HOMOTOPY THEORY AND LINEAR ALGEBRA

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In this note we collect together a number of results about the geometry and homotopy theory of certain spaces related to linear algebra.

Some things from the bestiary and/or the notes for "Homology and Manifolds" should be moved here. Perhaps also the appendix on functional calculus from "Common subbundles".

1. Universes

Definition 1.1. We write $\mathbb{R}^\infty$ for the direct sum of countably many copies of $\mathbb{R}$, with basis $\{e_0, e_1, \ldots\}$. We give this the obvious inner product, and regard $\mathbb{R}^n$ as a subspace in the obvious way. We write $\mathbb{R}^{\infty-n}$ for the orthogonal complement of $\mathbb{R}^n$, which is spanned by $\{e_k \mid k \geq n\}$.

A *universe* is a real vector space $\mathcal{U}$ of finite or countable dimension, equipped with an inner product. Any universe is isometrically isomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N} \cup \{\infty\}$, by a Gram-Schmidt argument. It will be technically convenient to require that the underlying set of $\mathcal{U}$ lies in the stage V_{ω_1} of the von Neumann hierarchy, where ω_1 is the first uncountable ordinal; this ensures that there is only a set of universes. With any of the usual definitions of $\mathbb{R}$, $\oplus$ and $\otimes$, we find that $\mathbb{R}$ is a universe and that the set of universes is closed under countable direct sums and under tensor products.

We say that a universe $\mathcal{U}$ is *infinite* if it has infinite dimension. We write $U \ll \mathcal{U}$ to indicate that U is a finite-dimensional subspace of $\mathcal{U}$. We topologise $\mathcal{U}$ as the colimit of its finite-dimensional subspaces; this is different from the metric topology if $\mathcal{U}$ is infinite. We write $\mathcal{L}^+$ for the category of universes; the morphisms $\mathcal{L}^+(\mathcal{U}, \mathcal{V})$ are the linear maps that preserve the inner product. We also write $\mathcal{L}$ for the full subcategory of infinite universes.

If $\mathcal{U}$ and $\mathcal{V}$ are finite then $\mathcal{L}^+(\mathcal{U}, \mathcal{V})$ has an evident topology making it a compact manifold. If $\mathcal{U}$ or $\mathcal{V}$ is infinite then we topologise $\mathcal{L}^+(\mathcal{U}, \mathcal{V})$ as

$$\mathcal{L}^+(\mathcal{U}, \mathcal{V}) = \varprojlim_{U \ll \mathcal{U}} \varinjlim_{V \ll \mathcal{V}} \mathcal{L}^+(U, V).$$

We also write

$$\begin{aligned}
\mathcal{L}^+(\mathcal{U}) &= \mathcal{L}^+(\mathcal{U},\mathcal{U}) \\
V_n(\mathcal{U}) &= \mathcal{L}^+(\mathbb{R}^n,\mathcal{U}) = \{(u_0, \ldots, u_{n-1}) \in \mathcal{U}^n \mid \langle u_i, u_j \rangle = \delta_{ij}\} \\
\mathcal{L}(n) &= \mathcal{L}((\mathbb{R}^\infty)^n, \mathbb{R}^\infty) \\
S^\infty &= V_1\mathbb{R}^\infty = \{u \in \mathbb{R}^\infty \mid \langle u, u \rangle = 1\}.
\end{aligned}$$

We next prove the well-known fact that many of these spaces are contractible.

Lemma 1.2. *The space $\mathcal{L}^+(\mathcal{U}, \mathcal{V})$ is a retract of the space $\mathcal{J}(\mathcal{U}, \mathcal{V})$ of injective linear maps from $\mathcal{U}$ to $\mathcal{V}$.*

Proof. After choosing bases, we may assume that $\mathcal{U} = \mathbb{R}^m$ for some $m \in \mathbb{N} \cup \{\infty\}$. This identifies $\mathcal{J}(\mathcal{U}, \mathcal{V})$ with the space X of linearly independent sequences $(v_i)_{i<m}$ in $\mathcal{V}$, and $\mathcal{L}^+(\mathcal{U}, \mathcal{V})$ of orthonormal sequences.

