bundle $E \xrightarrow{p} X$ with a given real structure on E such that the following maps are C_2-equivariant:

$$E \xrightarrow{p} X \tag{6.2.1}$$
$$E \times_X E \xrightarrow{+} E \tag{6.2.2}$$
$$\mathbb{C} \times E \xrightarrow{\cdot} E \tag{6.2.3}$$

Equivalently, we insist that τ should give a conjugate linear map $E_x \longrightarrow E_{\overline{x}}$ for each $x \in X$. The real bundles over X form a category in an evident way, and we write $KR(X)$ for the Grothendieck group. Suppose that the action of C_2 on X is trivial. Given a bundle E of $\mathbb{R}$-vector spaces over X, the bundle $\mathbb{C} \otimes E$ with conjugation $z \otimes e \mapsto \overline{z} \otimes e$ is a real bundle in this new sense. This construction gives an equivalence of categories, showing that $KR(X) = KO^0 X$ in this case.

We need some standard real spaces:

$$R^{p,q} = \mathbb{R}^p \oplus i\mathbb{R}^q \subset \mathbb{C}^{p+q} \tag{6.2.4}$$
$$B^{p,q} = \text{unit ball in } R^{p,q} \tag{6.2.5}$$
$$S^{p,q} = \text{unit sphere in } R^{p,q} \tag{6.2.6}$$

The action of C_2 is by complex conjugation. We can now define

$$KR^{p,q}(X,Y) = KR(X \times B^{p,q}, Y \times B^{p,q} \cup X \times S^{p,q}) \tag{6.2.7}$$

The tensor product gives a bigraded external multiplication in the obvious way. If we consider the reduced canonical complex line bundle as an element of

$$KR^{1,1} = KR(B^{1,1}, S^{1,1}) = KR(\mathbb{C}P^1) \tag{6.2.8}$$

then the induced multiplication map

$$KR^{p,q}(X,Y) \longrightarrow KR^{p+1,q+1}(X,Y) \tag{6.2.9}$$

turns out to be iso. Atiyah gives a proof of this using Fourier series. Using this, we define $KR^p(X,Y)$ for all $p \in \mathbb{Z}$, so that $KR^{p,q} \simeq KR^{p-q}$. This gives a C_2-equivariant cohomology theory.

We next define $KR_r^p(X) = KR^p(X \times S^{r,0})$ for $r \geq 0$. Atiyah shows that for $r = 1, 2$ or 4, this is periodic with period $2r$. If we restrict to the case when X has trivial action, then

$$KR^*(X) = KO^*(X) \tag{6.2.10}$$

$$KR_1^*(X) = KU^*(X) \tag{6.2.11}$$

$$KR_2^*(X) = KT^*(X) \tag{6.2.12}$$

$$KR_4^*(X) = KO^*(X) \oplus KSp^*(X) \tag{6.2.13}$$

Moreover $KO^*(X)$ is periodic, with period 8. One can check from Atiyah's proof that the periodicity map is just multiplication by ν, μ or λ as appropriate. Equivalently, these elements are invertible in the relevant homotopy rings.

CHAPTER 7

Proofs and Justifications

In this chapter we discuss how one might justify the statements made in chapters 1 to 3. Proofs for chapter 1 are only sketched, and are drawn mostly from the literature. For chapters 2 and 3, we assume the results of chapter 1 and the machinery described in chapter 4, but otherwise give complete proofs.

It will be convenient to make the following definitions:

$$AU_* = \mathbb{Z}[\nu^{\pm 1}] \tag{7.0.14}$$
$$AT_* = \mathbb{Z}[\alpha, \gamma, \mu^{\pm 1}]/(\alpha^2, 2\alpha, \alpha\gamma, \gamma^2) \tag{7.0.15}$$
$$AO_* = \mathbb{Z}[\alpha, \beta, \lambda^{\pm 1}]/(\alpha^3, 2\alpha, \alpha\beta, \beta^2 - 4\lambda) \tag{7.0.16}$$

Of course, we claim that there are isomorphisms $AU_* \xrightarrow{\simeq} KU_*$ etc.

7.1. Maps and Diagrams

The commutativity of diagram 1.3.1 is mostly obvious from the definitions of the maps involved in terms of bundles. For a complete list of the commutativity statements claimed, see the code for the program `Claims.m` The cases in which the composite is $1 + c$ are not so obvious – compare [**1**, chapter 3]. We shall also take as read the various ring and module properties of these maps.

Consider the following octahedral diagram of finite Real spectra:

$$\begin{tikzcd}
& & S^{20}/\emptyset \arrow[dr] \arrow[ddll, bend right=60] & & \\
& B^{20}/S^{20} \arrow[ur, "\circ"] \arrow[dl, "\circ"'] & & B^{20}/\emptyset \arrow[ll] \arrow[dl] & \\
B^{10}S^{10}/S^{10}S^{10} \arrow[rr] \arrow[rrrr, bend right, "\circ"'] & & B^{20}/S^{10}B^{10} \arrow[ul] \arrow[rr, "\circ"] & & S^{10}B^{10}/\emptyset \arrow[ul] \arrow[uull, bend right=60]
\end{tikzcd} \tag{7.1.1}$$

Here we have used the definitions 6.2.5 and 6.2.6 of B^{pq} and S^{pq}, and written just XY for the cartesian product $X \times Y$. The degree zero maps (those without circles) are all inclusions or projections. If we take the smash product of this diagram with a space X (with trivial action of C_2), and then apply KR^*, we get long exact sequences corresponding to diagram 1.3.2. Following Atiyah, I leave the details, the identification of the maps, and the recovery of the cofibration 1.3.2 itself to the reader. Some help is available in [**2**].

We next need to justify (at least some of) table 1.4.1, which gives the effect of various maps on the homotopy rings. We assume the material on Clifford modules for this. The first step is to identify ν^2, μ, β and θ as the same bundle in various different guises. This can be done by direct geometrical constructions which we omit. The same bundle serves as the periodicity generator in $KR^{2,2}$, if we let C_2 act on $\mathbb{H}$ by $z+jw \mapsto \overline{z}+j\overline{w} = -j(z+jw)j$. Next, we need to show that $\theta \otimes_{\mathbb{H}} \theta = \lambda$; this is discussed in section 7.3 below. This (together with the KO-module structure on KSp) shows that $KSp \simeq \Sigma^4 KO$. We can now use Clifford module methods and KO-linearity to fill in all the maps in the table which do not involve KT. More *ad hoc* arguments have to be given for the remaining maps. The task is made easier by using diagram 1.3.1 and the vanishing of composites in diagram 1.3.2. The hardest bit is the proof that $\delta_O(\mu) = \alpha$ – one could first prove that $KO_1 = \mathbb{F}\alpha$, and use exactness, although a proof by explicit construction would be more in keeping with our general approach. Again, we omit details.

Consider diagram 1.3.3. I claimed in section 1.3 that there was a morphism of diagrams from 1.3.2 to it, given by:

$$\begin{array}{llll} \theta\colon & \Sigma^4 KO & \to & KSp \\ \mu\colon & \Sigma^4 KT & \to & KT \\ \nu^2\colon & \Sigma^4 KU & \to & KU \end{array} \tag{7.1.2}$$

I define $\delta_{Sp}\colon KT \to \Sigma^3 KSp$ in the only possible way consistent with this, viz. $\delta_{Sp} = \theta\delta_O\mu^{-1}$. There are then various other commutativity constraints to check, but they follow easily from table 1.4.1 and KO-linearity. We thus have something isomorphic to a cofibre diagram, so it qualifies as a cofibre diagram itself.

7.2. Homotopy Rings

We have shown, or at least indicated how to show, that there are maps $AU_* \to KU_*$ etc.; we need to prove that these maps are iso.

We have also shown that the homotopy rings fit into various commutative and exact sequences (given by diagrams 1.3.1 to 1.3.3) and that the homotopy elements mentioned above are mapped according to table 1.4.1.

It is trivial to compute the following :

$$KU_0 = KT_0 = KO_0 = \mathbb{Z} \tag{7.2.1}$$

By considering clutching functions, we also have

$$KU_1 = \pi_0 U = 0 \tag{7.2.2}$$

$$KO_1 = \pi_0 O = \mathbb{F}\alpha \tag{7.2.3}$$

As KU_* is 2-periodic, we find that $AU_* \to KU_*$ is iso. Consider the diagram :

$$\begin{tikzcd} AU_{*+1} \arrow[r, "\nu^{-1}(1-c)"] \arrow[d] & AU_{*-1} \arrow[r, "\delta_T"] \arrow[d] & AT_* \arrow[r, "f_T"] \arrow[d] & AU_* \arrow[r, "\nu^{-1}(1-c)"] \arrow[d] & AU_{*-2} \arrow[d] \\ KU_{*+1} \arrow[r, "\nu^{-1}(1-c)"'] & KU_{*-1} \arrow[r, "\delta_T"'] & KT_* \arrow[r, "f_T"'] & KU_* \arrow[r, "\nu^{-1}(1-c)"'] & KU_{*-2} \end{tikzcd} \tag{7.2.4}$$

The bottom row is exact because it comes from a cofibration, and we can check that the top row is also exact. As $AU_* \to KU_*$ is iso, the 5-lemma tells us that $AT_* \to KT_*$ is also iso.

To prove that $AO_* \to KO_*$ is iso, we consider the exact sequence

$$KO_{*+1} \xrightarrow{l_O} KT_{*+1} \xrightarrow{\delta_O} KO_{*-2} \xrightarrow{\alpha^2} KO_* \xrightarrow{l_O} KT_* \xrightarrow{\delta_O} KO_{*-3} \tag{7.2.5}$$

We have the following picture :

$$\begin{array}{ccccccccccc}
0? & \to & \mathbb{Z}\mu^{-1}\gamma & \to & \mathbb{Z}\lambda^{-1}\beta? & \twoheadrightarrow & 0? & \to & 0 & \to & 0? \\
\mathbb{Z} & \xrightarrow{\simeq} & \mathbb{Z} & \xrightarrow{0} & 0? & \to & 0? & \to & \mathbb{Z}\mu^{-1}\gamma & \to & \mathbb{Z}\lambda^{-1}\beta? \\
\mathbb{F}\alpha & \xrightarrow{\simeq} & \mathbb{F}\alpha & \xrightarrow{0} & 0? & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{\simeq} & \mathbb{Z} & \xrightarrow{0} & 0? \\
\mathbb{F}\alpha^2? & \to & 0 & \to & 0? & \xrightarrow{0} & \mathbb{F}\alpha & \xrightarrow{\simeq} & \mathbb{F}\alpha & \xrightarrow{0} & 0? \\
0? & \xrightarrow{0} & \mathbb{Z}\gamma & \rightarrowtail & \mathbb{Z} & \twoheadrightarrow & \mathbb{F}\alpha^2? & \to & 0 & \to & 0? \\
\mathbb{Z}\beta? & \to & \mathbb{Z}\mu & \twoheadrightarrow & \mathbb{F}\alpha & \xrightarrow{0} & 0? & \xrightarrow{0} & \mathbb{Z}\gamma & \rightarrowtail & \mathbb{Z} \\
0? & \to & \mathbb{F}\mu\alpha & \to & \mathbb{F}\alpha^2? & \to & \mathbb{Z}\beta? & \to & \mathbb{Z}\mu & \twoheadrightarrow & \mathbb{F}\alpha \\
0? & \to & 0 & \to & 0? & \rightarrowtail & 0? & \to & \mathbb{F}\mu\alpha & \to & \mathbb{F}\alpha^2? \\
0? & \to & \mathbb{Z}\mu\gamma & \to & \mathbb{Z}\beta? & \twoheadrightarrow & 0? & \to & 0 & \to & 0? \\
\mathbb{Z}\lambda & \xrightarrow{\simeq} & \mathbb{Z}\mu^2 & \xrightarrow{0} & 0? & \to & 0? & \to & \mathbb{Z}\mu\gamma & \to & \mathbb{Z}\beta? \\
\mathbb{F}\lambda\alpha & \xrightarrow{\simeq} & \mathbb{F}\mu^2\alpha & \xrightarrow{0} & 0? & \xrightarrow{0} & \mathbb{Z}\lambda & \xrightarrow{\simeq} & \mathbb{Z}\mu^2 & \xrightarrow{0} & 0? \\
\mathbb{F}\lambda\alpha^2? & \to & 0 & \to & 0? & \xrightarrow{0} & \mathbb{F}\lambda\alpha & \xrightarrow{\simeq} & \mathbb{F}\mu^2\alpha & \xrightarrow{0} & 0? \\
0? & \xrightarrow{0} & \mathbb{Z}\mu^2\gamma & \rightarrowtail & \mathbb{Z}\lambda & \twoheadrightarrow & \mathbb{F}\lambda\alpha^2? & \to & 0 & \to & 0?
\end{array}$$

This uses the information about KT_* which we have just proved, the periodicity, and our determination of KO_{8n} and KO_{8n+1}. In particular, this shows that certain maps are iso, which forces others to vanish or be epi or mono, as marked on the diagram.

From the third, fourth and fifth rows respectively, we deduce that $KO_{-2} = 0$, $KO_{-1} = 0$ and $KO_2 = \mathbb{F}\alpha^2$. Putting this back in, we get :

$$\begin{array}{ccccccccccc}
0? & \to & \mathbb{Z}\mu^{-1}\gamma & \twoheadrightarrow & \mathbb{Z}\lambda^{-1}\beta? & \to & 0 & \to & 0 & \to & 0? \\
\mathbb{Z} & \xrightarrow{\simeq} & \mathbb{Z} & \xrightarrow{0} & 0? & \to & 0 & \to & \mathbb{Z}\mu^{-1}\gamma & \xrightarrow{\simeq} & \mathbb{Z}\lambda^{-1}\beta? \\
\mathbb{F}\alpha & \xrightarrow{\simeq} & \mathbb{F}\alpha & \to & 0 & \to & \mathbb{Z} & \xrightarrow{\simeq} & \mathbb{Z} & \xrightarrow{0} & 0? \\
\mathbb{F}\alpha^2 & \to & 0 & \to & 0 & \to & \mathbb{F}\alpha & \xrightarrow{\simeq} & \mathbb{F}\alpha & \to & 0 \\
0? & \xrightarrow{0} & \mathbb{Z}\gamma & \rightarrowtail & \mathbb{Z} & \twoheadrightarrow & \mathbb{F}\alpha^2 & \to & 0 & \to & 0 \\
\mathbb{Z}\beta? & \rightarrowtail & \mathbb{Z}\mu & \twoheadrightarrow & \mathbb{F}\alpha & \xrightarrow{0} & 0? & \xrightarrow{0} & \mathbb{Z}\gamma & \rightarrowtail & \mathbb{Z} \\
0? & \to & \mathbb{F}\mu\alpha & \xrightarrow{\simeq} & \mathbb{F}\alpha^2 & \xrightarrow{0} & \mathbb{Z}\beta? & \rightarrowtail & \mathbb{Z}\mu & \twoheadrightarrow & \mathbb{F}\alpha \\
0 & \to & 0 & \to & 0? & \xrightarrow{\simeq} & 0? & \to & \mathbb{F}\mu\alpha & \xrightarrow{\simeq} & \mathbb{F}\alpha^2 \\
0 & \to & \mathbb{Z}\mu\gamma & \twoheadrightarrow & \mathbb{Z}\beta? & \to & 0 & \to & 0 & \to & 0? \\
\mathbb{Z}\lambda & \xrightarrow{\simeq} & \mathbb{Z}\mu^2 & \xrightarrow{0} & 0? & \to & 0 & \to & \mathbb{Z}\mu\gamma & \twoheadrightarrow & \mathbb{Z}\beta? \\
\mathbb{F}\lambda\alpha & \twoheadrightarrow & \mathbb{F}\mu^2\alpha & \to & 0 & \to & \mathbb{Z}\lambda & \xrightarrow{\simeq} & \mathbb{Z}\mu^2 & \xrightarrow{0} & 0? \\
\mathbb{F}\lambda\alpha^2 & \to & 0 & \to & 0 & \rightarrowtail & \mathbb{F}\lambda\alpha & \twoheadrightarrow & \mathbb{F}\mu^2\alpha & \to & 0 \\
0? & \xrightarrow{0} & \mathbb{Z}\mu^2\gamma & \rightarrowtail & \mathbb{Z}\lambda & \twoheadrightarrow & \mathbb{F}\lambda\alpha^2 & \to & 0 & \to & 0
\end{array}$$

From the 6th row we deduce that $KO_3 = 0$. Feeding this into the 8th row, we see that $KO_5 = 0$. Finally, the 6th and 10th rows show that $KO_4 = \mathbb{Z}\beta$. This completes the proof that $AO_* \to KO_*$ is iso.

7.3. Derivation of Relations in Homology

In this section we prove the relations stated in sections 2.1–2.5. We start with a little geometry. Suppose $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$.

LEMMA 7.3.1. The scalar extension map

$$\mathbb{R}P^d \longrightarrow \mathbb{K}P^d \tag{7.3.1}$$

is homotopic to the map

$$\mathbb{R}P^d \longrightarrow \mathbb{R}P^d/\mathbb{R}P^{d-1} = S^d \simeq \mathbb{K}P^1 \longrightarrow \mathbb{K}P^d \tag{7.3.2}$$

PROOF. Think of

$$\begin{aligned} \mathbb{R}P^{d-1} &\quad \text{as} \quad P_{\mathbb{R}}\mathbb{K} && (7.3.3)\\ \mathbb{R}P^{d} &\quad \text{as} \quad P_{\mathbb{R}}(\mathbb{K}\oplus\mathbb{R}) && (7.3.4)\\ \mathbb{K}P^{1} &\quad \text{as} \quad P_{\mathbb{K}}(\mathbb{K}\otimes_{\mathbb{R}}\mathbb{R}\oplus\mathbb{K}) && (7.3.5)\\ \mathbb{K}P^{d} &\quad \text{as} \quad P_{\mathbb{K}}(\mathbb{K}\otimes_{\mathbb{R}}\mathbb{K}\oplus\mathbb{K}) && (7.3.6) \end{aligned}$$

Any real-linear embedding $\mathbb{K}\oplus\mathbb{R} \to \mathbb{K}\otimes_{\mathbb{R}}\mathbb{K}\oplus\mathbb{K}$ induces a map $\mathbb{R}P^d \longrightarrow \mathbb{K}P^d$ and the space of such embeddings is connected so any two such maps are homotopic. The two maps in question arise in this way from the two embeddings

$$\begin{aligned} (z,t) &\mapsto (1\otimes z, t) && (7.3.7)\\ (z,t) &\mapsto (z\otimes 1, t) && (7.3.8) \end{aligned}$$

□

PROOFS FOR SECTION 2.1.
The first thing we deduce from the above lemma is that the scalar extension maps $z_{\mathbb{R},k}$ to 0 if $k < d$ and to $z_{\mathbb{K},d}$ if $k = d$. Extending this inductively using the coproduct formula (2.1.7) (or by working in cohomology) we prove (2.1.3). The circle product formula (2.1.12) is the only one consistent with the coproduct. This is all very standard, of course.

Given $x \in \pi_n E$, it is clear from the definitions that $e^{\circ n} \circ [x] \in \mathrm{H}_n\, \underline{E}_0$ is the image under $x\colon S^n \to \underline{E}_0$ of the fundamental class. The homotopy element $\alpha\colon S^1 = \mathbb{R}P^1 \to BO$ classifies the reduced canonical line bundle, and so carries the class $z_{\mathbb{R},1} * [-1] = \overline{z}_{\mathbb{R},1}$ as claimed in (2.1.13). The proof of (2.1.14) is essentially the same.

For (2.1.15) we need a little Clifford module theory. Write ζ for the canonical reduced bundle over $\mathbb{R}P^4$ and π for the projection $\mathbb{R}P^4 \to \mathbb{H}P^1$; the above shows that $\pi^*\beta \simeq \mathbb{H}\otimes_{\mathbb{R}}\zeta$. Recall that the fourth negative definite Clifford algebra C_4 is $M_2\mathbb{H}$; the module over it corresponding to β is just $V = \mathbb{H}^2$ with $M_2\mathbb{H}$ acting by left multiplication. This action commutes with the right action of $\mathbb{H}$, giving the bundle β a right symplectic structure. We can twist this by conjugation in $\mathbb{H}$ to get a left structure and then form the external tensor product $\beta\otimes_{\mathbb{H}}\beta$ over $\mathbb{H}P^1\wedge\mathbb{H}P^1 \simeq S^8$. This corresponds to the Clifford module $V\otimes_{\mathbb{H}} V$ over C_8. This module has real dimension 16 and so coincides with $\pm\lambda$. The proof in [**3**] that $\beta^2 = 4\lambda$ can be imitated to show that the sign is positive. Thus, over $\mathbb{H}P^1\wedge\mathbb{R}P^4$:

$$\beta\otimes_{\mathbb{R}}\zeta \simeq \beta\otimes_{\mathbb{H}}(\mathbb{H}\otimes_{\mathbb{R}}\zeta) \simeq \beta\otimes_{\mathbb{H}}\pi^*\beta \simeq (1\wedge\pi)^*\lambda \tag{7.3.9}$$

The corresponding classifying maps $\mathbb{H}P^1\wedge\mathbb{R}P^4 \to BO$ are therefore homotopic. Considering the adjoint maps $\mathbb{R}P^4 \longrightarrow \mathrm{map}(\mathbb{H}P^1, BO) \simeq \Omega^4 BO$ we find that

$$\mathbb{R}P^4 \xrightarrow{\zeta} BO \xrightarrow{\circ[\beta]} \Omega^4 BO \tag{7.3.10}$$

agrees with

$$\mathbb{R}P^4 \xrightarrow{\pi} \mathbb{H}P^1 \xrightarrow{e^{\circ 4}\circ[\lambda]} \Omega^4 BO \tag{7.3.11}$$

Evaluating these maps on the top homology class, we find that $[\beta]\circ\overline{z}_{\mathbb{R},4} = e^{\circ 4}\circ[\lambda]$ as claimed.

To prove (2.1.16) we quote from [**19**] the fact that the Kudo-Araki operations in $\mathrm{H}_*(\mathbb{Z}\times BO)$ satisfy

$$Q^n[1] = [1]z_n \tag{7.3.12}$$

From this we deduce (for $n > 0$)

$$\begin{array}{rcll} (e^{\circ n})^2 & = & Q^n(e^{\circ n}) & \text{as } e^{\circ n} \text{ has dimension } n \\ & = & e^{\circ n}\circ Q^n[1] & \text{as } Q^n \text{ commutes with homology suspension} \\ & = & e^{\circ n}\circ([1]z_n) & \\ & = & e^{\circ n}\circ z_n & \text{by standard Hopf ring manipulations} \end{array} \tag{7.3.13}$$

as claimed. □

Proof of Proposition 2.2.1.
We next consider the relations (2.2.1)–(2.2.10) offered for $\mathrm{H}\mathbb{F}_*\,\underline{KO}_*$. As mentioned in section 2.1, we can deduce all these from proposition 2.1.1, together with our knowledge of various maps and their action in homotopy. To prove (2.2.5) we start with (2.1.14), take the circle product with $[\nu]$ and apply the forgetful map $f_U\colon KU \to KO$. If instead we take the circle product by $[\nu^2]$ and apply f_U, we find

$$[\beta]\circ\overline{z}_2 = 0 \tag{7.3.14}$$

$$[\beta]\circ z_2 = [\beta]\circ([1]\overline{z}_2) = [\beta]([\beta]\circ\overline{z}_2) = 0 \tag{7.3.15}$$

as claimed in (2.2.7). For (2.2.10), we have

$$(e^{\circ 3})^2 = e^{\circ 3}\circ z_3 = e\circ(e^{\circ 2}\circ z_2)\circ z_1 = e\circ(e^{\circ 2})^2\circ z_1 \tag{7.3.16}$$

This vanishes as the circle product of a primitive (e) and a decomposable ($(e^{\circ 2})^2$) always does. The other relations in the list are clear or already dealt with. □

Deduction of Relations in Theorem 2.2.2.
We next have to deduce from proposition 2.2.1 all the relations implicit in (2.2.11). Taking the circle product of (2.2.8) by $[\lambda]\circ z(t)$ (which is grouplike) we find that

$$e\circ[\lambda]\circ z_1\circ z(t) = (e^2)\circ[\lambda]\circ z(t) = (e\circ[\lambda]\circ z(t))^2 = \sum_k (e\circ[\lambda]\circ z_k)^2 t^{2k} \tag{7.3.17}$$

whence

$$e\circ[\lambda]\circ z_{2k-1} = (e\circ[\lambda]\circ z_{k-1})^2 \tag{7.3.18}$$

$$e^{\circ 2}\circ[\lambda]\circ z_{2k-1} = e\circ(e\circ[\lambda]\circ z_{k-1})^2 = 0 \tag{7.3.19}$$

Similarly, the circle product of (2.2.9) by $[\lambda]\circ z(t)$ gives

$$e^{\circ 2}\circ[\lambda]\circ z_{4k-2} = (e^{\circ 2}\circ[\lambda]\circ z_{2k-2})^2 \tag{7.3.20}$$

$$e^{\circ 3}\circ[\lambda]\circ z_{4k-2} = 0 \tag{7.3.21}$$

And from (2.2.10) we obtain

$$(e^{\circ 3}\circ[\lambda]\circ z_{4k})^2 = 0 \tag{7.3.22}$$

In $\mathrm{H}\mathbb{F}_*(\mathbb{Z}\times BSp)$ we have

$$[\beta]\circ\overline{z}_1 = [\beta]\circ e\circ[\alpha] = [\beta\alpha]\circ e = [0]\circ e = 0 \tag{7.3.23}$$

$$[\beta]\circ z_1 = [\beta]\circ([1]\overline{z}_1) = [\beta]([\beta]\circ\overline{z}_1) = 0 \tag{7.3.24}$$

$$[\beta]\circ z_{2k+1} = [\beta]\circ z_1\circ z_{2k} = 0 \tag{7.3.25}$$

Similarly (using (2.2.7))

$$[\beta] \circ z_{4k+2} = [\beta] \circ z_2 \circ z_{4k} = 0 \tag{7.3.26}$$

In $\mathbb{HF}_*(U/Sp)$ we have

$$(e \circ [\beta] \circ z(t))^2 = e^2 \circ [\beta] \circ z(t) = e \circ z_1 \circ [\beta] \circ z(t) = 0 \tag{7.3.27}$$

In $\mathbb{HF}_*(O/U)$ we have

$$[\alpha^2] \circ \overline{z}_1 = [\alpha^2] \circ e \circ [\alpha] = e \circ [\alpha^3] = e \circ [0] = 0 \tag{7.3.28}$$

$$\overline{[\alpha^2]} \circ z_{2k+1} = ([\alpha^2] - [0]) \circ ([1]\overline{z}_1) \circ z_{2k} = 0 \tag{7.3.29}$$

$$(\overline{[\alpha^2]} \circ z(t))^2 = ([\alpha]^2 - [0]^2) \circ z(t) = ([0] - [0]) \circ z(t) = 0 \tag{7.3.30}$$

Note also that the relation $e^{\circ 2} \circ [\beta] = \overline{[\alpha^2]} \circ \overline{z}_{2k}$ can be circled with $z(t)$ to obtain a relation expressing all the elements $e^{\circ 2} \circ [\beta] \circ z_{4k}$ in terms of the generators $\overline{[\alpha^2]} \circ z_{2k}$. Moving to $\mathbb{HF}_*(O)$, we find

$$e \circ \overline{[\alpha^2]} = (e \circ [\alpha]) \circ \overline{[\alpha]} = \overline{[\alpha]} \circ \overline{z}_1 \tag{7.3.31}$$

□

This proves that the object described in table (2.2.11) at least admits a sensible map to $\mathbb{HF}_* \underline{KO}_*$. We shall prove in section 7.6 that it is an isomorphism. We repeat the same scheme for KT.

Proof of Proposition 2.3.1
The relation (2.3.4) is the image of (2.1.13) under l_O. For (2.3.5), we apply δ_T to (2.1.14) :

$$e^{\circ 2} \circ [\gamma] = \delta_T(e^{\circ 2} \circ [\nu]) = \delta_T(\overline{z}_2 \circ [1]) = \overline{z}_2 \circ \delta_T[1] = [\alpha] \circ \overline{z}_2 \tag{7.3.32}$$

Similarly,

$$[\gamma] \circ \overline{z}_2 = \delta_T([\nu] \circ \overline{z}_2) = \delta_T(e^{\circ 2} \circ [\nu^2]) = e^{\circ 2} \circ [\alpha] \circ [\mu] = (e^2) \circ [\mu] = (e \circ [\mu])^2 \tag{7.3.33}$$

In both the above equations we have used KO-linearity to suppress the distinction between the various families of z's. □

Deduction of Relations in Theorem 2.3.2.
As with KO, we now deduce from proposition 2.3.1 the rest of the relations implicit in theorem 2.3.2. In $\mathbb{HF}_*(\mathbb{Z} \times BT)$ we have :

$$([\mu] \circ \overline{z}_1)^2 = [\mu] \circ \alpha \circ e^2 = [\gamma] \circ \alpha \circ \overline{z}_2 = [\gamma\alpha] \circ \overline{z}_2 = [0] \circ \overline{z}_2 = 0 \tag{7.3.34}$$

$$([\mu] \circ z_1 \circ z(t))^2 = z(t) \circ (([\mu] \circ \overline{z}_1)^2 [\mu]^2) = 0 \tag{7.3.35}$$

$$([\mu] \circ z_{2k+1})^2 = 0 \tag{7.3.36}$$

In $\mathbb{HF}_*(\Omega^2 T)$, the relation

$$e \circ [\mu] \circ z_{2k-1} = (e \circ [\mu] \circ z_{k-1})^2 \tag{7.3.37}$$

is proved in the same way as the analogous one in KO. Also

$$[\gamma] \circ z_1 = [\gamma]([\gamma] \circ \alpha \circ e) = 0 \tag{7.3.38}$$

In $\mathrm{H}\mathbb{F}_*(\Omega T)$, we have

$$(e \circ [\gamma] \circ z(t))^2 = e^2 \circ [\gamma] \circ z(t) = e^{\circ 2} \circ [\alpha] \circ [\gamma] \circ z(t) = 0 \tag{7.3.39}$$

All the remaining relations are just as in the KO case. □

Proofs for Section 2.4.
In KU we need only remark that

$$e^2 = m_O(e^2) = m_O(e^{\circ 2} \circ [\alpha]) = e^{\circ 2} \circ [0] = 0 \tag{7.3.40}$$

and deduce that

$$(e \circ z(t))^2 = e^2 \circ z(t) = 0 \tag{7.3.41}$$

□

Proofs for Section 2.5.
In KSp we have

$$z_{\mathbb{H}}(t) = n_O([1] \circ z_{\mathbb{R}}(t)) = (n_O[1]) \circ z(t) = [\lambda^{-1}\beta\theta] \circ z(t) \tag{7.3.42}$$

□

Justification of Table 2.7.1.
First recall that as $\psi(e) = e \otimes [0] + [0] \otimes e$ and $[0] \circ x = \epsilon(x)$, we find that $e \circ (xy) = (e \circ x)\epsilon(y) + \epsilon(x)(e \circ y)$. In particular, $e \circ x^2 = 0$, and $e \circ ([a]x) = e \circ x$ if $\epsilon(x) = 0$.

We shall also need some cases of the circle product $\overline{z}_k \circ \overline{z}_l$. Firstly, as $x \mapsto z(t) \circ x$ is a homomorphism, we find that

$$\overline{z}_1 \circ z(t) = (z_1/[1]) \circ z(t) = (z_1 \circ z(t))/z(t) = \dot{z}(t)/z(t) = q(t)/t \tag{7.3.43}$$

Also, as $\psi \overline{z}_1 = \overline{z}_1 \otimes [0] + [0] \otimes \overline{z}_1$,

$$\begin{aligned}
\overline{z}_1 \circ z(t) &= \overline{z}_1 \circ ([1]\overline{z}_{\mathbb{R}}(t)) \\
&= (\overline{z}_1 \circ [1])([0] \circ \overline{z}(t)) + ([0] \circ [1])(\overline{z}_1 \circ \overline{z}(t)) \\
&= \overline{z}_1 + \overline{z}_1 \circ z(t)
\end{aligned}$$

It follows that $\overline{z}_1 \circ z_k = \overline{z}_1 \circ \overline{z}_k = q_{k+1}$ when $k > 0$.

From the relation $z(s) \circ z(t) = z(s+t)$ we deduce

$$\overline{z}(s) \circ \overline{z}(t) = \frac{z(s)}{[1]} \circ \frac{z(t)}{[1]} = \frac{z(s+t)/z(s)}{z(t)/[1]} = \overline{z}(s+t)/\overline{z}(s)\overline{z}(t) \tag{7.3.44}$$

It follows that

$$\overline{z}(s)^{\circ 2} = \sum \overline{z}_k^2 s^{2k} = \overline{z}(s)^{-2} \tag{7.3.45}$$

By expanding the above we obtain :

$$
\begin{aligned}
\overline{z}_1 \circ \overline{z}_1 &= \overline{z}_1^2 && (7.3.46)\\
\overline{z}_1 \circ \overline{z}_2 &= \overline{z}_1^3 + \overline{z}_1\overline{z}_2 + \overline{z}_3 && (7.3.47)\\
\overline{z}_1 \circ \overline{z}_3 &= \overline{z}_1^4 && (7.3.48)\\
\overline{z}_1 \circ \overline{z}_4 &= \overline{z}_1^5 + \overline{z}_1^3\overline{z}_2 + \overline{z}_1\overline{z}_2^2 + \overline{z}_1^2\overline{z}_3 + \overline{z}_2\overline{z}_3 + \overline{z}_1\overline{z}_4 + \overline{z}_5 && (7.3.49)\\
\overline{z}_1 \circ \overline{z}_5 &= \overline{z}_1^6 + \overline{z}_1^2\overline{z}_2^2 + \overline{z}_3^2 && (7.3.50)\\
\overline{z}_2 \circ \overline{z}_2 &= \overline{z}_1^4 + \overline{z}_2^2 && (7.3.51)\\
\overline{z}_2 \circ \overline{z}_3 &= \overline{z}_1^5 + \overline{z}_1^3\overline{z}_2 + \overline{z}_1^2\overline{z}_3 && (7.3.52)\\
\overline{z}_2 \circ \overline{z}_4 &= \overline{z}_1^6 + \overline{z}_2^3 + \overline{z}_1\overline{z}_2\overline{z}_3 + \overline{z}_3^2 + \overline{z}_1^2\overline{z}_4 + \overline{z}_2\overline{z}_4 + \overline{z}_1\overline{z}_5 + \overline{z}_6 && (7.3.53)\\
\overline{z}_3 \circ \overline{z}_3 &= \overline{z}_1^6 + \overline{z}_3^2 && (7.3.54)\\
& && (7.3.55)
\end{aligned}
$$

Using this and the relations derived earlier in this section, it is not hard to fill in the table. □

7.4. Primitives and Duality

In this section we investigate the Hopf algebra $A_* = P[\overline{z}_{k+1} \mid k \geq 0]$ from a purely algebraic standpoint. In section 7.6 we will give a Hopf ring proof that it coincides with $H\mathbb{F}_*BO$, so the dual coincides with $H\mathbb{F}^*BO$. This will justify the statements in section 2.6.

For any countably infinite index set I, we let

$$P_I = \lim_{\longleftarrow} JP[x_i \mid i \in J] \tag{7.4.1}$$

Here J runs over the finite subsets of I. To describe the inverse system, we are required to specify a map $P[x_i \mid i \in K] \to P[x_i \mid i \in J]$ for each pair $J \subset K$ of such sets. It is the obvious map, which sends x_i to zero if $i \in K \setminus J$. The symmetric group Σ_I of permutations of I acts on P_I and we denote the ring of invariants by P_I^Σ. It is of course a polynomial algebra on generators σ_k^I, where

$$\prod_I (1 + x_i t) = \sum_k \sigma_k^I t^k \tag{7.4.2}$$

Any bijection $\phi\colon I \to J$ induces an isomorphism $P_I^\Sigma \simeq P_J^\Sigma$, which is independent of ϕ. The map

$$A_* = P[\overline{z}_{k+1}] \longrightarrow P_I^\Sigma \tag{7.4.3}$$

$$\overline{z}(t) \mapsto \prod_I (1 + x_i t) = \sum_k \sigma_k^I t^k \tag{7.4.4}$$

commutes with these isomorphisms and is itself iso. The evident isomorphism

$$P_{I \sqcup J} \longrightarrow P_I \otimes P_J \tag{7.4.5}$$

induces the usual coproduct on A_*, for which $\psi\overline{z}(t) = \overline{z}(t) \otimes \overline{z}(t)$. There is a unique power series

$$p(\underline{f}) = p(f_1, f_2, \dots) = \sum_\alpha p_\alpha f^\alpha \in A_*[\![f_k \mid k > 0]\!] \simeq P_I^\Sigma[\![f_k]\!] \tag{7.4.6}$$

such that

$$\prod_I f(x_i) = p(\underline{f}) \tag{7.4.7}$$

for any series $f(x)=\sum_k f_k x^k$ with $f_0=1$. By considering $f(x)=1+tx$, we see that $p_{ke_1}=\overline{z}_k$. On the other hand, working mod the ideal $(f_k \mid k>0)^2$ we find that

$$\sum_{k>0} p_{e_k} f_k = \sum_I \sum_{k>0} x_i^k f_k \tag{7.4.8}$$

Also

$$q(t)=\sum_{k>0} q_k t^k = t\mathrm{dlog}z(t)/\mathrm{d}t = \sum_I t\mathrm{dlog}(1+tx_i)/\mathrm{d}t = \sum_I \sum_{k>0} x_i^k t^k \tag{7.4.9}$$

so

$$p_{e_k}=q_k \tag{7.4.10}$$

From the description of the coproduct above, it is clear that

$$\psi p(\underline{t}) \;=\; p(\underline{t})\otimes p(\underline{t}) \tag{7.4.11}$$

$$\psi p_\alpha \;=\; \sum_{\alpha=\beta+\gamma} p_\beta\otimes p_\gamma \tag{7.4.12}$$

We define elements $\overline{z}'_k\in A^*$ by the relation

$$\langle \overline{z}'_k,\overline{z}^\alpha\rangle = \begin{cases} 1 & \text{if } \alpha=ke_1 \\ 0 & \text{otherwise}\end{cases} \tag{7.4.13}$$

Suppose we choose an element $0\in I$. We can then define a functional $P_I\to\mathbb{F}$ which sends x_0^k to 1 and all other monomials to 0. One can check easily that this extends the definition of $\overline{z}'_k$ on $A_*=P_I^\Sigma < P_I$. By using this extension, we find that

$$\langle \overline{z}'_k, p(\underline{t})\rangle = t_k \tag{7.4.14}$$

$$\langle \overline{z}'(t), p(\underline{t})\rangle = p(t,t^2,t^3,\dots) \tag{7.4.15}$$

Hence, using $\psi p(\underline{t})=p(\underline{t})\otimes p(\underline{t})$

$$\langle (\overline{z}')^\alpha, p(\underline{t})\rangle = t^\alpha \tag{7.4.16}$$

$$\langle (\overline{z}')^\alpha, p_\beta\rangle = \delta_{\alpha,\beta} \tag{7.4.17}$$

This shows in particular that the elements $(\overline{z}')^\alpha\in A^*$ are linearly independent, and so give a basis by counting dimensions. In other words, A^* is a polynomial algebra generated by the elements $\overline{z}'_k$. Similarly, the elements p_α form a basis for A_*.