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Given a sequence $\underline{v} \in X$, we define an orthonormal sequence $\gamma(\underline{u}) \in Y$ by the usual Gram-Schmidt procedure:

$$\begin{aligned} v_0 &= u_0/\|u_0\| \\ \tilde{v}_i &= u_i - \sum_{j<i} \langle u_i, v_j \rangle v_j \\ v_i &= \tilde{v}_i/\|\tilde{v}_i\| \\ \gamma(u_0, u_1, \ldots) &= (v_0, v_1, \ldots). \end{aligned}$$

More abstractly, $\underline{v}$ is the unique orthonormal sequence such that $\operatorname{span}(v_j \mid j < i) = \operatorname{span}(u_j \mid j < i)$ and $\langle u_i, v_i \rangle > 0$ for all i. It is easy to check that γ gives a continuous retraction $X \to Y$. $\square$

Proposition 1.3. *If $\mathcal{V}$ is infinite then $\mathcal{L}^+(\mathcal{U}, \mathcal{V})$ is contractible for all $\mathcal{U}$. In particular, the spaces $\mathcal{L}^+(\mathcal{V})$, $V_n(\mathcal{V})$, $\mathcal{L}(n)$ and S^∞ are contractible.*

Proof. We may assume that $\mathcal{V} = \mathbb{R}^\infty$ and that $\mathcal{U} = \mathbb{R}^n$ for some $n \leq \infty$. We will show that the space $F := \mathcal{J}(\mathbb{R}^n, \mathbb{R}^\infty)$ is contractible; it will follow from Lemma 1.2 that $\mathcal{L}^+(\mathcal{U}, \mathcal{V})$ is contractible.

First define $d(t)\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ by $d(t)(e_i) = (1-t)e_i + te_{2i}$. If $v \in \mathbb{R}^\infty \setminus \{0\}$ then, by considering $\max\{i \mid v_i \neq 0\}$, we see that $d(t)(v) \neq 0$ for all t. Thus, we have $d(t) \in \mathcal{J}(\mathbb{R}^\infty, \mathbb{R}^\infty)$, and we can define a homotopy $h\colon I \times F \to F$ by $h(t, f) = d(t) \circ f$.

Next, we define $j\colon \mathbb{R}^n \to \mathbb{R}^\infty$ by $j(e_i) = e_{2i+1}$, and we define another homotopy $k\colon I \times F \to F$ by

$$k(t, f) = (1-t)d(1) \circ f + tj\colon \mathbb{R}^n \to \mathbb{R}^\infty.$$

(This is an injective linear map, as required, because the images of $d(1)$ and j have trivial intersection.)

By composing h and k, we see that the identity map of F is homotopic to the constant map with value j. $\square$

It is not hard to see that the composition map

$$\mathcal{L}^+(\mathcal{U}, \mathcal{V}) \times \mathcal{L}^+(\mathcal{V}, \mathcal{W}) \to \mathcal{L}^+(\mathcal{U}, \mathcal{W})$$

is continuous, and thus that $\mathcal{L}^+(\mathcal{U})$ is a topological monoid. If $\mathcal{U}$ is infinite then it is rather far from being a group, and a number of our results seem very strange if one forgets this. However, it does contain a useful subgroup:

Definition 1.4. For any universe $\mathcal{U}$, we define the orthogonal group of $\mathcal{U}$ as

$$O(\mathcal{U}) = \{f \in \mathcal{L}^+(\mathcal{U}) \mid f|_{U^\perp} = 1 \text{ for some } U \ll \mathcal{U}\} = \varinjlim_{U \ll \mathcal{U}} O(U).$$

We also write $O(n) = O(\mathbb{R}^n)$ for $n \in \mathbb{N} \cup \{\infty\}$. If $\mathcal{U}$ is finite then it is clear that $O(\mathcal{U})$ is a topological group, and the same conclusion holds in general by passage to colimits.

Definition 1.5. Let $\mathcal{V}$ be another universe, and let $f\colon \mathcal{U} \to \mathcal{V}$ be a linear isometry. If $g \in O(\mathcal{U})$ then there is some $W \ll \mathcal{U}$ such that $g = 1$ on $W^\perp$. We can define $f_*(g)\colon fW \to fW$ by $fw \mapsto fgw$, and extend this to an automorphism of $\mathcal{V}$ by defining it to be 1 on $(fW)^\perp$. This is easily seen to be independent of the choice of W, and to give a continuous map $\mathcal{L}(\mathcal{U}, \mathcal{V}) \times O(\mathcal{U}) \to O(\mathcal{V})$ which behaves as a homomorphism in the second variable. In other words, this makes the construction $\mathcal{U} \mapsto O(\mathcal{U})$ into a continuous functor from $\mathcal{L}^+$ to the category of topological groups. If f is an isomorphism we just have $f_*(g) = fgf^{-1}$ but this does not make sense for general f.

Lemma 1.6. *For any universes $\mathcal{U}$ and $\mathcal{V}$, the evident map $\mathcal{L}^+(\mathcal{U}, \mathcal{V}) \to F(\mathcal{U}, \mathcal{V})$ is a closed inclusion.*

Proof. The left hand side is $\varprojlim_{U \ll \mathcal{U}} \mathcal{L}^+(U, \mathcal{V})$ and the right hand side is $\varprojlim_{U \ll \mathcal{U}} F(U, \mathcal{V})$, and inverse limits of regular monomorphisms are regular mono, so it is enough to check that $\mathcal{L}^+(U, \mathcal{V}) \to F(U, \mathcal{V})$ is a closed inclusion. In fact, by Lemma **??**, it is enough to check that the composite $\mathcal{L}^+(U, \mathcal{V}) \to F(U, \mathcal{V}) \to F(S(U)_+, \mathcal{V})$ is a closed inclusion. To do this, choose a sequence $V_0 < V_1 < \ldots \ll \mathcal{V}$ with $\mathcal{V} = \varinjlim_n V_n$. Observe that $\mathcal{L}^+(U, V_n)$ is compact, so the injections $\mathcal{L}^+(U, V_n) \to \mathcal{L}^+(U, V_{n+1})$ and $\mathcal{L}^+(U, V_n) \to F(S(U)_+, V_n)$ are closed inclusions. Moreover, $F(S(U)_+,-)$ preserves closed inclusions, so the following square is a pullback of closed inclusions:

$$\begin{tikzcd}
\mathcal{L}^+(U,V_n) \arrow[r, tail] \arrow[d, tail] & \mathcal{L}^+(U,V_{n+1}) \arrow[d, tail] \\
F(S(U)_+,V_n) \arrow[r, tail] & F(S(U)_+,V_{n+1})
\end{tikzcd}$$

By applying Lemma **??** and Lemma **??**, we conclude that the map $\mathcal{L}^+(U,\mathcal{V}) \to F(S(U)_+,\mathcal{V})$ of colimits is a closed inclusion as required. ☐

Lemma 1.7. *For any universes $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$, the evident map $\mathcal{L}^+(\mathcal{U},\mathcal{V}) \to \mathcal{L}^+(\mathcal{U},\mathcal{V}\oplus\mathcal{W})$ is a closed inclusion.*

Proof. Consider the following square.

$$\begin{tikzcd}
\mathcal{L}^+(\mathcal{U},\mathcal{V}) \arrow[r] \arrow[d, tail] & \mathcal{L}^+(\mathcal{U},\mathcal{V}\oplus\mathcal{W}) \arrow[d, tail] \\
F(\mathcal{U},\mathcal{V}) \arrow[r, tail] & F(\mathcal{U},\mathcal{V}\oplus\mathcal{W})
\end{tikzcd}$$

The vertical maps are closed inclusions by Lemma 1.6. The bottom horizontal map is a split monomorphism and thus a closed inclusion. The claim follows by Lemma **??**. ☐

Lemma 1.8. *The map $\epsilon\colon O(\infty) \to S^\infty$ sending g to $g(e_0)$ is a trivial bundle with fibre isomorphic to $O(\infty)$ (so in particular, it admits a section).*

Proof. For any $u \in \mathbb{R}^\infty \setminus \{0\}$ we let $\tau(u) \in O(\infty)$ be the reflection in the hyperplane orthogonal to u, so $\tau(u)(v) = v - 2\langle u,v\rangle u/\langle u,u\rangle$. If $u,v \in S^\infty$ and $u \neq v$ then it is easy to see that $\tau(u-v)$ exchanges u and v. Next, we let $\rho\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ be the shift map, given by $\rho(e_k) = e_{k+1}$ for all k. If $u \in S^\infty$ it is easy to see that $\rho(u) \in S^\infty$ and that $u \neq \rho(u) \neq e_0$. We can thus define $\sigma(u) = \tau(u-\rho(u))\tau(e_0-\rho(u)) \in O$ and note that $\sigma(u)(e_0) = u$, so σ is a section of ϵ. We also observe that the map $\rho_*\colon O(\infty) \to O(\infty)$ (as in Definition 1.5) gives a homeomorphism

$$O(\infty) \to O(e_0^\perp) = \{g \in O(\infty) \mid g(e_0) = e_0\}.$$

Now define $\phi\colon S^\infty \times O(\infty) \to O(\infty)$ by $\phi(u,g) = \sigma(u)\rho_*(g)$, so ϕ is a homeomorphism and $\epsilon\phi(u,g) = u$. This gives the required trivialisation. ☐

1.1. **Rotating subspaces.** Let A and B be finite-dimensional subspaces of a universe $\mathcal{U}$, with $\dim(A) = \dim(B)$. If A and B are sufficiently close then we will have $A \cap B^\perp = 0$ so the orthogonal projection $\pi\colon B \to A$ is an isomorphism; this gives a fairly canonical way to identify B with A, which has been used for many purposes. Unfortunately it does not give an *isometry* from A to B, which is inconvenient for us, so we will give a slightly different construction.