Using the expression above for ψp_α we see that the filtration of A_* by copowers of the augmentation ideal is :

$$\begin{aligned} F_sA_* &= \text{annihilator}((I^*)^{s+1})\subseteq A_* \\ &= \ker(A_*\xrightarrow{\psi} A_*^{\otimes(s+1)} \longrightarrow I_*^{\otimes(s+1)}) \\ &= \mathbb{F}\{p_\alpha \mid |\alpha|\le s\}\end{aligned}$$

In particular, we find that the module of primitives $\mathrm{Prim}(A_*)=F_1A_*\cap I$ is just $\mathbb{F}\{p_{e_{k+1}}\}=\mathbb{F}\{q_{k+1}\}$.

Consider the associated graded algebra

$$G_*A_* = \bigoplus_s F_s/F_{s+1} \tag{7.4.18}$$

It is spanned by the images of the elements p_α in $G_{|\alpha|}A_{\|\alpha\|}$, which we shall still denote by p_α. By considering the terms in $\psi((\overline{z}')^\alpha)$ which are of the least possible order of decomposability, we find that

$$p_\alpha p_\beta = \left(\prod_k (\alpha_k,\beta_k)\right) p_{\alpha+\beta} \qquad (\mathrm{mod}\ F_{|\alpha|+|\beta|+1}) \tag{7.4.19}$$

It follows that $p(\underline{s})p(\underline{t}) = p(\underline{s}+\underline{t})$ in G_*A_* , and hence that

$$G_*A_* = \bigotimes_k D[q_k] \tag{7.4.20}$$

$$q^{[\alpha]} = p_\alpha \tag{7.4.21}$$

The Frobenius map $F\colon A_* \to A_{2*}$ sending x to x^2 is a Hopf algebra homomorphism. The Hopf algebra cokernel is evidently $E[\overline{z}_{k+1}]$. One sees easily that the dual map in A^* (called the Verschiebung and written V) sends $\overline{z}'_{2k}$ to $\overline{z}'_k$ and $\overline{z}'_{2k+1}$ to 0. The Hopf algebra kernel of V (i.e. . the equaliser of the maps $(V \otimes 1)\circ\psi$ and $1\otimes\eta$ from A^* to $A^*\otimes A^*$) is thus $P[\overline{z}'_{2k+1}]$. This is therefore the dual of $E[\overline{z}_k]$. It follows that the images of $\{p_\alpha \mid \forall k\ \alpha_{2k} = 0\}$ form a basis for $E[\overline{z}_k]$, and that the primitives in this algebra are precisely the elements q_{2k+1}.

We obtain precisely equivalent results for (our candidates for) $\mathrm{H}\mathbb{F}_*BU$ and $\mathrm{H}\mathbb{F}_*BSp$ after multiplying degrees by two or four as appropriate.

The circle product formulae 2.6.10 and 2.6.11 were proved above while justifying table 2.7.1. The further formulae 2.6.12 and 2.6.13 follow easily.

7.5. Operations

We next discuss the formulae for the Steenrod and Kudo-Araki operations given in section 2.8. The formula (2.8.1) can be proved by expanding out and comparing with the usual formula with binomial coefficients. A formal power series proof is more interesting, however. One knows that

$$\mathrm{H}\mathbb{F}^*\mathbb{K}P^\infty = P[z'] \qquad\qquad |z'| = d \tag{7.5.1}$$

and

$$Sq(s)z' = z' + s^d {z'}^2 \tag{7.5.2}$$

for instability reasons. We recall that the product in cohomology and coproduct in homology interact as follows :

$$\langle x, uv\rangle = \langle \psi x, u\otimes v\rangle = \sum \langle x_0, u\rangle\langle x_1, v\rangle \tag{7.5.3}$$

where $\psi x = \sum x_0 \otimes x_1$. As $\psi z(t) = z(t)\otimes z(t)$ and $\langle z(t), z'\rangle = t$, it follows easily that

$$\langle z(t), f(z')\rangle = f(t) \tag{7.5.4}$$

for any power series f (and any parameter t). Also, as $Sq(s)$ is a ring map in cohomology, $Sq(s)f(z') = f(z' + s^d{z'}^2)$. We deduce that

$$\langle z(s^dt^2+t), f(z')\rangle = f(s^dt^2+t) = \langle z(t), f(s^d{z'}^2+z')\rangle = \langle z(t), Sq(s)f(z')\rangle \tag{7.5.5}$$

which implies that $z(t)Sq(s) = z(s^dt^2+t)$ as claimed.

All our statements about $\mathcal{A}^*$ and $\mathcal{A}_*$ are standard, except perhaps for the way we express the pairing. From the definitions it is clear that

$$\langle Sq(s), \xi(t)\rangle = t + t^2 s \tag{7.5.6}$$

Putting this together with the coproduct $\psi Sq(s) = Sq(s)\otimes Sq(s)$ we find that

$$\langle Sq(s), f(\xi(t))\rangle = f(t+t^2s) \tag{7.5.7}$$

as claimed.

To understand the coaction of $\mathcal{A}_*$ on $\mathrm{H}\mathbb{F}_*\mathbb{R}P^\infty$, we shall define algebraically a map

$$\alpha\colon \mathrm{H}\mathbb{F}_*\mathbb{R}P^\infty \to \mathcal{A}_*\otimes \mathrm{H}\mathbb{F}_*\mathbb{R}P^\infty \tag{7.5.8}$$

and then check firstly that it is a valid coaction and secondly that it is the right one. The definition is :

$$\alpha z(t^d) = (1 \otimes z)(\xi(t)^d \otimes 1) \tag{7.5.9}$$

This is a coaction because

$$(\psi \otimes 1)\alpha z(t^d) = (1 \otimes 1 \otimes z)(1 \otimes \xi \otimes 1)(\xi(t)^d \otimes 1 \otimes 1) = (1 \otimes \alpha)\alpha z(t^d) \tag{7.5.10}$$

It therefore induces a genuine action of the Steenrod algebra, by the following formula (in which $\alpha x = \sum a' \otimes x''$)

$$xR = \langle \alpha x, R\rangle := \sum \langle R, a'\rangle x'' \tag{7.5.11}$$

We need only check that this is correct on the generators. We find

$$z(t^d)Sq(s) = \langle Sq(s), (1 \otimes z)(\xi(t)^d \otimes 1)\rangle = z(t^d + t^{2d}s^d) \tag{7.5.12}$$

as required.

We also need to verify the formal power series version of the Wu formula :

$$Sq(st)\overline{z}'(s + t^2 s) = \overline{z}'(ts + t^2 s)\overline{z}'(s + ts) \tag{7.5.13}$$

We do this by pulling back to

$$\mathrm{H}\mathbb{F}^*(\mathbb{R}P^\infty)^n = P[z'_{(0)}, \dots z'_{(n-1)}] \tag{7.5.14}$$

$$\overline{z}'(t) \mapsto \prod_k (1 + \overline{z}'_{(k)}t) \tag{7.5.15}$$

This is injective through a range of dimensions increasing with n. As $|\overline{z}'_{(k)}| = 1$, we have

$$\begin{aligned} Sq(s)\overline{z}'_{(k)} &= \overline{z}'_{(k)} + s{\overline{z}'_{(k)}}^2 \\ Sq(st)(1 + \overline{z}'_{(k)}(s + t^2 s)) &= 1 + \overline{z}'_{(k)}(s + t^2 s) + {\overline{z}'_{(k)}}^2(ts^2 + t^3 s^2) \\ &= (1 + \overline{z}'_{(k)}(ts + t^2 s))(1 + \overline{z}'_{(k)}(s + ts)) \end{aligned}$$

from which the result follows.

Our first expression (2.8.17) for the Kudo-Araki operations can be checked by expanding and comparing with the formulae in [**19**] although a formal power series proof is really called for. The second formula (2.8.18) follows from the first by substituting $t + t^2 \mapsto t$ and using $z(s) \circ z(t) = z(s + t)$. By taking $t = 0$ we obtain (2.8.19).

7.6. Completeness of Generators and Relations

In this section we prove that the generators and relations offered for the various mod 2 Hopf rings are complete, so that the descriptions offered in sections 2.2–2.4 are correct. For the reasons discussed in the introduction, we present a fairly complete calculation of the homology using only Hopf ring methods. We also give full details of a number of spectral sequence arguments. The methods used appear to be well known to experts, but there is a dearth of fully explained examples in the literature.

Proof of Theorem 2.4.2.
By considering the definition of the structure map $\Sigma(\mathbb{Z} \times BU) \to U$ of the KU-spectrum, we find that the composite

$$S^1 \times \mathbb{C}P^\infty \longrightarrow \Sigma\mathbb{C}P^\infty_+ \longrightarrow \Sigma(\mathbb{Z} \times BU) \longrightarrow U \tag{7.6.1}$$

sends a point (z, L) to the linear map with eigenvalues z on L and 1 on $L^\perp$. In homology, it sends $\Sigma z_{\mathbb{C}}(t)$ to $e \circ z_{\mathbb{C}}(t)$. The usual inductive argument based on the Serre spectral sequence of the fibration $U(n) \to U(n+1) \to S^{2n+1}$ shows that $\mathrm{H}\mathbb{F}_*U = E[e \circ z_{\mathbb{C},2k}]$. We next give a Hopf ring based calculation of $\mathrm{H}\mathbb{F}_*BU$. We shall now think of U as the (-1)-space, so that $\mathrm{H}\mathbb{F}_*U = E[[\nu] \circ z_{\mathbb{C},2k}]$. We need to analyse the Rothenberg-Steenrod spectral sequence

$$\mathrm{Tor}^{H_*U}_{s,t}(\mathbb{F},\mathbb{F}) \Longrightarrow \mathrm{H}\mathbb{F}_{t+s}BU \tag{7.6.2}$$

The E^2-term is just

$$D[\sigma(e \circ [\nu] \circ z_{2k})] = E[[\nu] \circ z_{(2k,l)} \mid k,l \geq 0] \tag{7.6.3}$$

where we write $z_{(2k,l)}$ for $\sigma(e \circ z_{2k})^{[2^l]} = B_{2^l}(e \circ z_{2k})$. To understand the differentials, we map in the Rothenberg-Steenrod spectral sequence

$$\mathbb{F} \oplus \mathbb{F}\{B_{k+1}(e \circ [\nu])\} = \mathrm{Tor}^{H_*U(1)}_{**}(\mathbb{F},\mathbb{F}) \Longrightarrow \mathrm{H}\mathbb{F}_*\mathbb{C}P^\infty = \mathbb{F} \oplus \mathbb{F}\{\overline{z}_{\mathbb{C},2k+2}\} \tag{7.6.4}$$

This collapses for obvious reasons. Using

$$d^r(B(s)(e \circ [\nu] \circ z_{\mathbb{C}}(t))) = d^r(B(s)(e) \circ [\nu] \circ z_{\mathbb{C}}(t)) = d^r(B(s)(e)) \circ [\nu] \circ z_{\mathbb{C}}(t) = 0 \tag{7.6.5}$$

we find that all differentials in the original spectral sequence vanish. In particular, the primitive elements $\sigma(e \circ [\nu] \circ z_{2k})$ on the 1-line all survive to E^∞. We recall that the 1-line at E^∞ is isomorphic to the bottom filtration of $\mathrm{H}\mathbb{F}_*BU$ with $\sigma(x) = \langle x \rangle$ corresponding to $e \circ x$. It follows that $\langle e \circ [\nu] \circ z_{\mathbb{C},2k} \rangle = \overline{z}_{\mathbb{C},2} \circ z_{\mathbb{C},2k} = q_{\mathbb{C},2k+2}$. This shows that the map $A_* = P[\overline{z}_{\mathbb{C},2k+2}] \to \mathrm{H}\mathbb{F}_*BU$ is injective on primitives. This implies in the usual way that the map is injective everywhere, as a nonzero element of minimal degree in the kernel would have to be primitive. The Poincaré series of $A_* = \bigotimes_{k \geq 0} P[\overline{z}_{\mathbb{C},2k+2}]$ and $E^\infty = \bigotimes_{l \geq 0} D[\langle e \circ [\nu] \circ z_{\mathbb{C},2l} \rangle]$ are both equal to $\prod_{k \geq 0}(1 - t^{2k+2})^{-1}$. The Poincaré series of $\mathrm{H}\mathbb{F}_*BU$ agrees with that of E^∞. This implies that the map $A_* \longrightarrow \mathrm{H}\mathbb{F}_*BU$ must be iso.

It is of interest to look a little more closely at the E^∞ term and the associated filtration of H_*BU. Write I^* for the augmentation ideal in H^*BU. As $\langle F_1A_*, (I^*)^2 \rangle = 0$ and $\psi F_s \subset \sum_{s=t+u} F_t \otimes F_u$ we see that $\langle F_sA_*, (I^*)^{s+1} \rangle = 0$. Thus

$$F_sA_* \subseteq \mathrm{ann}((I^*)^{s+1}) = \mathbb{F}\{p_\alpha \mid |\alpha| \leq s\} \tag{7.6.6}$$

and by counting dimensions we must have equality. In other words, the filtration is precisely by copowers of the unit coideal.

I claim moreover that the induced map

$$\theta_0 \colon G_*A_* = D[q_{2k+2}] \longrightarrow D[\sigma(e \circ [\nu] \circ z_{2k})] = E^\infty \tag{7.6.7}$$

is the obvious one, which maps $p_\alpha = q^{[\alpha]}$ to $\prod_k B_{\alpha_k}(e \circ [\nu] \circ z_{2k-2})$, and which we refer to temporarily as θ_1. Note that both θ's are isomorphisms of bigraded bicommutative Hopf algebras. We recall that the endomorphisms of such an object form an Abelian group; the difference ϕ between $\theta_1^{-1}\theta_0$ and the identity is given by $\phi(x) = \sum \theta_1^{-1}\theta_0(x')\chi(x'')$ where $\psi(x) = \sum x' \otimes x''$. As $\theta_i(q_{2k}) = \sigma(e \circ [\nu] \circ z_{2k-2})$ for $i = 0,1$, we see that $\phi(q_{2k}) = 0$. If we know that $\phi(q^{[\beta]}) = 0$ for $|\beta| < |\alpha|$ then we deduce that $\phi(q^{[\alpha]})$ is primitive. As the primitives are concentrated on the 1-line, this shows that $\phi(q^{[\alpha]}) = 0$. It follows that $\phi = \eta\epsilon$ which is the zero element of our Abelian group, so $\theta_0 = \theta_1$ as claimed.

We have seen that $\mathrm{H}\mathbb{F}_*(0 \times BU) = P[\overline{z}_{2k+2}]$. It follows that

$$\mathrm{H}\mathbb{F}_*(\mathbb{Z} \times BU) = \mathrm{H}\mathbb{F}_*(0 \times BU) \otimes \mathbb{Z}[\pi_0 KU] = P[z_{2k}][-1] \tag{7.6.8}$$

as claimed.

As we take Bott periodicity for granted, it is not really necessary to consider the spectral sequence passing from $\mathbb{Z}\times BU$ to U, but it is easy and interesting to see how it works.

$$\mathrm{Tor}^{H_*(\mathbb{Z}\times BU)}_{s,t}(\mathbb{F},\mathbb{F}) = E[\sigma(z_{2k})] \Longrightarrow \mathrm{H}\mathbb{F}_{s+t}(U) \tag{7.6.9}$$

The E^2-page is generated by elements $\sigma(z_{2k})$ on the 1-line which must be permanent cycles as there is nowhere for the differentials to go. It follows that the differentials are zero everywhere. As E^∞ is generated by elements representing $e\circ z_{2k}$ and we know that $(e\circ z_{2k})^2 = 0$, we have an epimorphism $E[e\circ z_{2k}] \to \mathrm{H}\mathbb{F}_*(U)$. This must be iso by counting dimensions. □

Proof of Theorem 2.2.2.
We next look at the spectrum KO. Write η for the unreduced canonical line bundle over $\mathbb{R}P^\infty$, and α for the reduced canonical line bundle over $\mathbb{R}P^1 = S^1$. We first note that the composite

$$\mathbb{R}P^\infty \xrightarrow{\eta} 1\times BO \subseteq \mathbb{Z}\times BO \xrightarrow{\circ[\alpha]} O \tag{7.6.10}$$

is (essentially by definition) the clutching function of the bundle $\alpha\otimes\eta$ over $\Sigma\mathbb{R}P^\infty_+$. One checks that this is just the usual map $\mathbb{R}P^\infty \longrightarrow O$ which sends a line L to the reflection across $L^\perp$. Again, it follows from the usual argument by induction over the subgroups $O(n)$ and the Serre spectral sequence that our description of $\mathrm{H}\mathbb{F}_*\,\underline{KO}_{-1} = \mathrm{H}\mathbb{F}_*O$ is correct.

We next consider the Rothenberg-Steenrod spectral sequence

$$\mathrm{Tor}^{H_*O}_{s,t}(\mathbb{F},\mathbb{F}) \Longrightarrow \mathrm{H}\mathbb{F}_{t+s}BO \tag{7.6.11}$$

We argue essentially as in the unitary case. The E^2-term is just $D[\sigma([\alpha]\circ z_k)]$ and the differentials vanish by comparison with the spectral sequence

$$\mathbb{F}\oplus\mathbb{F}\{B_{k+1}(\overline{[\alpha]})\} = \mathrm{Tor}^{H_*O(1)}_{**}(\mathbb{F},\mathbb{F}) \Longrightarrow \mathrm{H}\mathbb{F}_*\mathbb{R}P^\infty = \mathbb{F}\oplus\mathbb{F}\{\overline{z}_{k+1}\} \tag{7.6.12}$$

Just as before, we show that the map $P[\overline{z}_{k+1}] \to \mathrm{H}\mathbb{F}_*BO$ is iso and thus that

$$\mathrm{H}\mathbb{F}_*(\mathbb{Z}\times BO) = P[z_k][-1] \tag{7.6.13}$$

as claimed.

We next consider the spectral sequence

$$\mathrm{Tor}^{H_*(\mathbb{Z}\times BO)}_{s,t}(\mathbb{F},\mathbb{F}) = E[\sigma(z_k)] \Longrightarrow \mathrm{H}\mathbb{F}_{t+s}(U/O) \tag{7.6.14}$$

As in the complex case, The E^2-page is generated by the 1-line so all the differentials vanish. As E^∞ is generated by elements representing $e\circ z_k$ and we know that $e\circ z_{2k-1} = (e\circ z_{k-1})^2$, it follows that the map $P[e\circ z_{2k}] \longrightarrow \mathrm{H}\mathbb{F}_*(U/O)$ is epi. Using

$$\prod_l (1+t^{(2k+1)2^l}) = \sum_m t^{(2k+1)m} = (1-t^{2k+1})^{-1} \tag{7.6.15}$$

We see that the Poincaré series of the source and target agree, so the map is iso as required.