Definition 1.9. For any universe $\mathcal{U}$, we define

$$\begin{aligned}
G(\mathcal{U}) &= \{A \mid A \ll \mathcal{U}\} = \text{ the Grassmannian of } \mathcal{U} \\
N(\mathcal{U}) &= \{(A,B) \in G(\mathcal{U})^2 \mid \dim(A) = \dim(B) \text{ and } A \cap B^\perp = 0\}.
\end{aligned}$$

If U is finite then $G(U)$ is a compact manifold, and in general we topologise $G(\mathcal{U})$ as $\varinjlim_{U \ll \mathcal{U}} G(U)$. It is easy to see that $N(\mathcal{U})$ is a neighbourhood of the diagonal in $G(\mathcal{U})^2$.

Proposition 1.10. *There is a natural continuous map $\rho\colon N(\mathcal{U}) \to O(\mathcal{U})$ such that $\rho(A,B)A = B$ and $\rho(B,A) = \rho(A,B)^{-1}$. (Naturality is to be interpreted using Definition 1.5.)*

The proof will follow after a lemma.

Lemma 1.11. *Let A and B be subspaces of a finite universe U, and suppose that* $\dim(A) = \dim(B)$. *Let $\alpha,\beta \in End(U)$ be the orthogonal projections onto A and B. Then $1-(\alpha-\beta)^2$ is a positive self-adjoint operator on U, and the following are equivalent:*

(a) $A \cap B^\perp = 0$

(b) $A^\perp \cap B = 0$
(c) $\|(\alpha - \beta)^2\| < 1$
(d) $1 - (\alpha - \beta)^2$ *is invertible.*

Proof. First note that $\alpha - \beta$ is self-adjoint (because α and β are) so we can choose an orthonormal basis of eigenvectors e_i with eigenvalues t_i say. As α is a projector we have $0 \leq \langle \alpha e_i, e_i \rangle \leq 1$ and similarly for β, so

$$t_i = \langle \alpha e_i, e_i \rangle - \langle \beta e_i, e_i \rangle \in [-1, 1],$$

so $1 - t_i^2 \in [0, 1]$. It follows that $\xi := 1 - (\alpha - \beta)^2$ is a positive self-adjoint operator. Given all this it is clear that (c)$\Leftrightarrow$(d).

Next let $j \colon A \to U$ and $k \colon B \to U$ be the inclusions, and write j^* and k^* for their adjoints. Then $A^\perp \cap B$ is the kernel of $j^* k$, so it vanishes iff $j^* k$ is iso, iff the adjoint map $k^* j$ is iso, iff $A \cap B^\perp = 0$. Thus (a)$\Leftrightarrow$(b).

Next note that $\xi = (1 - \alpha + \beta)(1 + \alpha - \beta)$ is invertible iff the two factors are invertible. On $A \cap B^\perp$ we have $\alpha = 1$ and $\beta = 0$ so $1 - \alpha + \beta = 0$. Thus if ξ is invertible we have $A \cap B^\perp = 0$ and similarly $A^\perp \cap B = 0$. Conversely, suppose that $1 - \alpha + \beta$ is not invertible, so $(1 - \alpha)u = -\beta u = v$ say for some $u \in U \setminus \{0\}$. Clearly $v \in \operatorname{image}(1 - \alpha) \cap \operatorname{image}(\beta) = A^\perp \cap B$. If $v \neq 0$ then $A^\perp \cap B \neq 0$. On the other hand, if $v = 0$ then $u \in \ker(1 - \alpha) \cap \ker(\beta) = A \cap B^\perp$ so $A \cap B^\perp \neq 0$ by the first paragraph. A similar argument works if $1 + \alpha - \beta$ is not invertible. This completes the proof. $\square$

Proof of proposition 1.10. Suppose that $(A, B) \in N(\mathcal{U})$. Choose $U \in \mathcal{U}$ with $A + B \leq U$. Let $\alpha, \beta \colon U \to U$ be the orthogonal projections onto A and B, and put

$$\delta = (\alpha - \beta)^2 = \alpha + \beta - \alpha\beta - \beta\alpha.$$

As $\|\delta\| < 1$ we can define $\chi = \sqrt{1 - \delta}$ by the usual power series expansion. This is clearly self-adjoint and invertible. We also write $\gamma = \alpha\beta - \beta\alpha$, which clearly has $\gamma^* = -\gamma$. We see by direct calculation that α and β commute with δ and thus with χ and χ^{-1}. It follows that any expression in α and β commutes with δ, χ and χ^{-1}; in particular this is true of γ. We also check directly that $\gamma^2 = -\delta(1 - \delta) = -\delta\chi^2$. Now put $\rho = \chi - \gamma\chi^{-1}$. We have

$$\rho\rho^* = (\chi - \gamma\chi^{-1})(\chi + \gamma\chi^{-1}) = \chi^2 - \gamma^2\chi^{-2} = (1 - \delta) + \delta = 1,$$

so $\rho \in O(U)$. We also calculate that

$$\begin{aligned} \rho\alpha\chi = \rho\chi\alpha &= (1 - \delta - \gamma)\alpha \\ &= (1 - \alpha - \beta + \alpha\beta + \beta\alpha - \alpha\beta + \beta\alpha)\alpha \\ &= \beta\alpha, \end{aligned}$$

so $(1 - \beta)\rho\alpha = (1 - \beta)\beta\alpha\chi^{-1} = 0$. As $A = \operatorname{image}(\alpha)$ and $B = \ker(1 - \beta)$ we see that $\rho(A) \leq B$, and thus $\rho(A) = B$ by dimension count.

Now define $\rho(A, B) \colon \mathcal{U} \to \mathcal{U}$ to be ρ on U and 1 on $U^\perp$. This is easily seen to be natural and independent of the choice of U, and to depend continuously on A and B.

If we exchange A and B then δ and χ are unchanged and γ becomes $-\gamma$, so ρ becomes $\chi + \gamma\chi^{-1}$ which we have seen is inverse to the original ρ. Thus $\rho(B, A) = \rho(A, B)^{-1}$ as claimed. $\square$

Remark 1.12. A more abstract point of view is as follows. Let M be a manifold with a transitive action of a compact Lie group G, and suppose we have chosen an invariant inner product on G. Given sufficiently close points $x, y \in M$, there will be a unique element v close to zero in the Lie algebra of G such that $\exp(v).x = y$ and v is orthogonal to the Lie algebra of the stabiliser of x. We can then define $\rho(x, y) = \exp(v)$ to get a map from a neighbourhood of the diagonal in $M \times M$ to G, with the property that $\rho(x, y).x = y$. Note also that the map $t \mapsto \exp(tv).x$ gives a geodesic from x to y.

In particular, this applies when $G = O(W)$ and M is the Grassmannian of n-planes in W and the inner product on $LG \leq \operatorname{End}(W)$ is given by $\langle \alpha, \beta \rangle = \operatorname{trace}(\alpha\beta)$. One can check that the resulting map ρ in this case is the same as that defined more explicitly above. We will record some of the formulae necessary to check this, using the notation of the above proof. Let $\sqrt{\delta}$ be the unique positive self-adjoint square root of δ and let $\phi = \frac{1}{2}\sin^{-1}(\sqrt{\delta})$ be defined by the obvious power series. Then $\delta = \sin^2(2\phi)$ and $\chi = \cos(2\phi)$. If ϕ is invertible we can define $i = 2\gamma/\sin(4\phi)$ and we find that $i^2 = -1$ and $-i\alpha i = 1 - \alpha$, so i gives a complex structure on W. We then find that $2i\phi$ is orthogonal to the Lie algebra of $O(V) \times O(V^\perp)$ (because it carries

V to $V^\perp$ and *vice versa*) and that $\exp(2i\phi) = \rho(U, V)$. We also find that the orthogonal projection onto the space $\exp(2i\phi t).U$ is given by

$$\pi_t = \frac{\sin(4\phi(1-t))}{\sin(4\phi)}\alpha + \frac{\sin(4\phi t)}{\sin(4\phi)}\beta - \frac{\sin(2\phi t)\sin(2\phi(1-t)}{\cos(2\phi)}.$$

In particular, one checks that

$$\pi_{1/2} = (2\chi)^{-1}(\alpha + \beta + \chi - 1).$$

If ϕ is not invertible we merely remark that $f(z) = 4z/\sin(4z)$ is a power series in z with radius of convergence π so we can still interpret $f(\phi)\gamma$ as an endomorphism of W, which is just $2i\phi$ in the invertible case. We again have $\rho(U, V) = \exp(f(\phi)\gamma)$ and everything goes through much as before.

It is an illuminating exercise to work through all the formulae when $W = \mathbb{R}^2$ and U and V are the lines at angles $+\theta$ and $-\theta$ to the x-axis.

We now digress slightly to prove another useful result using the same ideas.

Proposition 1.13. *Fix integers a, b, n with $0 \leq a, b \leq n$, and put $d = \min(a, b, n-a, n-b)$. Let $\mathcal{C}$ be the category of triples (A, B, V), where V is an n-dimensional complex universe, and A and B are subspaces of dimensions a and b. Then $\pi_0\mathcal{C}$ is naturally identified with the simplex Δ_d.*

Proof. Firstly, we identify Δ_d with

$$\{x \in I^d \mid 0 \leq x_1 \leq \cdots \leq x_d \leq 1\} \subset I^d,$$

and we note that the resulting map

$$\Delta_d \to I^d \to I^d/\Sigma_d = \mathrm{SP}^d(I)$$

is a homeomorphism. We can regard $\mathrm{SP}^d(I)$ as a subset of $\mathbb{Z}[I]$, and thus of $\mathbb{Q}[I]$ (the free rational vector space on the set I). We will define an invariant $\phi\colon \pi_0\mathcal{C} \to \mathbb{Z}[I]$ and show that it actually gives a bijection $\pi_0\mathcal{C} \to \mathrm{SP}^d(I)$.