We can analyse the spectral sequence

$$\mathrm{Tor}^{H_*(U/O)}_{s,t}(\mathbb{F},\mathbb{F}) = E[\sigma(e\circ z_{2k})] \Longrightarrow \mathrm{H}\mathbb{F}_{t+s}(Sp/U) \tag{7.6.16}$$

in essentially the same way, to obtain

$$\mathrm{H}\mathbb{F}_*(Sp/U) = P[e^{\circ 2}\circ z_{4k}] \tag{7.6.17}$$

In the spectral sequence

$$\mathrm{Tor}^{H_*(Sp/U)}_{s,t}(\mathbb{F},\mathbb{F}) = E[\sigma(e^{\circ 2}\circ z_{4k})] \Longrightarrow \mathrm{H}\mathbb{F}_{t+s}(Sp) \tag{7.6.18}$$

we again find that all differentials vanish. In this case we use the fact that $(e^{\circ 3} \circ z_{4k})^2 = 0$ to conclude that

$$\mathrm{H}\mathbb{F}_*(Sp) = E[e^{\circ 3} \circ z_{4k}] \tag{7.6.19}$$

Our next spectral sequence is

$$\mathrm{Tor}^{H_*(Sp)}_{s,t}(\mathbb{F},\mathbb{F}) = D[\sigma(e^{\circ 3} \circ z_{4k})] \Longrightarrow \mathrm{H}\mathbb{F}_{t+s}(BSp) \tag{7.6.20}$$

We treat this in a similar way to the sequence $O \Longrightarrow BO$, by mapping in the spectral sequence

$$\mathrm{Tor}^{H_* S^3}_{s,t} = \mathrm{Tor}^{H_* Sp(1)}_{s,t} = D[\sigma(e^{\circ 3})] \Longrightarrow \mathrm{H}\mathbb{F}_* \mathbb{H}P^\infty \tag{7.6.21}$$

which clearly collapses. The image in the target is spanned by the elements $[\theta^{-1}] \circ \overline{z}_{\mathbb{H},4k} = [\lambda^{-1}\beta] \circ \overline{z}_{4k}$. Using $d^r B_k(e^{\circ 3}) = 0$ and taking the circle product with $z(t)$, we find that all the differentials in the spectral sequence for $Sp \Longrightarrow BSp$ also vanish. The map $P[[\lambda^{-1}\beta] \circ \overline{z}_{4k+4}] \to \mathrm{H}\mathbb{F}_* BSp$ is thus injective on primitives, hence injective, hence iso by counting dimensions. It follows that

$$\mathrm{H}\mathbb{F}_*(\mathbb{Z} \times BSp) = P[[\lambda^{-1}\beta] \circ z_{4k}][-\lambda^{-1}\beta] \tag{7.6.22}$$

as claimed.

For notational convenience, we move down eight spaces to $\underline{KO}_{-4}$ where we find that $\mathrm{H}\mathbb{F}_*(\mathbb{Z} \times BSp) = P[[\beta] \circ z_{4k}][-\beta]$

The step from $\mathbb{Z} \times BSp$ to U/Sp is the same as from Sp/U to Sp, giving

$$\mathrm{H}\mathbb{F}_*(U/Sp) = E[e \circ [\beta] \circ z_{4k}] \tag{7.6.23}$$

To understand the next spectral sequence

$$\mathrm{Tor}^{H_*(U/Sp)}_{s,t}(\mathbb{F},\mathbb{F}) = D[\sigma(e \circ [\beta] \circ z_{4k})] \Longrightarrow \mathrm{H}\mathbb{F}_{s+t}(SO/U) \tag{7.6.24}$$

we compare it with the complex analogue

$$\mathrm{Tor}^{H_*(U)}_{s,t}(\mathbb{F},\mathbb{F}) = D[\sigma(e \circ [\nu^2] \circ z_{2k})] \Longrightarrow P[[\nu] \circ \overline{z}_{2k+2}] = \mathrm{H}\mathbb{F}_{s+t} BU \tag{7.6.25}$$

We can map the unitary version to the orthogonal one by the map $f_U : KU \to KO$ which forgets the complex structure. Using $f_U(\nu^2) = \beta$ and KO-linearity, we see that $\sigma(e \circ [\nu^2] \circ z_{4k}) \mapsto \sigma(e \circ [\beta] \circ z_{4k})$ and $\sigma(e \circ [\nu^2] \circ z_{4k+2}) \mapsto 0$. The map is thus epi at E^2 and the source spectral sequence collapses so the target one does also. It follows that we have an epimorphism at E^∞ and therefore also (by induction over the filtration) in the abutment of the spectral sequences. Thus $\mathrm{H}\mathbb{F}_*(SO/U)$ is generated by the images of the elements $[\nu] \circ \overline{z}_{2k}$ for $k > 0$, that is by the elements $[\alpha^2] \circ \overline{z}_{2k}$. We know that these elements square to zero so we have an epimorphism

$$E[[\alpha^2] \circ \overline{z}_{2k+2}] \twoheadrightarrow \mathrm{H}\mathbb{F}_*(SO/U) \tag{7.6.26}$$

By comparing with the Poincaré series derived from our E^∞-page, we deduce that this is an isomorphism, and hence that

$$\mathrm{H}\mathbb{F}_*(O/U) = \mathrm{H}\mathbb{F}_*(SO/U) \otimes \mathbb{Z}[\mathbb{F}\alpha^2] = E[\overline{[\alpha^2]} \circ z_{2k}] \tag{7.6.27}$$

as claimed.

To close the circle, we consider the spectral sequence

$$\mathrm{Tor}^{H_*(O/U)}_{s,t}(\mathbb{F},\mathbb{F}) = D[\sigma(\overline{[\alpha^2]} \circ z_{2k})] \Longrightarrow \mathrm{H}\mathbb{F}_{s+t}(SO) \tag{7.6.28}$$

Taking the circle product by $[\alpha]$ gives an epimorphism from the collapsing sequence for $O \Longrightarrow BO$ to this one, which must therefore also collapse. We know that $(\overline{[\alpha]} \circ z_k)^2 = 0$ from the relations we proved earlier, so we have a map $E[\overline{[\alpha]} \circ z_{k+1}] \to \mathrm{H}\mathbb{F}_* SO$. The primitives $\sigma(\overline{[\alpha^2]} \circ z_{2k})$ in E^∞ correspond to $e \circ \overline{[\alpha^2]} \circ z_{2k} = [\alpha] \circ q_{2k+1}$ so the elements $[\alpha] \circ q_{2k+1}$ are nonzero. It follows that our map is injective on primitives, hence injective, hence iso. Of course we knew that anyway. $\square$

Proof of Theorem 2.3.2.
We now turn to KT. We attack this by considering the fibrations coming from the stable cofibrations in section 1.3. See section 5.4 for generalities about this sort of situation. Firstly, we have a fibration

$$Sp/U \xrightarrow{\alpha^2} \mathbb{Z}\times BO \xrightarrow{l_O} \mathbb{Z}\times BT \tag{7.6.29}$$

$$P[e^{\circ 2}\circ z_{4k}] \xrightarrow{\circ[\alpha^2]} P[z_k][-1] \xrightarrow{l_O} \mathrm{H}\mathbb{F}_*(\mathbb{Z}\times BT) \tag{7.6.30}$$

As the fibre is connected, the local coefficient system in the Serre spectral sequence is trivial (see 5.4). We have

$$[\alpha^2]\circ e^{\circ 2}\circ z(t) = \overline{z}_1^2\circ z(t) = q(t)^2/t^2 = \dot{z}(t)^2/z(t)^2 \tag{7.6.31}$$

We see from this that the left hand map is injective and that the ideal generated by the image in positive dimension is the same as that generated by the elements z_{2k+1}^2. It follows that the right hand map is epi with kernel precisely that ideal, so

$$\mathrm{H}\mathbb{F}_*(\mathbb{Z}\times BT) = P[z_{2k}][-1]\otimes E[z_{2k+1}] \tag{7.6.32}$$

as claimed.

We also have a fibration

$$\Omega^2 T \xrightarrow{\delta_O} \mathbb{Z}\times BO \xrightarrow{\alpha^2} O/U \tag{7.6.33}$$

The action in π_0 is as follows :

$$\mathbb{Z}\gamma \longrightarrow \mathbb{Z} \longrightarrow \mathbb{F}\alpha^2 \tag{7.6.34}$$

$$\gamma\mapsto 2 \qquad\qquad 1\mapsto \alpha^2 \tag{7.6.35}$$

We therefore have a subfibration of connected spaces :

$$(\Omega^2 T)' \longrightarrow BO \longrightarrow SO/U \tag{7.6.36}$$

$$\mathrm{H}\mathbb{F}_*(\Omega^2 T') \xrightarrow{\delta_O} P[\overline{z}_{k+1}] \xrightarrow{\circ[\alpha^2]} E[[\alpha^2]\circ\overline{z}_{2k+2}] \tag{7.6.37}$$

The local coefficients are again simple, as $\pi_1(O/U) = \pi_3(KO) = 0$. The right hand map is visibly epi, so the left hand map is mono with image the Hopf algebra kernel of the right hand one. This kernel is easily seen to contain the subalgebra $B_* = P[q_{2k+1}, \overline{z}_{2k+2}^2]$. As $q_{2k+1} = \overline{z}_{2k+1} \pmod{\text{decomposables}}$, we have

$$P[\overline{z}_{k+1}] = P[\overline{z}_{2k+2}]\otimes P[q_{2k+1}] \tag{7.6.38}$$

It follows that $P[\overline{z}_{k+1}]//B_* \to E[[\alpha^2]\circ\overline{z}_{2k+2}]$ is iso, and hence (via the Milnor-Moore theorem 4.1.1) that B_* is all of the kernel. An easy calculation shows that $\delta_O(e\circ[\mu]\circ z_{2k}) = q_{2k+1}$ and $\delta_O([\gamma]\circ z_{4k}) = z_{2k}^2$. Putting back the extra components, we find that

$$\mathrm{H}\mathbb{F}_*(\Omega^2 T) = P[e\circ[\mu]\circ z_{2k}, [\gamma]\circ z_{4k}][-\gamma] \tag{7.6.39}$$

as claimed.

To compute $\mathrm{H}\mathbb{F}_*(\Omega T)$, we return to the Rothenberg-Steenrod spectral sequence :

$$\mathrm{Tor}_{s,t}^{H_*(\Omega^2 T)}(\mathbb{F},\mathbb{F}) = E[\sigma(e\circ[\mu]\circ z_{2k}), \sigma([\gamma]\circ z_{4k})] \Longrightarrow \mathrm{H}\mathbb{F}_{s+t}(\Omega T) \tag{7.6.40}$$

This is generated on the 1-line and therefore collapses. The relations $e^{\circ 2}\circ z_{4k-2} = (e^{\circ 2}\circ z_{2k-2})^2$ and $(e\circ[\gamma]\circ z_{4k})^2 = 0$ solve the algebra extension problems, giving

$$\mathrm{H}\mathbb{F}_*(\Omega T) = P[e^{\circ 2}\circ[\mu]\circ z_{4k}]\otimes E[e\circ[\gamma]\circ z_{4k}] \tag{7.6.41}$$

as claimed.

Finally, we have

$$\mathrm{Tor}_{s,t}^{H_*(\Omega T)}(\mathbb{F},\mathbb{F}) = E[\sigma(e^{\circ 2}\circ[\mu]\circ z_{4k})]\otimes D[\sigma(e\circ[\gamma]\circ z_{4k})] \Longrightarrow \mathrm{H}\mathbb{F}_*(T) \tag{7.6.42}$$

Yet again, this collapses, as everything is generated by the 1-line and the image under δ_T of the sequence for $U \Longrightarrow \mathbb{Z}\times BU$. The usual arguments give an epimorphism and then an isomorphism from the candidate algebra to the actual one, so

$$\mathrm{H}\mathbb{F}_*(T) = E[\overline{[\alpha]}\circ z_{2k}, e^{\circ 3}\circ[\mu]\circ z_{4k}] \tag{7.6.43}$$

as claimed. □

This completes the proof of the completeness of the stated generators and relations for the mod 2 Hopf rings for KO, KT and KU.

7.7. Bockstein Homology

Proof of Proposition 2.9.1.
Where we assert (implicitly) that the Bockstein vanishes, this is clear from the given data. We next observe that the following decompositions respect the Bockstein action :

$$\mathrm{H}\mathbb{F}_*(O) = E[\overline{[\alpha]}]\otimes\bigotimes_{k\geq 0}E[[\alpha]\circ z_{2k+1},[\alpha]\circ z_{2k+2}] \tag{7.7.1}$$

$$\mathrm{H}\mathbb{F}_*(\mathbb{Z}\times BO) = \mathbb{F}[\mathbb{Z}]\otimes\bigotimes_{k\geq 0}P[z_{2k+1},z_{2k+2}] \tag{7.7.2}$$

$$\mathrm{H}\mathbb{F}_*(\mathbb{Z}\times BT) = \mathbb{F}[\mathbb{Z}]\otimes\bigotimes_{k\geq 0}E[z_{2k+1}]\otimes P[z_{2k+2}] \tag{7.7.3}$$

In each case it is elementary to compute the homology of each factor, and by applying the Künneth theorem we obtain :

$$\begin{aligned}
\mathrm{H}(\mathrm{H}\mathbb{F}_*(O),\beta) &= E[\overline{[\alpha]}]\otimes E[[\alpha]\circ(z_{2k+1}z_{2k+2})\mid k\geq 0]\\
\mathrm{H}(\mathrm{H}\mathbb{F}_*(\mathbb{Z}\times BO),\beta) &= P[z_{2k}^2\mid k\geq 0][-1]\\
\mathrm{H}(\mathrm{H}\mathbb{F}_*(\mathbb{Z}\times BT),\beta) &= P[z_{2k}^2\mid k\geq 0][-1]\otimes E[z_{2k+1}z_{2k+2}\mid k\geq 0]
\end{aligned} \tag{7.7.4}$$

This is as claimed for these cases, except that we have $z_{2k+1}z_{2k+2}$ instead of q_{4k+3}. Let us write $z_{od}(t) = \sum_k z_{2k+1}t^{2k+1}$, and work in $\mathrm{H}\mathbb{F}_*(\mathbb{Z}\times BT) = P[z_{2k}][-1]\otimes E[z_{2k+1}]$, so that $z_{od}(t)^2 = 0$. We write temporarily

$$g_{4k+3} = z_{2k+1}z_{2k+2} \tag{7.7.5}$$

$$g(t) = \sum_{k\geq 0} g_{4k+3}t^{4k+3} \tag{7.7.6}$$

$$f(t) = \sum_{0\leq k<l} z_{2k}z_{2l}t^{2k+2l-1} \tag{7.7.7}$$

$$\tag{7.7.8}$$

We have

$$q(t) = \frac{t\dot{z}(t)}{z(t)} = \frac{z_{od}(t)}{z_{ev}(t) + z_{od}(t)} = \sum_{k>0} \frac{z_{od}(t)^k}{z_{ev}(t)^k} = \frac{z_{od}(t)}{z_{ev}(t)} \tag{7.7.9}$$

$$\beta z_{ev}(t) = t z_{od}(t) \tag{7.7.10}$$

$$\beta q(t) = 0 = q(t)^2 \tag{7.7.11}$$

$$q(t) z_{ev}(t)^2 = z_{ev}(t) z_{od}(t) = g(t) + \beta f(t) \tag{7.7.12}$$

$$q(t) = \frac{g(t)}{z_{ev}(t)^2} + \beta\left(\frac{f(t)}{z_{ev}(t)^2}\right) \tag{7.7.13}$$

$$\tag{7.7.14}$$

Taking into account the fact that $z_{ev}(t)^2$ is concentrated in degrees divisible by 4, the above shows that q_{4k+1} is a β-boundary and that g_{4k+3} lies in the image of $P[z_{2l}^2][-1] \otimes E[q_{4k+3}]$ in $\mathrm{H}(\mathrm{H}\mathbb{F}_*(\mathbb{Z}\times BT), \beta)$. As $\mathrm{H}(\mathrm{H}\mathbb{F}_*(\mathbb{Z}\times BT), \beta) = P[z_{2l}^2][-1] \otimes E[g_{4k+3}]$, we deduce that

$$P[z_{2l}^2][-1] \otimes E[q_{4k+3}] \longrightarrow \mathrm{H}(\mathrm{H}\mathbb{F}_*(\mathbb{Z}\times BT), \beta) \tag{7.7.15}$$

is epi. By comparing Poincaré series (which we know from 7.7.4), it must be iso, which gives the claim. The same argument shows that we actually have

$$[\alpha] \circ g_{4k+3} \equiv [\alpha] \circ q_{4k+3} \in \mathrm{H}(\mathrm{H}\mathbb{F}_*(O), \beta) \tag{7.7.16}$$

We next consider $\mathrm{H}\mathbb{F}_*(U/O) = P[e \circ z_{2k}]$. We have an isomorphism

$$P[e \circ z_{2k}] \xrightarrow{\circ[\alpha]} P[q_{2k+1}] \subset P[\overline{z}_k] = \mathrm{H}\mathbb{F}_* BO \tag{7.7.17}$$

$$e \circ z_{2k} \mapsto q_{2k+1} \tag{7.7.18}$$

We can either observe from this or prove directly from the definitions that

$$\beta q_{2k} = 0 \qquad\qquad \beta q_{2k+1} = q_{2k} \tag{7.7.19}$$

By duality we obtain an epimorphism

$$P[\overline{z}'_k] \twoheadrightarrow P[e \circ z_{2k}]^* \tag{7.7.20}$$

I claim that this kills $(\overline{z}'_k)^2$. Indeed, we have

$$\psi \overline{z}'(t)^2 = \overline{z}'(t)^2 \otimes \overline{z}'(t)^2 \tag{7.7.21}$$

$$\langle \overline{z}'(t)^2, xy \rangle = \langle \overline{z}'(t)^2, x \rangle \langle \overline{z}'(t)^2, y \rangle \tag{7.7.22}$$

$$\langle \overline{z}'(t)^2, q(s) \rangle = \langle \overline{z}'(t) \otimes \overline{z}'(t), q(s) \otimes 1 + 1 \otimes q(s) \rangle = 0 \tag{7.7.23}$$

$$\langle \overline{z}'(t)^2, P[q_{2k+1}] \rangle = 0 \tag{7.7.24}$$

which gives the claim. We thus have an epimorphism

$$E[\overline{z}'_k] \twoheadrightarrow P[e \circ z_{2k}]^* \tag{7.7.25}$$

which is iso by counting dimensions. By reading equation 2.8.16 modulo t^2, we find that the dual Bockstein action is

$$\beta \overline{z}'_{2k} = \overline{z}'_{2k+1} + \overline{z}'_1 \overline{z}'_{2k} \tag{7.7.26}$$

$$\beta \overline{z}'_{2k+1} = \overline{z}'_{2k+1} \overline{z}'_1 \tag{7.7.27}$$

If we pass to the quotient by $\overline{z}'_1$ and use the Künneth theorem we find that

$$\mathrm{H}(E[\overline{z}'_k \mid k > 1], \beta) = E[\overline{z}'_{2k} \overline{z}'_{2k+1} \mid k > 0] \tag{7.7.28}$$

Note that these generators are still β-cycles in $E[\overline{z}'_k \mid k > 0]$, and consider the following short exact sequence of complexes (with differential β).

$$E[\overline{z}'_k \mid k > 1] \xrightarrow{\overline{z}_{1'}} E[\overline{z}'_k \mid k > 0] \longrightarrow E[\overline{z}'_k \mid k > 1] \tag{7.7.29}$$

We find that

$$\mathrm{H}(E[\overline{z}'_k \mid k > 1], \beta) = E[\overline{z}'_1] \otimes E[\overline{z}'_{2k}\overline{z}'_{2k+1} \mid k > 0] \tag{7.7.30}$$

Write

$$\begin{aligned} h_1 &= e && (7.7.31)\\ h_{4k+1} &= e \circ z_{4k} + (e \circ z_{2k-1})(e \circ z_{2k}) \in \mathrm{H}\mathbb{F}_{4k+1}(U/O) && (7.7.32) \end{aligned}$$

For notational convenience we shall confuse this with its image in $\mathrm{H}\mathbb{F}_*BO$:

$$h_{4k+1} = q_{4k+1} + q_{2k}q_{2k+1} \in \mathrm{H}_{4k+1}(BO) \tag{7.7.33}$$

We find that

$$\beta h_{4k+1} = 0 \tag{7.7.34}$$

$$h^2_{4k+1} = \beta(q_{8k+3} + q_{4k}q_{4k+3}) \tag{7.7.35}$$

$$\psi(h_{4k+1}) = h_{4k+1} \otimes 1 + 1 \otimes h_{4k+1} + \beta(q_{2k+1} \otimes q_{2k+1}) \tag{7.7.36}$$

We thus have a Hopf algebra map

$$E[h_{4k+1}] \longrightarrow \mathrm{H}(P[q_k], \beta) \simeq (E[\overline{z}'_1] \otimes E[\overline{z}'_{2k+2}\overline{z}'_{2k+3}])^* \tag{7.7.37}$$

In $E[h_{4k+1}]$, the generators (and thus only the generators) are primitive. A straightforward calculation gives

$$\begin{aligned} \langle h_1, \overline{z}'_1 \rangle &= 1 && (7.7.38)\\ \langle h_{4k+1}, \overline{z}'_{2k}\overline{z}'_{2k+1} \rangle &= 1 && (7.7.39) \end{aligned}$$

so the primitives (and thus everything) are mapped injectively. By counting dimensions we find that

$$\mathrm{H}(\mathrm{H}\mathbb{F}_*(U/O), \beta) = E[e] \otimes E[h_{4k+5}] \tag{7.7.40}$$

as claimed.