As before, we let α and β be the orthogonal projectors on to A and B, and we let $\chi = \chi(\alpha, \beta)$ be the unique positive self-adjoint square root of $(\alpha - \beta)^2$. We note that χ commutes with α and β, and has all eigenvalues in I. It is also clear that

$$\chi(\alpha, \beta) = \chi(\beta, \alpha) = \chi(1-\alpha, 1-\beta) = \chi(1-\beta, 1-\alpha).$$

We put

$$\xi(\alpha, \beta) = \chi(\alpha, 1-\beta) = \chi(1-\alpha, \beta) = \chi(\beta, 1-\alpha) = \chi(1-\beta, \alpha).$$

Observe that

$$\begin{aligned}
\chi^2 + \xi^2 &= (\alpha - \beta)^2 + (\alpha + \beta - 1)^2 \\
&= \alpha^2 - \alpha\beta - \beta\alpha + \beta^2 + \alpha^2 + \beta^2 + 1 + \alpha\beta + \beta\alpha - 2\alpha - 2\beta \\
&= \alpha - \alpha\beta - \beta\alpha + \beta + \alpha + \beta + 1 + \alpha\beta + \beta\alpha - 2\alpha - 2\beta \\
&= 1.
\end{aligned}$$

The n-tuple of eigenvalues of χ (repeated with appropriate multiplicities) defines a poin $\mathrm{spec}(\chi) \in \mathrm{SP}^n(I) \subset \mathbb{Q}[I]$. We define

$$\phi[A, B, V] = (\mathrm{spec}(\chi) - |a-b|.[1] - |a+b-n|.[0])/2 \in \mathbb{Q}[I].$$

To analyse this in more detail, we let $\nu(s)$ denote the multiplicity of s as an eigenvalue of ξ (which is the same as the multiplicity of $t = \sqrt{1-s^2}$ as an eigenvalue of χ). Put $V_s = \ker(\xi - s)$, so $V = \bigoplus_s V_s$ and $\dim(V_s) = \nu(s)$. As α and β commute with ξ we see that they preserve this splitting, so $A = \bigoplus_s A_s$ and $B = \bigoplus_s B_s$, where $A_s = A \cap V_s$ and $B_s = B \cap V_s$. Let α_s and β_s be the restrictions of α and β to V_s.

We next claim that $\nu(1) \geq |a-b|$. It will suffice to prove this when $a \geq b$. We then have $\dim(A \cap B^\perp) \geq a - b = |a-b|$, and on $A \cap B^\perp$ we have $\alpha = 1$ and $\beta = 0$ so $\chi = \sqrt{(1-0)^2} = 1$ as required. By applying the same logic to the triple $(A, B^\perp, V)$ we see that $\nu(0) \geq |a+b-n|$. It follows that $2\phi[A, B, V] \in \mathbb{N}[I] \subset \mathbb{Q}[I]$.

Now suppose that $0 < s < 1$ and put $V_s^+ = \ker(\alpha - \beta - s)$ and $V_s^- = \ker(\alpha - \beta + s)$. As $(\alpha - \beta)^2 = \chi^2$ acts as s^2 on V_s we see that $V_s = V_s^+ \oplus V_s^-$. Put $t = \sqrt{1 - s^2}$. We claim that relative to the above splitting we have

$$\alpha_s = \frac{1}{2}\begin{pmatrix} 1+s & t\mu \\ t\mu^{-1} & 1-s \end{pmatrix} \qquad \beta_s = \frac{1}{2}\begin{pmatrix} 1-s & t\mu \\ t\mu^{-1} & 1+s \end{pmatrix}$$

for some isometric isomorphism $\mu\colon V_s^- \to V_s^+$. Indeed, we certainly have $\alpha_s - \beta_s = \left(\begin{smallmatrix} s & 0 \\ 0 & -s \end{smallmatrix}\right)$, and it follows easily from this and self-adjointness that

$$\alpha_s = \frac{1}{2}\begin{pmatrix} \sigma+s & t\mu \\ t\mu^* & \rho-s \end{pmatrix} \qquad \beta_s = \frac{1}{2}\begin{pmatrix} \sigma-s & t\mu \\ t\mu^* & \rho+s \end{pmatrix}$$

for some $\sigma\colon V_s^+ \to V_s^+$ and $\rho\colon V_s^- \to V_s^-$ and $\mu\colon V_s^- \to V_s^+$, with σ and ρ self-adjoint. We must show that σ and ρ are identity maps and that μ is an isometric isomorphism. **I have this claim on a piece of paper but without a proof. It probably follows from the idempotence of α and β but I don't see a good way to organise the calculation at the moment. Given this, we see that $\nu(s) = 2\dim(V_s^+)$ is even for $0 < s < 1$. We still need a separate argument to show that $\nu(0) - |a + b - n|$ and $\nu(1) - |a - b|$ are even.** ☐

Remark 1.14. One can identify $\pi_0\mathcal{C}$ with $(G_a(\mathbb{C}^n) \times G_b(\mathbb{C}^n))/U(n)$, which gives a topology. Presumably $\pi_0\mathcal{C}$ is actually homeomorphic to Δ_d, but I have not attempted to check any details.