We can immediately deduce the result for

$$\mathrm{H}\mathbb{F}_*(\Omega^2 T) = P[e \circ z_{2k}] \otimes P[[\mu^{-1}\gamma] \circ z_{4k}][-\mu^{-1}\gamma] \tag{7.7.41}$$

as β acts trivially on the second factor. $\square$

7.8. The Torsion Free Hopf Ring for KU

Our analysis of the mod 2 Hopf ring for KU can be done integrally in much the same way. We find

$$\begin{aligned} \mathrm{H}_*(\mathbb{C}P^\infty) &= \mathbb{Z}\{y_{2k} \mid k \geq 0\} \\ \mathrm{H}_*(\mathbb{Z}\times BU) &= P[[\nu] \circ y_{2k}][-\nu] \\ \mathrm{H}_*(U) &= E[e \circ [\nu] \circ y_{2k}] \\ \mathrm{H}_*(\mathbb{Z}\times BU) &= P[y_{2k}][-1] \qquad y_0 = [1] \end{aligned} \tag{7.8.1}$$

$$\begin{aligned} y(t) &= \sum_k y_{2k} t^k && (7.8.2) \\ \psi y(t) &= y(t) \otimes y(t) && (7.8.3) \\ y(s) \circ y(t) &= y(s+t) && (7.8.4) \\ e^{\circ 2} \circ y(t) &= [\nu^{-1}] \circ \mathrm{dlog}\, y(t)/\mathrm{d}t && (7.8.5) \\ \rho y(t) &= z_{\mathbb{C}}(t) && (7.8.6) \end{aligned}$$

Perhaps the fourth of these requires a little comment. We know that $e^{\circ 2} \circ [\nu] = \overline{y}_2 = y_2/[1]$ – the proof in section 7.3 works integrally. By reading the equation $y(s) \circ y(t) = y(s+t)$ modulo s^2, we deduce that $y_2 \circ y(t) = \dot{y}(t)$ and hence that

$$\overline{y}_2 \circ y(t) = (y_2 \circ y(t))/([1] \circ y(t)) = \dot{y(t)}/y(t) = \mathrm{dlog}\, y(t)/\mathrm{d}t \tag{7.8.7}$$

We also need to understand the complex conjugation map. From table 1.4.1 we see that $c([\nu]) = [-\nu]$. Taking the circle product by $e^{\circ 2}$, we find that

$$c(\overline{y}_2) = e^{\circ 2} \circ c[\nu] = e^{\circ 2} \circ [-1] \circ [\nu] = -e^{\circ 2} \circ [\nu] = -\overline{y}_2 \tag{7.8.8}$$

Also, $c[n] = [n]$ as c is a ring map. Thus $c(y_2) = [1]c(\overline{y}_2) = -y_2$. Given this, the formula $cy(t) = y(-t)$ is forced by compatibility with the coproduct.

7.9. The Torsion Free Hopf Ring for KO

LEMMA 7.9.1.

$$\begin{aligned} m_O \colon \overline{\mathrm{H}}_* \underline{KO}'_n &\longrightarrow \mathrm{H}_* \underline{KU}'_n \qquad \text{is mono} && (7.9.1) \\ f_U \colon \mathrm{H}[\tfrac{1}{2}]_* \underline{KU}'_n &\longrightarrow \mathrm{H}[\tfrac{1}{2}]_* \underline{KO}'_n \qquad \text{is epi} && (7.9.2) \end{aligned}$$

PROOF. We know that the composite

$$KO \xrightarrow{m_O} KU \xrightarrow{f_U} KO \tag{7.9.3}$$

is just multiplication by 2. Let k be a coefficient ring for which $\mathrm{H}_*(\underline{KO}'_n; k)$ is free, and therefore a Hopf algebra. (Recall that $\underline{KO}'_n$ is the base component of $\underline{KO}_n$). The induced map is as follows :

$$x \mapsto [2] \circ x = \sum x' x'' \qquad \Big(\text{where } \psi x = \sum x' \otimes x'' \Big) \tag{7.9.4}$$

In particular, the induced maps on the subgroup of primitives and on the indecomposable quotient are both just multiplication by 2. As the homology algebra is free and connected, a map which is mono on primitives is mono and a map which is epi on indecomposables is epi. Thus, if 2 is invertible in k then our map is an isomorphism.

If we take $k = \mathbb{F}_p$, then $\mathrm{H}_*(\underline{KO}'_n; k)$ is certainly free. If p is odd, we deduce that $\mathrm{H}_*(\underline{KO}'_n; k)$ embeds in $\mathrm{H}_*(\underline{KU}'_n; k)$ and so has trivial Bockstein action, whence $\mathrm{H}_*(\underline{KO}_n; \mathbb{Z})$ has no p-torsion. We deduce that we can take $k = \mathbb{Z}[\frac{1}{2}]$ and still get a Hopf algebra. This shows that with $\mathbb{Z}[\frac{1}{2}]$ as coefficients, m_O is mono and f_U is epi. As $\overline{\mathrm{H}} \to \mathrm{H}[\frac{1}{2}]$ is mono, the result follows. □

Composing the other way around, we have

$$f_U \circ m_O = 1 + c \tag{7.9.5}$$

This means that $\mathbb{C} \otimes_{\mathbb{R}} \zeta \simeq \zeta \oplus c\zeta$. In homology, the Whitney sum turns into the $*$-product and the implicit diagonal (which gives two copies of ζ on the right hand side) corresponds to ψ. This implies

$$(1+c)_* x = \sum x' c_*(x'') \qquad \Big(\text{where } \psi x = \sum x' \otimes x''\Big) \tag{7.9.6}$$

This can of course be proven more rigorously by writing down a suitable commutative diagram. To use this, we apply

$$\begin{aligned} c([\nu^k]) &= [(-1)^k] \circ [\nu^k] && (7.9.7)\\ c(y(t)) &= y(-t) && (7.9.8)\\ [\pm 1] \circ y(t) &= y(t)^{\pm 1} && (7.9.9) \end{aligned}$$

giving

$$\begin{aligned} m_O({}_k y(t)) &= [\nu^k] \circ (y(t)y(-t)) && k \text{ even} && (7.9.10)\\ m_O({}_k y(t)) &= [\nu^k] \circ (y(t)/y(-t)) && k \text{ odd} && (7.9.11) \end{aligned}$$

where ${}_k y(t) = f_U([\nu^k] \circ y(t))$ as before. We obtain:

$$k \text{ even:} \left\{ \begin{array}{lcl} m_O({}_k y_{4l}) & = & 2[\nu^k] \circ ([1] y_{4l}) \text{ mod decomposables} \\ m_O({}_k y_{4l+2}) & = & 0 \end{array} \right. \tag{7.9.12}$$

$$k \text{ odd:} \left\{ \begin{array}{lcl} m_O({}_k y_{4l+4}) & = & 0 \text{ mod decomposables} \\ m_O({}_k y_{4l+2}) & = & 2[\nu^k] \circ \overline{y}_{4l+2} \text{ mod decomposables} \end{array} \right. \tag{7.9.13}$$

As $y(t)$ is grouplike, so is ${}_k y(t)$. This gives equation (3.4.8). Splitting this into parts in dimension congruent to 0 or 2 mod 4, we obtain (3.4.10) and (3.4.9).

The action of the double suspension on ${}_{(k+1)}y(t)$ follows directly from that on $y(t)$. Consider the double suspension of the relation ${}_k\breve{y}(t)^2 - {}_k\hat{y}(t)^2 = 1$:

$$\begin{aligned} 0 &= e^{\circ 2} \circ 1 = e^{\circ 2} \circ ({}_k\breve{y}(t)^2 - {}_k\hat{y}(t)^2) \\ &= 2e^{\circ 2} \circ {}_k\breve{y}(t) \;\; \epsilon({}_k\breve{y}(t)) - 2e^{\circ 2} \circ {}_k\hat{y}(t) \;\; \epsilon({}_k\hat{y}(t)) \\ &= 2e^{\circ 2} \circ {}_k\breve{y}(t) \end{aligned} \tag{7.9.14}$$

This shows that $e^{\circ 2} \circ {}_k\breve{y}(t) \equiv 0$.

For (3.4.13) and (3.4.14) we have :

$$2e \circ [\alpha] = e \circ [2] \circ [\alpha] = e \circ [2\alpha] = e \circ [0] = 0 \tag{7.9.15}$$

For (3.4.15), we note that $f_U \colon KU \to KO$ is a KO-module map. Here KO acts on KU via $m_O \colon KO \to KU$ and $m_O(\alpha) = 0$. Thus

$$[\alpha] \circ {}_k y(t) = [\alpha] \circ f_U([\nu^k] \circ y(t)) = f_U([0] \circ [\nu^k] \circ y(t)) = [0] = 1 \tag{7.9.16}$$

Similarly, as $m_O(\beta) = 2\nu^2$ we have

$$[\beta] \circ {}_k y(t) = f_U([2\nu^2] \circ [\nu^k] \circ y(t)) = [2] \circ {}_{(k+2)} y(t) = {}_{(k+2)} y(t)^2 \tag{7.9.17}$$

Finally, using $m_O({}_l y(t)) = [\nu^l] \circ (y(t)y(-t)^\epsilon$ (where $\epsilon = (-1)^l$) we deduce

$$\begin{aligned} {}_k y(s) \circ {}_l y(t) &= f_U([\nu^k] \circ y(s) \circ [\nu^l] \circ (y(t)y(-t)^\epsilon)) \\ &= f_U([\nu^{k+l}] \circ y(s+t)) f_U([\nu^{k+l}] \circ y(s-t))^\epsilon \\ &= {}_{(k+l)} y(s+t) \; {}_{(k+l)} y(s-t)^\epsilon \end{aligned}$$

We have already said enough to show that the earlier description of $\overline{\mathrm{H}}_* \, \underline{KO}_*$ gives the right answer after inverting 2. To complete the proof, we need only put ourselves in a position to apply theorem 5.1.1. The next step is to collect some facts about the reduction mod 2. Using the KO-linearity of f_U, we have :

$$\rho({}_k y(t^2)) = f_U([\nu^k] \circ m_O(z(t))) = [f_U(\nu^k)] \circ z(t) \tag{7.9.18}$$

In particular :

$$\rho({}_0y(t^2)) = [2] \circ z(t) = z(t)^2 \tag{7.9.19}$$

$$\rho(e \circ {}_0y(t^2)) = e \circ z(t)^2 = 2e \circ z(t) = 0 \tag{7.9.20}$$

$$\rho({}_1y(t^2)) = [\alpha^2] \circ z(t) \tag{7.9.21}$$

$$\rho(e \circ {}_1y(t^2)) = e \circ [\alpha^2] \circ z(t) = [\alpha] \circ \overline{z}_1 \circ z(t) = [\alpha] \circ q(t)/t \tag{7.9.22}$$

$$\rho({}_2y(t^2)) = [\beta] \circ z(t) \tag{7.9.23}$$

$$\rho(e \circ {}_2y(t^2)) = e \circ [\beta] \circ z(t) \tag{7.9.24}$$

$$\rho({}_3y(t^2)) = [0] \circ z(t) = 1 \tag{7.9.25}$$

The Bockstein vanishes on $\mathrm{H}\mathbb{F}_*(Sp/U)$ so the integral homology is torsion free. As $\rho({}_3\hat{y}(t)) = 0$ we can uniquely divide ${}_3\hat{y}_{4k+2}$ by 2 in $\mathrm{H}_*(Sp/U)$. If we define

$$y_{ev}(t) = \sum_k y_{4k} t^{2k} \tag{7.9.26}$$

$$y_{od}(t) = \sum_k y_{4k+2} t^{2k+1} \tag{7.9.27}$$

Then

$$\begin{aligned} m_O({}_3\hat{y}(t)/2) &= [\nu^3] \circ \left(\frac{y(t)}{y(-t)} - \frac{y(-t)}{y(t)} \right) /4 \\ &= [\nu^3] \circ \frac{y_{ev} y_{od}}{y_{ev}^2 - y_{od}^2} \end{aligned}$$

We need to understand $\rho({}_3\hat{y}_{4k+2}/2)$. Just for this calculation, we write $z_{ev}(t) = \sum_k z_{\mathbb{C},4k} t^{2k}$ and $z_{od}(t) = \sum_k z_{\mathbb{C},4k+2} t^{2k+1}$, so $q_{\mathbb{C}}(t) = z_{od}(t)/z_{ev}(t)$. We then have

$$\begin{aligned} \rho m_O({}_3\hat{y}(t^2)/2) &= [\nu^3] \circ \frac{z_{ev}(t) z_{od}(t)}{z_{\mathbb{C}}(t)^2} \\ &= [\nu^3] \circ (q_{\mathbb{C}}(t) + q_{\mathbb{C}}(t)^2) \\ &= [\nu^4] \circ (te^{\circ 2} + (te^{\circ 2})^2) \circ z_{\mathbb{C}}(t) \end{aligned}$$

One checks easily that $m_O \colon Sp/U \to \mathbb{Z} \times BU$ is mono in $\mathrm{H}\mathbb{F}_*$, so we must have

$$\rho({}_3\hat{y}(t^2)/2) = [\lambda] \circ (t^2 e^{\circ 2} + (t^2 e^{\circ 2})^2) \circ z(t) = \sum_k e^{\circ 2} \circ [\lambda] \circ z_{4k} t^{4k+2} \tag{7.9.28}$$

The second equality here follows from (2.2.9).

$$\rho(e \circ {}_3\hat{y}(t^2)/2) = t^2 [\lambda] \circ e^{\circ 3} \circ z(t) = \sum_k e^{\circ 3} \circ [\lambda] \circ z_{4k} t^{4k+2} \tag{7.9.29}$$

We can now deal with all the spaces in the KO spectrum except for U/O. We know enough to construct a map from the structure described by table 3.4.1 to $\overline{\mathrm{H}}_* \underline{KO}_*$. We know that it is iso after inverting 2 by comparison with KU, as discussed at the beginning of this section. The analysis of ρ above, combined with the Bockstein homology calculations of section 2.9 and theorem 5.1.1, assure us that it is iso mod 2 and therefore iso on the nose.

To deal with O/U, we have to organise our argument just slightly differently. As $\rho_0 y(t^2) = z(t)^2$, we find that $\rho(e \circ {}_0y(t)) = e \circ z(t)^2 = 0$ and thus that $e \circ {}_0y(t)$ is divisible (nonuniquely) by 2. Consider the following commutative diagram :

$$\begin{tikzcd}
e \circ {}_0y_{4k} \arrow[r, mapsto] & 2e \circ y_{4k} & \\
E[e \circ {}_0y_{4k}/2] \arrow[r, tail, "m_O"'] \arrow[d, "\rho"] & E[e \circ y_{2k}] \arrow[d, "\rho"] & e \circ y_{2k} \arrow[d, mapsto] \\
E[e] \otimes E[h_{4k+1}] \arrow[r, tail, "m_O"] & E[e \circ z_{2k}] & e \circ z_{2k} \\
h_{4k+1} \arrow[r, mapsto] & e \circ z_{4k} &
\end{tikzcd} \tag{7.9.30}$$

The terms on the bottom line are the Bockstein homology groups. It is easily seen that the maps act as shown. We conclude that $\rho(e \circ {}_0y_{4k}/2) = h_{4k+1} \bmod \operatorname{im}(\beta)$ and thus that the Bockstein spectral sequence collapses, so

$$\overline{\mathrm{H}}_*(U/O) = E[e \circ {}_0y_{4k}/2] \tag{7.9.31}$$

and the torsion is annihilated by 2. This shows that the torsion subgroup is precisely the image of the Bockstein $\tilde{\beta} \colon \mathrm{H}\mathbb{F} \to \mathrm{H}$. This satisfies $\rho\tilde{\beta} = \beta$. Using this we see that we can specify a lifting of $e \circ {}_0y_{4k}/2$ from $\overline{\mathrm{H}}$ to H by requiring that $\rho(e \circ {}_0y_{4k}/2) = h_{4k+1}$ exactly, rather than just mod the image of β. I know of no geometric construction of these elements, however. Having got this far, we can complete the argument for U/O in the same way as for the other spaces.

7.10. The Torsion Free Hopf Ring for KT

We next consider the self-conjugate case. Recall the definition

$${}_kx(t) = \delta_T([\nu^k] \circ y(t)) \tag{7.10.1}$$

The ring map $f_T \colon KT \to KU$ makes KU into a KT-module spectrum, and it sends μ to ν^2. With respect to this structure, $\delta_T \colon \Sigma KU \to KT$ is KT-linear. It follows that

$$[\mu] \circ {}_kx(t) = [\mu] \circ \delta_T([\nu^k] \circ y(t)) = \delta_T([\nu^{k+2}] \circ y(t)) = {}_{k+2}x(t) \tag{7.10.2}$$

Also

$$e^{\circ 2} \circ {}_{(k+1)}x(t) = \delta_T([\nu^k] e^{\circ 2} \circ [\nu] \circ y(t)) = \delta_T([\nu^k] \circ \mathrm{dlog}(y(t))/\mathrm{d}t) = \mathrm{dlog}({}_kx(t))/\mathrm{d}t \tag{7.10.3}$$

From diagram (1.3.2) we see that

$$\delta_T \circ m_\nu^{-1} \circ (1 - c) = 0 \tag{7.10.4}$$

As ν is a unit and $c(\nu) = -\nu$ we have $m_\nu^{-1} \circ (1-c) = (1+c) \circ m_\nu^{-1}$ and thus $\delta_T = -\delta_T \circ c$. It follows that

$$ {}_kx(t) = (-\delta_T \circ c)_*([\nu^k] \circ y(t)) = [-1] \circ \delta_T([(-\nu)^k] \circ y(-t)) = [(-1)^{k+1}] \circ {}_kx(-t) \tag{7.10.5}$$

As $[-1] \circ {}_kx(t) = {}_kx(t)^{-1}$, this justifies equations (3.5.1).