1.2. The topology of S^∞.

We next prove the following slightly surprising result.

Proposition 1.15. *S^∞ is homeomorphic to $\mathbb{R}^\infty$.*

To see this, let B_k be the closed ball of radius k in $\mathbb{R}^k$. Then B_k is compact and is contained in the interior of B_{k+1}, and it is easy to check that $\mathbb{R}^\infty = \varinjlim_k B_k$.

Similarly, put $S_k = \{(x_0, \dots, x_k) \in S^k \mid x_k \leq 1/2\}$. One checks that S_k is homeomorphic to B_k and is contained in the interior of S_{k+1}, and that $S^\infty = \varinjlim_k S_k$. We will also need to know that the pair (S_{k+1}, S_k) is homeomorphic to (B_{k+1}, B_k), or in other words that there is a homeomorphism $f\colon S_{k+1} \to B_{k+1}$ such that $f(S_k) = B_k$. One can either construct such an f directly, or appeal to a general result to be mentioned later. Given this, we need only prove the following result.

Proposition 1.16. *Let X be a space, and suppose that X is the colimit of a sequence of subspaces X_k such that each pair (X_{k+1}, X_k) is homeomorphic to (B_{k+1}, B_k) for all k. Then $X \simeq \mathbb{R}^\infty$.*

The proof depends on the following lemma.

Lemma 1.17. *Any homeomorphism $f\colon B_k \to B_k$ can be extended to give a homeomorphism $E(f)\colon B_{k+1} \to B_{k+1}$.*

Proof. Note that f automatically preserves the boundary ∂B_k, so it is an automorphism of the pair $(B_k, \partial B_k)$. Given any pair of spaces (Y, Z), let $E(Y, Z)$ be the unreduced suspension of $(Y \times \{0\}) \cup (Z \times [0, 1])$, which is clearly a functor of (Y, Z). It is easy to identify $E(B_k, \partial B_k)$ with B_{k+1}, and this gives the result.

More explicitly, a point $u \in B_{k+1}$ can be written as $u = r\cos(\theta)x + (k+1)\sin(\theta)e_k$ for some $r \in [0, k+1]$, $\theta \in [-\pi, \pi]$ and $x \in S^{k-1}$. We then have

$$E(f)(u) = \begin{cases} \cos(\theta)f(rx) + (k+1)\sin(\theta)e_k & \text{if } 0 \leq r \leq k \\ r\cos(\theta)f(kx)/k + (k+1)\sin(\theta)e_k & \text{if } k \leq r \leq k+1. \end{cases}$$

☐

Proof of Proposition 1.16. By assumption, we can choose homeomorphisms $f_k\colon B_k \to X_k$ such that the restriction of f_k gives a homeomorphism $B_{k-1} \to X_{k-1}$, which we denote by f'_k. Now define homeomorphisms $g_k\colon B_k \to X_k$ by $g_1 = f_1$ and $g_k = f_k \circ E((f'_k)^{-1} \circ g_{k-1})$ for $k > 1$. One checks that $g_k|_{B_{k-1}} = g_{k-1}$ for all k, so we can pass to colimits to get a homeomorphism $\mathbb{R}^\infty \to X$. ☐

We can use the same technique to show that many other spaces are homeomorphic to $\mathbb{R}^\infty$.

Proposition 1.18. *Let S be any finite subset of $\mathbb{R}^\infty$. Then $\mathbb{R}^\infty \setminus S$ is homeomorphic to $\mathbb{R}^\infty$.*

Proof. First, the homeomorphism type of $\mathbb{R}^\infty \setminus S$ depends only on the integer $n = |S|$; this is well-known for subsets of $\mathbb{R}^k$ when $k < \infty$, and the case $k = \infty$ follows immediately. We may thus assume that $S = \{me_0 \mid m \in \{0, \dots, n-1\}\}$. Now put

$$\begin{aligned} F_k &= [-n-k, n+k]^k \subset \mathbb{R}^k \\ G_k &= S + (-1/2^k, 1/2^k)^{k-1} \times (-1/2^k, \infty) \subset \mathbb{R}^k \\ X_k &= F_k \setminus G_k. \end{aligned}$$

Then X_k is a k-ball with n separate holes drilled part way into it, so it is again a k-ball. One can also check directly that X_k is contained in the interior of X_{k+1}. If K is a compact subspace of $\mathbb{R}^\infty \setminus S$ then there exists k such that $\mathrm{dist}(K, S) > 1/2^k$ and $\mathrm{dist}(K, 0) < k$ and $K \subset \mathbb{R}^k$, and it follows that $K \subset X_{k+1}$. Using this, we see that $\mathbb{R}^\infty \setminus S = \varinjlim_k X_k$.