Recall that we denote $l_O({}_ky(t)) = m_U([\nu^k] \circ y(t))$ simply by ${}_ky(t)$. To understand $[\mu] \circ {}_ky(t)$, we observe (see diagram (1.3.1)) that $m_U = f_{Sp} l_U = l_O f_U$. Recall also that the maps

$$\begin{array}{llcl} \theta\colon & \Sigma^4 KO & \to & KSp \\ \mu\colon & \Sigma^4 KT & \to & KT \\ \nu^2\colon & \Sigma^4 KU & \to & KU \end{array} \tag{7.10.6}$$

give a morphism between diagrams (1.3.2) and (1.3.3). In particular, this means that $m_\mu \circ f_{Sp} = l_O \circ m_\theta$ and $m_\theta \circ l_U = f_U \circ m_{\nu^2}$. It follows that $m_\mu \circ m_U = m_U \circ m_{\nu^2}$. (Note, however, that m_U is not a KT-module map – consider for example the action of m_γ in homotopy.) This shows that

$$[\mu] \circ {}_ky(t) = m_U([\nu^2] \circ [\nu^k] \circ y(t)) = {}_{(k+2)}y(t) \tag{7.10.7}$$

As δ_T is a KO-module map, we have

$$\begin{aligned} [\gamma] \circ {}_ky(t) &= \delta_T([\nu]) \circ {}_ky(t) = \delta_T([\nu] \circ m_O({}_ky(t))) \\ &= \delta_T([\nu^{k+1}] \circ (y(t)y(-t)^\epsilon)) = {}_{(k+1)}x(t)\,{}_{(k+1)}x(-t)^\epsilon \end{aligned} \tag{7.10.8}$$

where $\epsilon = (-1)^k$. On the other hand, we showed above that ${}_kx(-t) = {}_kx(t)^\epsilon$. Thus

$$[\gamma] \circ {}_ky(t) = {}_{(k+1)}x(t)^2 \tag{7.10.9}$$

As homology suspension is a derivation, we conclude that

$$e \circ [\gamma] \circ {}_ky(t) = 2e \circ {}_{(k+1)}x(t) \tag{7.10.10}$$

We next need formulae for the circle product. The expression for ${}_ky(s) \circ {}_ly(t)$ carries over from $\mathrm{H}_* \, \underline{KO}_*$, of course. Next,

$$\begin{aligned} {}_kx(s) \circ {}_ly(s) &= \delta_T([\nu^k] \circ y(s) \circ [\nu^l] \circ (y(t)y(-t)^\epsilon)) \\ &= \delta_T([\nu^{k+l}] \circ (y(s+t)y(s-t)^\epsilon)) \\ &= {}_{(k+l)}x(t+s)\ {}_{(k+l)}x(t-s)^\epsilon \end{aligned}$$

For the final case we use the KT-linearity of δ_T again. Note that $f_T\delta_T = 0$, so $f_T({}_lx(s)) = 1$.

$$ {}_kx(s) \circ {}_lx(s) = \delta_T([\nu^k] \circ y(s) \circ f_T({}_lx(s))) = \delta_T([0]) = 1 \tag{7.10.11}$$

We next see what we can say about the mod 2 reduction. First note that ${}_3y_{4k+2}/2$ is already defined in KO, and on taking the circle product by $[\mu^{-1}]$ we obtain elements which deserve to be called ${}_1y_{4k+2}/2$. With this convention, the first half of (3.5.14) is immediate. For terms involving x's, we have :

$$\rho({}_kx(t^2)) = [\delta_T(\nu^k)] \circ z(t) \tag{7.10.12}$$

$$\rho({}_0x(t^2)) = [\alpha] \circ z(t) \tag{7.10.13}$$

$$\rho(e \circ {}_0x(t^2)) = \overline{z}_1 \circ z(t) = q(t)/t \tag{7.10.14}$$

$$\rho({}_1x(t^2)) = [\gamma] \circ z(t) \tag{7.10.15}$$

This proves the rest of (3.5.14).

Proof of Theorem 2.2.2.
We again argue using theorem 5.1.1. As in the KO case, the data above give a map from our candidate structure to $\overline{\mathrm{H}}_* \, \underline{KT}_*$ which is iso mod 2. However, we need a

different argument to show that it is iso with coefficients $\mathbb{Z}[\frac{1}{2}]$. The real reason for this is that $2\alpha^2 = 0$, so that the cofibration

$$\Sigma^2 KO \xrightarrow{\alpha^2} KO \xrightarrow{l_O} KT \xrightarrow{\delta_O} \Sigma^3 KO \xrightarrow{\alpha^2} \Sigma KO \tag{7.10.16}$$

splits to give

$$KT[\tfrac{1}{2}] = KO[\tfrac{1}{2}] \oplus \Sigma^3 KO[\tfrac{1}{2}] \tag{7.10.17}$$

if we invert 2 geometrically. To turn this into a proper proof requires a little thought about the relation between localisations of spaces, spectra and homology groups. An alternative is to look at the Serre spectral sequences (with coefficients $\mathbb{Z}[\frac{1}{2}]$) of the fibrations derived from the above stable cofibration :

$$\mathbb{Z}\times BO \xrightarrow{l_O} \mathbb{Z}\times BT \xrightarrow{\delta_O} Sp \tag{7.10.18}$$

$$P[{}_0y_{4k}][-1] \to \mathrm{H}[\tfrac{1}{2}]_*(\mathbb{Z}\times BT) \to E[e \circ {}_{-1}y_{4k+2}] \tag{7.10.19}$$

$$U/O \xrightarrow{l_O} \Omega^2 T \xrightarrow{\delta_O} 2\mathbb{Z}\times BSp \tag{7.10.20}$$

$$E[e \circ {}_0y_{4k}] \to \mathrm{H}[\tfrac{1}{2}]_*(\Omega^2 T) \to P[{}_{-2}y_{4k}][-2\lambda^{-1}\beta] \tag{7.10.21}$$

$$Sp/U \xrightarrow{l_O} \Omega T \xrightarrow{\delta_O} U/Sp \tag{7.10.22}$$

$$P[{}_{-1}y_{4k+2}] \to \mathrm{H}[\tfrac{1}{2}]_*(\Omega T) \to E[e \circ {}_{-2}y_{4k}] \tag{7.10.23}$$

$$Sp \xrightarrow{l_O} T \xrightarrow{\delta_O} O/U \tag{7.10.24}$$

$$E[e \circ {}_{-2}y_{4k+2}] \to \mathrm{H}[\tfrac{1}{2}]_*(\mathbb{Z}\times BT) \to P[{}_{-3}y_{4k+2}] \otimes \mathbb{Z}[\mathbb{F}\alpha^2] \tag{7.10.25}$$

In each case the connecting homomorphism $\pi_1 B \to \pi_0 F$ vanishes so the local coefficients are simple. Using $\delta_O\delta_T = f_U\nu^{-1}$ we find that $\delta_O({}_kx(t)) = {}_{(k-1)}y(t)$. This implies that the edge map $\mathrm{H}[\frac{1}{2}]_*E \to \mathrm{H}[\frac{1}{2}]_*B$ is surjective, and thus that the spectral sequence collapses. Our conclusion follows easily. $\square$

APPENDIX A

Mathematica Code

The symbolic mathematics program *Mathematica* was very helpful in writing this thesis. I used it to develop my understanding of the algebra involved, to do calculations, test conjectures and check some of the more intricate statements. Much of this was done using short, ad hoc programs. I include as an appendix some of the better documented code. Amongst other things, this can be used to check the consistency of the diagrams in section 1.3 with table 1.4.1, as mentioned at the end of section 1.4. The standard reference for *Mathematica* is the manual [**27**]; for more information about the associated programming language see [**14**]. The documentation in the first file included below explains a number of standard Mathematica constructs and is intended to be comprehensible to a reader with little or no familiarity with the system. The documentation in the remaining files assumes some knowledge, however.

A.1. Formal Power Series

The following code is the file `Formal.m` , which sets up some basic tools for using formal power series.

A.2. Scalars and Linearity

The following code is the file `Scalar.m` , which sets up convenient methods for telling *Mathematica* that certain maps are linear or are ring homomorphisms.

```
BeginPackage["Scalar`"]

Needs["Mod`"]

ScalarQ::usage                 =
  "ScalarQ[x] = True if x is a number or formal variable etc."

AssertScalar::usage            =
  "AssertScalar[x] asserts that x is a scalar. The assertion has to be
   tied to a tag symbol. If x is not a symbol, use AssertScalar[x,tag]
   e.g. AssertScalar[ x[23] , x ]"

AssertLinear::usage            =
  "AssertLinear[f] asserts that f is a linear function"

AssertRingMap2::usage           =
  "AssertRingMap2[f] asserts that f is a homomorphism of Z/2-algebras"

AssertRingMap::usage           =
  "AssertRingMap[f] asserts that f is a ring homomorphism"

AssertMultilinear::usage       =
```

```
  "AssertMultilinear[f] asserts that f is multilinear and associative"

SquareQ::usage =
"SquareQ[x] = True iff x is a square mod 2"

Begin["Private`"]

Attributes[Until] = {HoldAll}

Until[body_, test_] := Module[ {t}, For[ t=False, !t, t=test, body ] ]

StripPattern[a_] := a /.
        b_PatternTest :> First[b]       /.
        b_Pattern     :> First[b]       /.
        b_Optional    :> First[b]

SymbolQ[x_] := (Head[x] === Symbol)

URules[x_] :=
Module[{hsl,usl},
  hsl = HeldPart[#,1]& /@ Cases[x,y_Pattern,Infinity];
  hsl = {#,ReleaseHold[Unique /@ Unevaluated /@ #]} & /@ hsl;
  hsl = Rule @@ # & /@ (hsl /. Hold->Literal)
]


SquareQ[x_Plus] := And @@ (SquareQ /@ (List @@ x))
SquareQ[x_Times] := And @@ (SquareQ /@ (List @@ x))
SquareQ[n_Integer] := True
SquareQ[x_^n_Integer] := EvenQ[n]
SquareQ[_] := False


(* ScalarQ[x] = True if x is a number or formal variable etc.          *)

{
ScalarQ[n_?NumberQ]             := True;
ScalarQ[x_?ScalarQ y_]          := ScalarQ[y];
ScalarQ[x_ y_]                  := False /; Not[ScalarQ[y]];
ScalarQ[x_^n_]                  := ScalarQ[x];
};

(* AssertScalar[x] asserts that x is a scalar. The assertion has to be  *)
(* tied to a tag symbol. If x is not a symbol, use AssertScalar[x,tag]  *)
(* e.g. AssertScalar[ x[23] , x ]                                       *)

AssertScalar[n_,tag_Symbol] := (tag/: ScalarQ[n] = True ;)

AssertScalar[n_Symbol] := AssertScalar[n,n]

(* AssertRingMap[f] asserts that f is a ring homomorphism               *)

AssertRingMap[f_] := AssertRingMap[f,1]
```

```
AssertRingMap[f_,u_]:=
Module[{head,uf,sf,ur,rplus,rtimes,rpow,rsc0,rsc1,x,n},
  ur = URules[f];
  uf = f /. ur;
  sf = StripPattern[uf];
  rplus  = Hold[RuleDelayed[Literal[{x_Plus}],Map[{},x]]];
  rplus  = ReplaceHeldPart[rplus,uf,{1,1,1,0}];
  rplus  = ReplaceHeldPart[rplus,sf,{1,2,1}];
  rtimes = Hold[RuleDelayed[Literal[{x_Times}],Expand[Map[{},x]]]];
  rtimes = ReplaceHeldPart[rtimes,uf,{1,1,1,0}];
  rtimes = ReplaceHeldPart[rtimes,sf,{1,2,1,1}];
  rpow   = Hold[RuleDelayed[Literal[{x_^n_}],Expand[{x}^n]]];
  rpow   = ReplaceHeldPart[rpow,uf,{1,1,1,0}];
  rpow   = ReplaceHeldPart[rpow,sf,{1,2,1,1,0}];
  rsc0   = Hold[RuleDelayed[Literal[{n_?ScalarQ}],n u]];
  rsc0   = ReplaceHeldPart[rsc0,uf,{1,1,1,0}];
  rsc1   = Hold[RuleDelayed[Literal[{n_?ScalarQ x_}],n {x}]];
  rsc1   = ReplaceHeldPart[rsc1,uf,{1,1,1,0}];
  rsc1   = ReplaceHeldPart[rsc1,sf,{1,2,2,0}];
  rr = ReleaseHold /@ { rplus, rtimes, rpow, rsc0, rsc1 };
  If[SymbolQ[f],
    DownValues[Release[f]] =
      Join[DownValues[Release[f]],rr],
  (* Else *)
    head = sf;
    While[Length[head]>0,head = Head[head]];
    If[SymbolQ[head],
      SubValues[Release[head]] =
        Join[SubValues[Release[head]],rr],
    (* Else *)
      Message[AssertRingMap::badtag,head];
    ]
  ];
]

AssertRingMap::badtag = "I regret that ‘1‘ appears to be an invalid tag";

(* AssertRingMap2[f] asserts that f is a ring homomorphism              *)

AssertRingMap2[f_] := AssertRingMap2[f,1]

AssertRingMap2[f_,u_]:=
Module[{head,uf,sf,ur,rplus,rtimes,rpow,rsc0,rsc1,x,n},
  ur = URules[f];
  uf = f /. ur;
  sf = StripPattern[uf];
  rplus  = Hold[RuleDelayed[Literal[{x_Plus}],Map[{},x]]];
  rplus  = ReplaceHeldPart[rplus,uf,{1,1,1,0}];
  rplus  = ReplaceHeldPart[rplus,sf,{1,2,1}];
  rtimes = Hold[RuleDelayed[Literal[{x_Times}],Expand[Map[{},x]]]];
  rtimes = ReplaceHeldPart[rtimes,uf,{1,1,1,0}];
  rtimes = ReplaceHeldPart[rtimes,sf,{1,2,1,1}];
  rpow   = Hold[RuleDelayed[Literal[{x_^n_}],modpE[{x}^n]]];
  rpow   = ReplaceHeldPart[rpow,uf,{1,1,1,0}];
```

```
  rpow   = ReplaceHeldPart[rpow,sf,{1,2,1,0}];
  rsc0   = Hold[RuleDelayed[Literal[{n_?ScalarQ}],n u]];
  rsc0   = ReplaceHeldPart[rsc0,uf,{1,1,1,0}];
  rsc1   = Hold[RuleDelayed[Literal[{n_?ScalarQ x_}],n {x}]];
  rsc1   = ReplaceHeldPart[rsc1,uf,{1,1,1,0}];
  rsc1   = ReplaceHeldPart[rsc1,sf,{1,2,2,0}];
  rr = ReleaseHold /@ { rplus, rtimes, rpow, rsc0, rsc1 };
  If[SymbolQ[f],
    DownValues[Release[f]] =
      Join[DownValues[Release[f]],rr],
  (* Else *)
    head = sf;
    While[Length[head]>0,head = Head[head]];
    If[SymbolQ[head],
      SubValues[Release[head]] =
        Join[SubValues[Release[head]],rr],
    (* Else *)
      Message[AssertRingMap2::badtag,head];
    ]
  ];
]

AssertRingMap2::badtag = "I regret that `1` appears to be an invalid tag";

(* AssertLinear[f] asserts that f is a linear function                *)

AssertLinear[f_]:=
Module[{head,uf,sf,ur,rplus,rscl,x,n},
  ur = URules[f];
  uf = f /. ur;
  sf = StripPattern[uf];
  rplus  = Hold[RuleDelayed[Literal[{x_Plus}],Map[{},x]]];
  rplus  = ReplaceHeldPart[rplus,uf,{1,1,1,0}];
  rplus  = ReplaceHeldPart[rplus,sf,{1,2,1}];
  rscl   = Hold[RuleDelayed[Literal[{n_?ScalarQ x_}],n {x}]];
  rscl   = ReplaceHeldPart[rscl,uf,{1,1,1,0}];
  rscl   = ReplaceHeldPart[rscl,sf,{1,2,2,0}];
  rr = ReleaseHold /@ { rplus, rscl };
  If[SymbolQ[f],
    DownValues[Release[f]] =
      Join[DownValues[Release[f]],rr],
  (* Else *)
    head = sf;
    While[Length[head]>0,head = Head[head]];
    If[SymbolQ[head],
      SubValues[Release[head]] =
        Join[SubValues[Release[head]],rr],
    (* Else *)
      Message[AssertLinear::badtag,head];
    ]
  ];
]

AssertLinear::badtag = "I regret that `1` appears to be an invalid tag";
```

```
(* AssertMultilinear[f] asserts that f is multilinear and associative   *)

AssertMultilinear[f_Symbol] :=
  ( SetAttributes[f,{Flat}];
    f[x___ , y_Plus, z___] :=
      Plus @@ (f @@ # & /@ Distribute[{x,y,z},Plus,List,List,List]);
    f[x___, n_?((ScalarQ[#] && # =!= 1)&) y_., z___] :=
      n f[x,y,z];
  )

End[]

EndPackage[]
```

A.3. Stable Homotopy Theory

The following code is the file `Stable.m` , which sets up some basic facts about spectra and maps between them.

```
(* This file contains general facts about spectra and stable homotopy   *)

BeginPackage["Stable`"]

Deg::usage      =
  "Deg[x] is the degree of x in some appropriate sense"
Source::usage   =
  "Source[f] is the source of a stable map f.  f:Source[f] -> Target[f]"
Target::usage   =
  "Target[f] is the target of a stable map f.  f:Source[f] -> Target[f]"
ZeroMap::usage  =
  "ZeroMap[G,F,n] is the null map G->F of degree n"
OneMap::usage   =
  "OneMap[G] is the identity map of G"
MultiplyBy::usage       =
  "MultiplyBy[x,G] is the self map of G given by multiplication by
   the homotopy element x in pi_* G"
Sigma::usage    =
  "Sigma[n,G] is the n'th suspension of the spectrum G"
Homotopy::usage =
  "Homotopy[f] gives the effect of a stable map f in homotopy"

DefineStableMap::usage  =
  "DefineStableMap[f,A,B] declares that f is a stable map of degree 0 from
   A to B. Options can be added e.g. DefineStableMap[f,A,B,MapDeg->3].
   declares that f is a map of degree 3 from A to B. Possible options are
   MapDeg,MapRingQ,MapTag,MapTeX."

MapDeg::usage           = ""
MapRingQ::usage         = ""
MapTag::usage           = ""
MapTeX::usage           = ""
RingMapQ::usage         = ""

Begin["Private`"]
```

```
(* Deg[x] is the degree of x in some appropriate sense                *)
Deg[n_Integer]          := 0
Deg[x_Times]            := Plus @@ ( Deg /@ (List @@ x))
Deg[x_^n_Integer]       := n Deg[x]
Deg[x_+y_]              := If[Deg[x]===Deg[y],Deg[x],Indeterminate]

Options[DefineStableMap] =
  { MapDeg -> 0, MapRingQ -> False, MapTag -> Null, MapTeX -> Null }

DefineStableMap[f_,s_,t_,opts___] :=
  Module[{tag,rm,tf},
    tag = MapTag /. {opts} /. Options[DefineStableMap];
    If[ tag == Null , tag = f ];
    Release[tag] /: Source[f] = s;
    Release[tag] /: Target[f] = t;
    Release[tag] /: Deg[f]    = MapDeg /. {opts} /. Options[DefineStableMap];
    rm = MapRingQ /. {opts} /. Options[DefineStableMap];
    If[ rm === True, (Release[tag] /: RingMapQ[f] = True)];
    tf = MapTeX /. {opts} /. Options[DefineStableMap];
    If[ tf =!= Null, (Release[tag] /: Format[f,TeXForm] = tf)];
  ]

(* ZeroMap[G,F,n] is the null map G->F of degree n *)

ZeroMap[G_]             := ZeroMap[G,G,0]

ZeroMap[G_,F_]          := ZeroMap[G,F,0]

DefineStableMap[
  ZeroMap[G_,F_,n_],G,F,
  MapDeg->n, MapTag->ZeroMap
]

ZeroMap/:       Composition[f_,ZeroMap[G_,F_,n_]] :=
        ZeroMap[G,Target[f],n+Deg[f]]

ZeroMap/:       Composition[ZeroMap[G_,F_,n_],f_] :=
        ZeroMap[Source[f],F,n+Deg[f]]

ZeroMap/:       f_ + ZeroMap[G_,F_,n_] := f /;
                Source[f] === G && Target[f] === F && Deg[f] === n

ZeroMap/:       n_? NumberQ z_ZeroMap := z

Format[ZeroMap[G_,F_,n_],TeXForm] := Subscripted["0"[G,F]]

(* OneMap[G] is the identity map of G *)

DefineStableMap[OneMap[G_],G,G,MapRingQ -> True, MapTag -> OneMap]

OneMap/:        Composition[f_,OneMap[G_]]      := f

OneMap/:        Composition[OneMap[G_],f_]      := f
```

```
Format[OneMap[G_],TeXForm] := Subscripted["1"[G]]

(* MultiplyBy[x,G] is the self map of G given by multiplication by      *)
(* the homotopy element x in pi_* G                                     *)

DefineStableMap[MultiplyBy[x_,G_],G,G, MapDeg->Deg[x], MapTag->MultiplyBy]

(* Sigma[n,G] is the n'th suspension of the spectrum G                  *)

Sigma[0,G_]                     := G

Sigma[G_]                       := Sigma[1,G]

Sigma[n_,Sigma[m_,G]]           := Sigma[n+m,G]

(* Sigma[n,f] is the n'th suspension of the map f                       *)

Homotopy[Sigma[n_,f_]]          := Homotopy[f]

Source[Sigma[n_,f_]]            := Sigma[n,Source[f]]

Target[Sigma[n_,f_]]            := Sigma[n,Target[f]]

Sigma[n_,ZeroMap[G_,F_,m_]]     :=
        ZeroMap[Sigma[n,G],Sigma[n,F],m]

Sigma[n_,OneMap[G_]]            := OneMap[Sigma[n,G]]

Sigma[n_,MultiplyBy[x_,G_]]     :=
        MultiplyBy[x,Sigma[n,G]]

Sigma[n_,f_Composition]         := Sigma[n,#]& /@ f

Format[Sigma[n_,x_],TeXForm]    := SequenceForm["\\Sigma"^n,x]

Source[n_Integer f_] := Source[f]

Target[n_Integer f_] := Target[f]

Source[f_+g_]        := If[Source[f]===Source[g],Source[f],Indeterminate]

Target[f_+g_]        := If[Target[f]===Target[g],Target[f],Indeterminate]

Source/:        Source[Composition[f_,g_]]      := Source[g]

Target/:        Target[Composition[f_,g_]]      := Target[f]

Deg/:           Deg[Composition[f_,g_]]         := Deg[f]+Deg[g]

(* Homotopy[f] gives the effect of a stable map f in homotopy           *)

Homotopy[f_][n_ a_]     := n Homotopy[f][a] /; NumberQ[n]
```

```
Homotopy[f_][n_]        := n Homotopy[f][1] /; (NumberQ[n] && n =!= 1)

Homotopy[f_][a_Plus]    := Homotopy[f] /@ a

Homotopy[f_][1]         := 1                        /; RingMapQ[f]

Homotopy[f_][a_Times]   := Expand[Homotopy[f] /@ a] /; RingMapQ[f]

Homotopy[f_][a_^n_]     := Expand[Homotopy[f][a]^n] /; RingMapQ[f]

Homotopy[f_+g_][a_]     := Homotopy[f][a]+Homotopy[g][a]

Homotopy[n_ f_][a_]     := n Homotopy[f][a] /; NumberQ[n]

Homotopy[Composition[f_,g_]][a_]:= Homotopy[f][Homotopy[g][a]]

Homotopy[MultiplyBy[a_,G_]][b_] := Expand[a b]

Homotopy[OneMap[G_]][a_]        := a

Homotopy[ZeroMap[E_,G_,n_]][a_] := 0

Format[Homotopy[f_],TeXForm] := "{\\pi_*}"[f]

End[]

EndPackage[]
```

A.4. Code For Hopf Rings

The following code is the file `Hopf.m` which sets up the various operations and identities for general Hopf rings.

```
(* This file contains definitions for general Hopf rings.               *)

(*
BeginPackage["Hopf`",
                (* Needs *) "Scalar`","Stable`"]
*)

Bracket::usage          =
  "Bracket[x] represents the zero dimensional homology class [x]"
Tensor::usage           =
  "Tensor is the tensor product"
Multiply::usage         =
  "Multiply is the star multiplication map"
Circ::usage             =
  "Circ is the circle product"
Epsilon::usage          =
  "Epsilon is the augmentation map"
Chi::usage              =
  "Chi is the Hopf Algebra antipode"
Psi::usage              =
  "Psi is the coproduct map"
Homology::usage         =
```

```
  "Homology[f] gives the effect of a stable map f in homology"
HopfTimes::usage = ""
e::usage =
  "e[G] is the fundamental class in H_1 G_1 for a spectrum G"

(* Begin["Private`"] *)

(* Bracket[x] represents the zero dimensional homology class [x]        *)

{
Bracket[0] := 1;
(**)
Bracket/:       Bracket[x_]^n_Integer           := Bracket[n x];
Bracket/:       Bracket[x_] Bracket[y_]         := Bracket[x + y];
(**)
Format[Bracket[x_]]         := SequenceForm["[",x,"]"];
Format[Bracket[x_],TeXForm] := SequenceForm["[",x,"]"];
};

(* Tensor is the tensor product                                         *)

{
Tensor[x___,Tensor[y___],z___] := Tensor[x,y,z];
Tensor[x___ , y_Plus, z___] :=
  Plus @@ (Tensor @@ # & /@ Distribute[{x,y,z},Plus,List,List,List]);
Tensor[x___, n_?((ScalarQ[#] && # =!= 1)&) y_., z___] :=
  n Tensor[x,y,z];
(**)
Tensor/: Tensor[x0_,z0_] Tensor[x1_,z1_]:= Tensor[x0 x1 , z0 z1];
Tensor/: Tensor[x_ ,z_ ]^n_Integer      := Tensor[x^n,z^n];
(**)
Format[Tensor[x__]]             := Infix[Tensor[x]," O "];
Format[Tensor[x__],TeXForm]     := Infix[Tensor[x]," \\otimes "];
};

(* Multiply is the star multiplication map                              *)

{
Multiply[x_Plus] := Multiply /@ x;
Multiply[x_Times] := Multiply /@ x;
Multiply[x_^n_] := Multiply[x]^n;
Multiply[n_?ScalarQ] := n;
Multiply[x_Tensor]              := Expand[Times @@ x];
};

(* Circ is the circle product                                           *)

{
(* Circ[x___ , y_Plus, z___] := Plus @@ (Circ @@ # & /@ Distribute[{x,y,z},Plus,List,List,Lis
Circ[x_Plus,y_] := Circ[#,y] & /@ x;
Circ[x_,y_Plus] := Circ[x,#] & /@ y;
Circ[x___, n_?((ScalarQ[#] && # =!= 1)&) y_., z___] :=
  n Circ[x,y,z];
(**)
```

```
Circ[1  , x_]                   := Epsilon[x];
Circ[Bracket[1], x_]            := x;
Circ[Bracket[x_],Bracket[y_]]   := Bracket[x y];
Circ[Bracket[n_Integer], x_]    :=
        Multiply[Circ[Tensor[Bracket[1],Bracket[n-1]],Psi[x]]] /; n>1 ;
Circ[Bracket[n_Integer], x_]    := Circ[Bracket[-n],Chi[x]] /; n<0 ;
(**)
Circ[x_Tensor, y_Tensor]        :=
        Inner[Circ, List @@ x , List @@ y ,Tensor];
Circ[x_ ,y_?ScalarQ z_] := y Circ[x,z];
Circ[x_ ,y_ z_ ]                := Multiply[Circ[Psi[x], Tensor[y,z]]];
Circ[y_ z_, x_]                 := Multiply[Circ[Psi[x], Tensor[y,z]]];
Circ[x_ ,y_^n_Integer]          :=
        Multiply[Circ[Psi[x], Tensor[y,y^(n-1)]]] /; n>1;
(**)
Format[Circ[x__]]               := Infix[Circ[x]," o "];
Format[Circ[x__],TeXForm]       := Infix[Circ[x]," \\circ "];
};

(* Epsilon is the augmentation map                                       *)

{
Epsilon[x_Plus] := Epsilon /@ x;
Epsilon[x_Times] := Epsilon /@ x;
Epsilon[x_^n_] := Epsilon[x]^n;
Epsilon[n_?ScalarQ] := n;
(**)
Epsilon[Bracket[_]]     := 1;
Epsilon[e[_]] := 0;
(**)
Epsilon[x_Tensor]       := Expand[Times @@ (Epsilon /@ (List @@ x))];
Epsilon[x_Circ]         := Expand[Times @@ (Epsilon /@ (List @@ x))];
(**)
Format[Epsilon,TeXForm]:="\\epsilon";
};

(* Chi is the Hopf Algebra antipode                                      *)

{
Chi[x_Plus] := Chi /@ x;
Chi[x_Times] := Chi /@ x;
Chi[x_^n_] := Chi[x]^n;
Chi[n_?ScalarQ] := n;
(**)
Chi[Bracket[x_]]         := Bracket[-x];
Chi[Circ[x_,y_]]         := Circ[Chi[x],y];
(**)
Format[Chi,TeXForm] := "\\chi";
};

(* Psi is the coproduct map                                              *)

{
Psi[x_Plus] := Psi /@ x;
```

```
Psi[x_Times] := modpE[Psi /@ x];
Psi[x_^n_] := Psi[x]^n;
Psi[n_?ScalarQ] := n Tensor[1,1];
(**)
Psi[Bracket[x_]]         := Tensor[Bracket[x],Bracket[x]];
(**)
Psi[x_Circ] := Psi /@ x;
(**)
Psi[0,x_] := Epsilon[x];
Psi[1,x_] := x;
Psi[2,x_] := Psi[x];
Psi[n_,x_] :=
  Psi[n-1,x] /. Tensor[y__,z_] :> Tensor[y,Psi[z]] /; n > 2;
Format[Psi,TeXForm] := "\\psi";
};

(* Homology[f] gives the effect of a stable map f in the homology *)
(* of the Omega spectrum *)

{
Homology[f_][x_Plus] := Homology[f] /@ x;
Homology[f_][x_Times] := Homology[f] /@ x;
Homology[f_][x_^n_] := Homology[f][x]^n;
Homology[f_][n_?ScalarQ] := n;
(**)
Homology[f_][Bracket[a_]]:=Bracket[Homotopy[f][a]] ;
Homology[f_][e[G_]] := e[Target[f]] /; Source[f] == G ;
Homology[f_][x_Tensor]  := Homology[f] /@ x ;
Homology[f_][x_Circ]    := Homology[f] /@ x /; RingMapQ[f];
Homology[f_+g_][x_]     :=
        (Psi[x] /. Tensor[y_,z_]->Expand[Homology[f][y]Homology[g][z]]);
Homology[n_Integer f_][x_]      := Circ[Bracket[n],Homology[f][x]];
Homology[Composition[f_,g_]][a_]:= Homology[f][Homology[g][a]];
Homology[MultiplyBy[a_,G_]][x_] := Circ[Bracket[a],x];
Homology[OneMap[G_]][x_]        := x;
Homology[ZeroMap[E_,G_,n_]][x_] := Epsilon[x];
};

HopfTimes[0,x_] := Epsilon[x];
HopfTimes[1,x_] := x;
HopfTimes[n_Integer,x_Plus] := HopfTimes[n,#] & /@ x;
HopfTimes[n_Integer,x_Times] := HopfTimes[n,#] & /@ x;
HopfTimes[m_Integer,x_^n_] := HopfTimes[m,x]^n;
HopfTimes[m_Integer,n_?ScalarQ] := n;
HopfTimes[n_Integer,x_] :=
  Expand[Multiply[Psi[x] /. Tensor[y_,z_] :> Tensor[y, HopfTimes[n-1,z]]]] /; n > 1

e/: Psi[e[G_]] := Tensor[e[G],1] + Tensor[1,e[G]]
e/: Chi[e[G_]] := e[G]
Deg[e[G_]] := 1

(*
End[]
EndPackage[]
```

```
*)
```

A.5. Specific Code for K-theories

The code in this section comes from the file `KTheory.m`. It tells *Mathematica* various specific things about the spectra KO, KU, KT and KSp. It also contains routines to check the commutativity and exactness of diagrams. These assume that all homotopy groups are cyclic, which is why they are in this section rather than a more general one.

```
(* KTheory.m Neil Strickland *)

(* This file contains specific information about K-theory spectra      *)
(* See my thesis for notation etc. *)

(* lbase, lord, and lspace are lists of bases and orders of homotopy    *)
(* groups and of spaces in omega spectra. The functions base, order and *)
(* space defined below extend these lists by periodicity.               *)
(* pgen[E] is an invertible element in \pi_8(E) which generates the     *)
(* periodicity                                                          *)

Needs["Stable`"]

Needs["Hopf2`"]

{
Mod40Q[n_Integer] := (Mod[n,4] == 0);
Mod41Q[n_Integer] := (Mod[n,4] == 1);
Mod42Q[n_Integer] := (Mod[n,4] == 2);
Mod43Q[n_Integer] := (Mod[n,4] == 3);
};

(**********************************************************************)

{
lbase[KO]  = {1,al,al^2,0,bt,0,0,0,lm};
lord[KO]   = {Infinity,2,2,1,Infinity,1,1,1,Infinity};
lspace[KO] = {ZxBO,UmO,SpmU,Sp,ZxBSp,UmSp,OmU,OO,ZxBO};
pgen[KO]   = lm;
};

{
Format[ZxBO ,TeXForm]   := "{\\Bbb Z}\\times BO";
Format[UmO  ,TeXForm]   := "U/O";
Format[SpmU ,TeXForm]   := "Sp/U";
Format[Sp   ,TeXForm]   := "Sp";
Format[ZxBSp,TeXForm]   := "{\\Bbb Z}\\times BSp";
Format[UmSp ,TeXForm]   := "U/Sp";
Format[OmU  ,TeXForm]   := "O/U";
Format[OO   ,TeXForm]   := "O";
};

{
Deg[al] = 1;
Deg[bt] = 4;
```

```
Deg[lm] = 8;
};

{
Format[al   ,TeXForm]   := "\alpha";
Format[bt   ,TeXForm]   := "\beta";
Format[lm   ,TeXForm]   := "\lambda";
};

{
al /: n_Integer al := If[OddQ[n],al,0];
al /: al^n_Integer := 0 /; n > 2;
bt /: bt^n_Integer := bt^Mod[n,2] (4 lm)^Quotient[n,2] /; n > 1;
bt /: al^n_. bt := 0;
};

{
z0 /:  n_Integer x_z0 := If[OddQ[n],x,0];
S  /: n_Integer x:S[z0,_] := If[OddQ[n],x,0];
};

Deg[z0[n_]] := n;

{
z0[0]  = Bracket[1];
S[z0,0]  = Bracket[1];
};

z0 /: Psi[z0[n_]] :=
  Module[{k},Sum[Tensor[z0[k],z0[n-k]],{k,0,n}]]

S /:  Psi[x:S[z0,_]] := Tensor[x,x]

z0 /: Chi[z0[n_]] :=
  (Chi[z0[n]] =
    Module[{k},
      Expand[Bracket[-1] Sum[z0[n-k] Chi[z0[k]],{k,0,n-1}]]
    ]
  )

S /:  Chi[x:S[z0,_]] := 1 / x

S /:  Circ[ Bracket[n_Integer], x:S[z0,_] ] := x^n

z0 /: Epsilon[z0[n_]] := 0

S /:  Epsilon[S[z0,_]] := 1

z0bar[n_] := z0[n] Bracket[-1]

S[z0bar,t_] := S[z0,t] Bracket[-1]

z0 /: Circ[z0[n_],z0[m_]] := Multinomial[n,m] z0[n+m]
```

```
S /:  Circ[S[zO,s_],S[zO,t_]] := S[zO,s+t]

Circ[e[KO],Bracket[al x_.]]  := Circ[Bracket[x],zObar[1]]
Circ[e[KO],Bracket[al^2 x_.]] := Circ[Bracket[al x],zObar[1]]
Circ[e[KO],e[KO],Bracket[bt x_.]] := Circ[Bracket[al^2 x],zObar[2]]
Circ[e[KO],e[KO],e[KO],e[KO]]  = Circ[Bracket[bt/lm],zObar[4]]

Circ[e[KO],zO[n_?OddQ]] := Circ[e[KO],zO[(n-1)/2]]^2
Circ[e[KO],e[KO],zO[n_?Mod42Q]] := Circ[e[KO],e[KO],zO[(n-2)/2]]^2
Circ[Bracket[bt x_.],zO[n_Integer]] := 0 /; Mod[n,4] != 0

Circ /: Circ[Bracket[bt a_.],e[KO]]^n_ := 0 /; n > 1
Circ /: Circ[Bracket[bt a_.],e[KO],x_zO]^n_ := 0 /; n > 1
Circ /: Circ[e[KO],e[KO],e[KO]]^n_ := 0 /; n > 1
Circ /: Circ[e[KO],e[KO],e[KO],x_zO]^n_ := 0 /; n > 1
Circ /: Circ[Bracket[al a_.],x_zO]^n_ := 0 /; n > 1
Circ /: Circ[Bracket[al^2 a_.],x_zO]^n_ := 0 /; n > 1
Circ /: Circ[Bracket[al^2 a_.],zO[n_?OddQ]] := 0

(**********************************************************************)

lbase[KT]  = {1,as,0,gm,mu,mu as,0,mu gm,mu^2}
lord[KT]   = {Infinity,2,1,Infinity,Infinity,2,1,Infinity,Infinity}
lspace[KT] = {BT,BBT,OT,T,BT,BBT,OT,T,BT}
pgen[KT]   = mu^2

Format[ZxBT ,TeXForm]   := "T"
Format[OOT  ,TeXForm]   := "\\Omega^2T"
Format[OT   ,TeXForm]   := "\\Omega T"
Format[T    ,TeXForm]   := "T"

Deg[as]  = 1
Deg[gm] = 3
Deg[mu] = 4

{
Format[as   ,TeXForm]   := "\alpha";
Format[gm   ,TeXForm]   := "\gamma";
Format[mu   ,TeXForm]   := "\mu";
};

as /: n_Integer as := If[OddQ[n],as,0]
as /: as^n_Integer := 0 /; n > 1
as /: as gm  = 0
gm /: gm^n_Integer  := 0 /; n > 1

zT /:  n_Integer x_zT := If[OddQ[n],x,0]
S  /: n_Integer x:S[zT,_] := If[OddQ[n],x,0]
zT /: zT[n_?OddQ]^m_ := 0 /; m > 1

Deg[zT[n_]] := n

zT[0]  = Bracket[1]
S[zT,0]  = Bracket[1]
```

```
zT /: Psi[zT[n_]] :=
  Module[{k},Sum[Tensor[zT[k],zT[n-k]],{k,0,n}]]

S /:  Psi[x:S[zT,_]] := Tensor[x,x]

zT /: Chi[zT[n_]] :=
  (Chi[zT[n]] =
    Module[{k},
      Expand[Bracket[-1] Sum[zT[n-k] Chi[zT[k]],{k,0,n-1}]]
    ]
  )

S /:  Chi[x:S[zT,_]] := 1 / x

S /:  Circ[ Bracket[n_Integer], x:S[zT,_] ] := x^n

zT /: Epsilon[zT[n_]] := 0

S /:  Epsilon[S[zT,_]] := 1

zTbar[n_] := zT[n] Bracket[-1]

S[zTbar,t_] := S[zT,t] Bracket[-1]

zT /: Circ[zT[n_],zT[m_]] := Multinomial[n,m] zT[n+m]

S /:  Circ[S[zT,s_],S[zT,t_]] := S[zT,s+t]

Circ[e[KT],Bracket[as]]    = zTbar[1]
Circ[Bracket[as],zT[n_?OddQ]] := 0
Circ[e[KT],e[KT],Bracket[gm]]  = Circ[Bracket[as],zTbar[2]]
Circ[Bracket[gm],zT[n_?OddQ]] := 0
Circ[Bracket[gm],zT[n_?Mod42Q]] :=
  Circ[Bracket[gm] Circ[e[KT],Bracket[mu]]^2 , zT[n-2]]
Circ[e[KT],zT[n_?OddQ]] := Circ[e[KT],zT[(n-1)/2]]^2
Circ[e[KT],e[KT],zT[n_?Mod42Q]] := Circ[e[KT],e[KT],zT[(n-2)/2]]^2
Circ /: Circ[e[KT],e[KT],e[KT]]^n_ := 0 /; n > 1
Circ /: Circ[e[KT],e[KT],e[KT],x_zO]^n_ := 0 /; n > 1

(*********************************************************************)

lbase[KU]  = {1,0,nu,0,nu^2,0,nu^3,0,nu^4}
lord[KU]   = {Infinity,1,Infinity,1,Infinity,1,Infinity,1,Infinity}
lspace[KU] = {ZxBU,U,ZxBU,U,ZxBU,U,ZxBU,U,ZxBU}
pgen[KU]   = nu^4

Format[ZxBU ,TeXForm]   := "{\\Bbb Z}\\times BU"
Format[U    ,TeXForm]   := "U"

Deg[nu] = 2

Format[nu   ,TeXForm]   := "\nu";
```

```
zU /:  n_Integer x_zU := If[OddQ[n],x,0]
zU /: zU[n_?OddQ] := 0
S  /: n_Integer x:S[zU,_] := If[OddQ[n],x,0]

Deg[zU[n_]] := n

zU[0]  = Bracket[1]
S[zU,0]  = Bracket[1]

zU /: Psi[zU[n_]] :=
  Module[{k},Sum[Tensor[zU[2k],zU[n-2k]],{k,0,n/2}]]

S /:  Psi[x:S[zU,_]] := Tensor[x,x]

zU /: Chi[zU[n_]] :=
  (Chi[zU[n]] =
    Module[{k},
      Expand[Bracket[-1] Sum[zU[n-2k] Chi[zU[2k]],{k,0,n/2-1}]]
    ]
  )

S /:  Chi[x:S[zU,_]] := 1 / x

S /:  Circ[ Bracket[n_Integer], x:S[zU,_] ] := x^n

zU /: Epsilon[zU[n_]] := 0

S /:  Epsilon[S[zU,_]] := 1

zUbar[n_] := zU[n] Bracket[-1]

S[zUbar,t_] := S[zU,t] Bracket[-1]

zU /: Circ[zU[n_],zU[m_]] := Multinomial[n,m] zU[n+m]

S /:  Circ[S[zU,s_],S[zU,t_]] := S[zU,s+t]

Circ[e[KU],e[KU]]  = Circ[Bracket[1/nu],zUbar[2]]
e /: e[KU]^n_Integer := 0 /; n > 1
Circ /: Circ[e[KU],x_zU]^n_Integer := 0 /; n > 1

(*************************************************************************)

lbase[KSp] = {bt th/lm,0,0,0,th,al th,al^2 th,0,bt th}
lord[KSp]  = {Infinity,1,1,1,Infinity,2,2,1,Infinity}
lspace[KSp]= {ZxBSp,UmSp,OmU,OO,ZxBO,UmO,SpmU,Sp,ZxBSp}
pgen[KSp]  = lm

Deg[th] = 4

Format[th   ,TeXForm]   := "\theta";

th /: th^n_Integer := th^Mod[n,2] lm^Quotient[n,2] /; n > 1
```

```
e[KSp] = e[KO]

(*********************************************************************)

lbase[Sigma[n_,G_]]      := Block[{i},Table[ base[G,i-n],{i,0,8}]]
lord[Sigma[n_,G_]]       := Block[{i},Table[order[G,i-n],{i,0,8}]]
lspace[Sigma[n_,G_]]     := Block[{i},Table[space[G,i+n],{i,0,8}]]
pgen[Sigma[n_,G_]]       := pgen[G]

base[G_,n_]     := pgen[G]^Quotient[n,8] lbase[G][[Mod[n,8]+1]]
order[G_,n_]    := lord[G][[Mod[n,8]+1]]
space[G_,n_]    := lspace[G][[Mod[n,8]+1]]

(*********************************************************************)
(* Properties of maps                                                *)

(* All maps considered are KO-linear *)

{
zz[KO]  = zO;
zz[KT]  = zT;
zz[KU]  = zU;
zz[KSp] = zO;
d[zO] = d[zT] = 1;
d[zU] = 2;
};

AssertKOLinear[f_Symbol] :=
  Module[{z0,z1,d0,d1},
    z0 = zz[Source[f]];
    d0 = d[z0];
    z1 = zz[Target[f]];
    d1 = d[z1];
    Homology[f][Release[z0][k_]] :=
      Release[Circ[Bracket[Homotopy[f][1]],z1[k]]];
    Homology[f][Circ[Release[z0][k_],x_]] :=
      Release[Circ[z1[k],Homology[f][x]]];
    Homology[f][Circ[Release[z0][k_],x_,y__]] :=
      Release[Circ[z1[k],Homology[f][Circ[x,y]]]];
    Which[
      d0 === d1,
       (Homology[f][S[Release[z0],t_]] :=
          Release[Circ[Bracket[Homotopy[f][1]],S[z1,t]]];
        Homology[f][Circ[S[Release[z0],t_],x_]] :=
          Release[Circ[S[z1,t],Homology[f][x]]];
        Homology[f][Circ[S[Release[z0],t_],x_,y__]] :=
          Release[Circ[S[z1,t],Homology[f][Circ[x,y]]]];),
      (d0 === 1 && d1 === 2),
       (Homology[f][S[Release[z0],t_]] :=
          Release[Circ[Bracket[Homotopy[f][1]],S[z1,Frobenius[t]]]];
        Homology[f][Circ[S[Release[z0],t_],x_]] :=
          Release[Circ[S[z1,Frobenius[t]],Homology[f][x]]];
        Homology[f][Circ[S[Release[z0],t_],x_,y__]] :=
          Release[Circ[S[z1,Frobenius[t]],Homology[f][Circ[x,y]]]];),
```

```
      (d0 === 2 && d1 === 1),
       (Homology[f][S[Release[z0],t_?SquareQ]] :=
          Release[Circ[Bracket[Homotopy[f][1]],S[z1,Frobenius[-1,t]]]];
        Homology[f][Circ[S[Release[z0],t_?SquareQ],x_]] :=
          Release[Circ[S[z1,Frobenius[-1,t]],Homology[f][x]]];
        Homology[f][Circ[S[Release[z0],t_?SquareQ],x_,y__]] :=
          Release[Circ[S[z1,Frobenius[-1,t]],Homology[f][Circ[x,y]]]];)
    ]
  ]


(* cc is the complex conjugation map                                        *)

DefineStableMap[cc,KU,KU,MapRingQ -> True, MapTeX -> "c"]

Homotopy[cc][nu]         = -nu

Homology[cc][zU[k_]] := zU[k]
Homology[cc][S[zU,t_]] := S[zU,t]

(* lU symplectifies complex bundles                                         *)

DefineStableMap[lU,KU,KSp,MapTeX -> "{l_U}"]

Homotopy[lU][nu]         = 0
Homotopy[lU][nu^2]       = 2 th
Homotopy[lU][nu^3]       = al^2 th
Homotopy[lU][nu^4]       = bt th
Homotopy[lU][nu^n_Integer]:=
  lm^Quotient[n,4] Homotopy[lU][nu^Mod[n,4]]
Homotopy[lU][1]          = bt th/lm

AssertKOLinear[lU]

(* lO complexifies real bundles, remembering the self-conjugacy             *)

DefineStableMap[lO,KO,KT,MapRingQ -> True, MapTeX ->"{l_O}"]

Homotopy[lO][al]         = as
Homotopy[lO][al^2]       = 0
Homotopy[lO][bt]         = 2 mu
Homotopy[lO][lm]         = mu^2

Homology[lO][zO[k_]] := zT[k]
Homology[lO][S[zO,t_]] := S[zT,t]

(* mU adds a complex bundle to its conjugate                                *)

DefineStableMap[mU,KU,KT,MapTeX -> "{m_U}"]

Homotopy[mU][nu^n_Integer:1]:=
        If[EvenQ[n],2 mu^(n/2),0]
Homotopy[mU][1]          = 2
```

```
AssertKOLinear[mU]

(* mT symplectifies the underlying complex bundle of a self-conjugate   *)
(* bundle                                                                *)

DefineStableMap[mT,KT,KSp,MapTeX -> "{m_T}"]

Homotopy[mT][as]         = 0
Homotopy[mT][gm]         = 0
Homotopy[mT][mu]         = 2 th
Homotopy[mT][mu as]      = 0
Homotopy[mT][mu gm]      = 0
Homotopy[mT][mu^n_Integer x_.]:=
        lm^Quotient[n,2] Homotopy[mT][mu^Mod[n,2] x] //. Relations
Homotopy[mT][1]          = bt th/lm

AssertKOLinear[mT]

(* mO complexifies real bundles                                          *)

DefineStableMap[mO,KO,KU,MapRingQ -> True, MapTeX -> "{m_O}"]

Homotopy[mO][al]         = 0
Homotopy[mO][bt]         = 2 nu^2
Homotopy[mO][lm]         = nu^4

Homology[mO][zO[k_]] := zU[k]
Homology[mO][S[zO,t_]] := S[zU,Frobenius[t]]

(* nO symplectifies real bundles                                         *)

DefineStableMap[nO,KO,KSp, MapTeX -> "{n_O}"]

Homotopy[nO][x_]        := x bt th/lm //. Relations

Homology[nO][x_] := Circ[x,Bracket[bt th/lm]]

(* fU forgets a complex structure                                        *)

DefineStableMap[fU,KU,KO, MapTeX -> "{f_U}"]

Homotopy[fU][1]          = 2
Homotopy[fU][nu]         = al^2
Homotopy[fU][nu^2]       = bt
Homotopy[fU][nu^3]       = 0
Homotopy[fU][nu^n_Integer:1 x_.]:=
        lm^Quotient[n,4] Homotopy[fU][nu^Mod[n,4] x] //. Relations

AssertKOLinear[fU]

(* fT forgets a self-conjugacy                                           *)

DefineStableMap[fT,KT,KU,MapRingQ -> True, MapTeX -> "{f_T}"]
```

```
Homotopy[fT][as]        = 0
Homotopy[fT][gm]        = 0
Homotopy[fT][mu]        = nu^2

Homology[fT][zT[k_]] := zU[k]
Homology[fT][S[zT,t_]] := S[zU,Frobenius[t]]

(* fSp forgets a symplectic structure on a complex bundle, but        *)
(* remembers the induced self-conjugacy                               *)

DefineStableMap[fSp,KSp,KT, MapTeX -> "{f_Sp}"]

Homotopy[fSp][x_. th]   := Homotopy[lO][x] mu

AssertKOLinear[fSp]

(* gT forgets both the self conjugacy and the complex structure on a  *)
(* self-conjugate bundle                                              *)

DefineStableMap[gT,KT,KO, MapTeX -> "{g_T}"]

Homotopy[gT][1]         = 2
Homotopy[gT][as]        = 0
Homotopy[gT][gm]        = 0
Homotopy[gT][mu]        = bt
Homotopy[gT][mu as]     = 0
Homotopy[gT][mu gm]     = 0
Homotopy[gT][mu^n_Integer x_.]:=
        lm^Quotient[n,2] Homotopy[gT][mu^Mod[n,2] x] //. Relations

AssertKOLinear[gT]

(* gSp forgets a symplectic structure on a complex bundle             *)

DefineStableMap[gSp,KSp,KU, MapTeX -> "{g_Sp}"]

Homotopy[gSp][x_. th]   := Homotopy[mO][x] nu^2

AssertKOLinear[gSp]

(* hSp forgets a symplectic structure on a real bundle                *)

DefineStableMap[hSp,KSp,KO, MapTeX -> "{h_Sp}"]

Homotopy[hSp][x_. th]   := x bt

AssertKOLinear[hSp]

(* dT is a boundary map                                               *)

DefineStableMap[dT,KU,KT,MapDeg -> 1, MapTeX -> "{\delta_T}"]

Homotopy[dT][1]         = as
Homotopy[dT][nu]        = gm
```

```
Homotopy[dT][nu^n_Integer x_.]:=
        mu^Quotient[n,2] Homotopy[dT][nu^Mod[n,2] x] //. Relations

AssertKOLinear[dT]

(* dO is a boundary map                                                  *)

DefineStableMap[dO,KT,KO,MapDeg -> -3, MapTeX -> "{\delta_O}"]

Homotopy[dO][1]          = 0
Homotopy[dO][as]         = 0
Homotopy[dO][gm]         = 2
Homotopy[dO][mu]         = al
Homotopy[dO][mu as]      = al^2
Homotopy[dO][mu gm]      = bt
Homotopy[dO][mu^n_Integer x_.]:=
        lm^Quotient[n,2] Homotopy[dO][mu^Mod[n,2] x] //. Relations

AssertKOLinear[dO]

(* dSp is a boundary map                                                 *)

DefineStableMap[dSp,KT,KSp,MapDeg -> -3, MapTeX -> "{\delta_Sp}"]

Homotopy[dSp][1]         = (al/lm) th
Homotopy[dSp][as]        = (al^2 /lm) th
Homotopy[dSp][gm]        = (bt /lm) th
Homotopy[dSp][mu]        = 0
Homotopy[dSp][mu as]     = 0
Homotopy[dSp][mu gm]     = 2 th
Homotopy[dSp][mu^n_Integer x_.]:=
        lm^Quotient[n,2] Homotopy[dSp][mu^Mod[n,2] x] //. Relations

AssertKOLinear[dSp]

(* mth multiplies by theta                                               *)

DefineStableMap[mth,KO,KSp,MapDeg -> 4, MapTeX -> "{m_\theta}"]

Homotopy[mth][x_]       := x th

AssertKOLinear[mth]

(* Relations in homotopy rings                                           *)
Relations=
{al^n_Integer           :> 0 /; n > 2,
 as^n_Integer           :> 0 /; n > 1,
 n_Integer al^m_.       :> If[EvenQ[n],0,al^m],
 n_Integer as^m_.       :> If[EvenQ[n],0,as^m],
 bt^n_Integer           :> (4 lm)^Quotient[n,2] bt^Mod[n,2] /; n>1,
 al^n_. bt              :> 0 /; n > 0,
 as gm                  :> 0,
 gm^n_Integer           :> 0 /; n > 1   }
```

```
(* MapEqualQ[f,g] is True iff f&g have the same effect on homotopy groups *)

MapEqualQ[f_,g_] := (Source[f]===Source[g]) &&
                    (Target[f]===Target[g]) &&
                    (Deg[f]===Deg[g]) &&
                    ( ((Homotopy[f] /@ lbase[Source[f]])//.Relations) ===
                      ((Homotopy[g] /@ lbase[Source[g]])//.Relations) )

(* VanishQ[f] is True iff f induces zero in homotopy                    *)
VanishQ[f_]     := ((Homotopy[f] /@ lbase[Source[f]])//.Relations ===
                        {0,0,0,0,0,0,0,0})

mod[k_,Infinity]        := Abs[k]
mod[k_,l_]              := Mod[k,l]

Image[f_]:=
  Module[{i,x,G,F,d,im},
    G = Source[f];
    F = Target[f];
    d = Deg[f];
    im = Table[  (Homotopy[f][ base[G,i-d] ]//.Relations) , {i,0,8}  ];
    For[i=0, i<9, i++,
      If[(x=base[F,i]) === 0,
        im[[i+1]] = 0,
      (* Else *)
        im[[i+1]] = mod[ im[[i+1]]/x , order[F,i] ]
      ]
    ];
  im
  ]

Kernel[f_]:=
  Module[{i,j,k,x,y,z,G,F,d,ker},
    G = Source[f];
    F = Target[f];
    d = Deg[f];
    ker = Table[ 0 , {i,1,9}  ];
    For[i=0, i<9, i++,
      x = base[G,i];
      y = (Homotopy[f][x] //. Relations);
      z = base[F,i+d];
      If[ x === 0,
        k = 0,
      (* Else *)
        If[ z === 0,
          k = 1,
        (* Else *)
          j = mod[y/z,n=order[F,i+d]];
          If[ j === 0,
            k = 1,
          (* Else *)
            If[ n === Infinity,
              k = 0,
            (* Else *)
```

```
              k = mod[n/j,order[G,i]]
            ]
          ]
        ]
      ];
      ker[[i+1]] = k
    ];
    ker
  ]

ExactQ[f_,g_] := (Image[f] === Kernel[g])

ExactTriangleQ[f_,g_,h_] := ExactQ[f,g] && ExactQ[g,h] && ExactQ[h,f]
```

A.6. Claims about Diagrams

This section contains the code from `Claims.m`, which is a list of claims in *Mathematica* notation about the commutativity and exactness of various diagrams of homotopy groups.

```
(* This file contains lists of statements about maps between K theory   *)
(* spectra. Definitions of the objects involved and Mathematica code    *)
(* implementing them are in the file KTheory.m                          *)

(* comm is a list of claimed commutativity statements                   *)

<<KTheory.m

comm = {
MapEqualQ[  mU,
            Composition[fSp,lU]],
MapEqualQ[  mU,
            Composition[lO,fU]],
MapEqualQ[  nO,
            Composition[mT,lO]],
MapEqualQ[  nO,
            Composition[lU,mO]],
MapEqualQ[  mO,
            Composition[fT,lO]],
MapEqualQ[  mT,
            Composition[lU,fT]],
MapEqualQ[  MultiplyBy[2,KO],
            Composition[fU,mO]],
MapEqualQ[  MultiplyBy[2,KSp],
            Composition[mT,fSp]],
MapEqualQ[  OneMap[KU]+cc,
            Composition[mO,fU]],
MapEqualQ[  gSp,
            Composition[fT,fSp]],
MapEqualQ[  gT,
            Composition[fU,fT]],
MapEqualQ[  hSp,
            Composition[gT,fSp]],
MapEqualQ[  hSp,
```

```
             Composition[fU,gSp]],
MapEqualQ[  Composition[lO,MultiplyBy[al,KO]],
             Composition[ dT,mO]],
MapEqualQ[  Composition[fU,MultiplyBy[1/nu,KU],fT],
             Composition[MultiplyBy[al,KO],dO]],
MapEqualQ[  Composition[MultiplyBy[1/nu,KU],OneMap[KU]-cc],
             Composition[mO,fU,MultiplyBy[1/nu,KU]]],
MapEqualQ[  Composition[fSp,MultiplyBy[al,KSp]],
             Composition[dT,gSp]],
MapEqualQ[  Composition[lU,MultiplyBy[1/nu,KU],fT],
             Composition[MultiplyBy[al,KSp],dSp]],
MapEqualQ[  Composition[gSp,lU,MultiplyBy[1/nu,KU]],
             Composition[MultiplyBy[1/nu,KU],OneMap[KU]-cc]],
MapEqualQ[  Composition[dSp,MultiplyBy[mu,KT]],
             Composition[mth,dO]],
MapEqualQ[  Composition[fSp,mth],
             Composition[MultiplyBy[mu,KT],lO]],
MapEqualQ[  Composition[lU,MultiplyBy[nu,KU]],
             Composition[mth,fU,MultiplyBy[1/nu,KU]]],
MapEqualQ[  Composition[gSp,mth],
             Composition[MultiplyBy[nu^2,KU],mO]],
MapEqualQ[  Composition[dT,MultiplyBy[nu^2,KU]],
             Composition[MultiplyBy[mu,KT],dT]],
MapEqualQ[  Composition[fT,MultiplyBy[mu,KT]],
             Composition[MultiplyBy[nu^2,KU],fT]],
MapEqualQ[  Composition[MultiplyBy[1/nu,KU],
               OneMap[KU]-cc,
               MultiplyBy[nu^2,KU]],
             Composition[MultiplyBy[nu,KU],OneMap[KU]-cc]]
}

(* ex is a list of claimed exactness statements                      *)

ex = {
ExactTriangleQ[lO,
        dO,
        MultiplyBy[al^2,KO]],
ExactTriangleQ[mO,
        Composition[fU,MultiplyBy[1/nu,KU]],
        MultiplyBy[al,KO]],
ExactTriangleQ[fT,
        Composition[MultiplyBy[1/nu,KU],OneMap[KU]-cc],
        dT],
ExactTriangleQ[fSp,
        dSp,
        MultiplyBy[al^2,KSp]],
ExactTriangleQ[gSp,
        Composition[lU,MultiplyBy[1/nu,KU]],
        MultiplyBy[al,KSp]]
}

(* The final verdict ...                                             *)
And @@ comm && And @@ ex
```

A.7. Results

The following is a transcript of a *Mathematica* session using the above code.

```
Mathematica 2.0 for HP 9000 RISC
Copyright 1988-91 Wolfram Research, Inc.
 -- X11 windows graphics initialized --

In[1]:= <<Claims.m

Out[1]= True

In[2]:= Quit

Process math finished
```

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