We still need to check that $(X_{k+1}, X_k) \simeq (B_{k+1}, B_k)$, but I think this should be formal. □

Proposition 1.19. *Let S be a finite subset of $\mathbb{R}^\infty$, and let $F_n(\mathbb{R}^\infty \setminus S)$ denote the configuration space of n-tuples of distinct points in $\mathbb{R}^\infty \setminus S$. Then $F_n(\mathbb{R}^\infty \setminus S)$ is homeomorphic to $\mathbb{R}^\infty$.*

Proof. Define $\pi \colon F_n(\mathbb{R}^\infty \setminus S) \to \mathbb{R}^\infty \setminus S$ by $\pi(x_1, \dots, x_n) = x_n$. This is well-known to be a fibre bundle projection, with fibres of the form $F_{n-1}(\mathbb{R}^\infty \setminus (S \cup \{x_n\}))$. By induction, we may assume that these fibres are homeomorphic to $\mathbb{R}^\infty$. We have seen that the base is homeomorphic to $\mathbb{R}^\infty$, which is a contractible CW complex. It follows that the bundle is trivialisable, so $F_n(\mathbb{R}^\infty \setminus S) \simeq \mathbb{R}^\infty \times \mathbb{R}^\infty \simeq \mathbb{R}^\infty$. □

Proposition 1.20. *The Stiefel manifold $V_n\mathbb{R}^\infty$ is homeomorphic to $\mathbb{R}^\infty$.*

Proof. Apply the same argument to the well-known fibre bundle $V_{n-1}\mathbb{R}^\infty \to V_n\mathbb{R}^\infty \to S^\infty$. (In the next section we will be more precise about how to trivialise this bundle, but for the moment we need only note that it is trivialisable.) □

Remark 1.21. The space $F_n\mathbb{R}^\infty$ is one of the standard models for $E\Sigma_n$, and $V_n\mathbb{R}^\infty$ is one of the standard models for $EO(n)$.

1.3. **The topology of $\mathcal{L}(1)$.** We now prove that $\mathcal{L}(1)$ is homeomorphic to a product of countably many copies of S^∞. This is needed to clear up a minor technical point later, and in any case it seems like an illuminating fact. We also prove that all orbits of the evident left action of the orthogonal group O on $\mathcal{L}(1)$ are dense, which will have a number of applications.

Let $F \colon \mathbb{R} \oplus \mathbb{R}^\infty \to \mathbb{R}^\infty$ be the obvious isomorphism, and define $T \colon \mathcal{L}(1) \to \mathcal{L}(1)$ by

$$T(g) = F \circ (1 \oplus g) \circ F^{-1}.$$

This clearly gives a homeomorphism of $\mathcal{L}(1)$ with $\{h \in \mathcal{L}(1) \mid h(e_0) = e_0\}$, and more generally T^n identifies $\mathcal{L}(1)$ with $\{h \in \mathcal{L}(1) \mid h|_{\mathbb{R}^n} = 1\}$.

We also choose a section σ of the evaluation map $\epsilon \colon O(\infty) \to S^\infty$ as in Lemma 1.8 and define

$$\begin{array}{ll} \sigma_n \colon \prod_{k<n} S^\infty \to O(\infty) & \qquad \sigma_n(v_0, \dots, v_{n-1}) = \sigma(v_0) \circ \dots \circ T^{n-1}\sigma(v_{n-1}) \\ \alpha_n \colon \prod_{k<n} S^\infty \to V_n\mathbb{R}^\infty & \qquad \alpha_n(\underline{v}) = \sigma_n(\underline{v})|_{\mathbb{R}^n}. \end{array}$$

Lemma 1.22. *The map α_n is a homeomorphism.*

Proof. The inverse β_n is defined recursively as follows. Suppose we have defined β_n. We identify $V_n\mathbb{R}^\infty = \mathcal{L}^+(\mathbb{R}^n, \mathbb{R}^\infty)$ with the set of n-frames in $\mathbb{R}^\infty$ in the obvious way. Let $\underline{u} = (u_0, \dots, u_{n-1})$ be an n-frame, so that $g = \sigma_n\beta_n(\underline{u}) \in O$ and $g(e_i) = u_i$ for $i < n$, so g induces an isomorphism $\{e_0, \dots, e_{n-1}\}^\perp \simeq \{u_0, \dots, u_{n-1}\}^\perp$. Now suppose that we have a vector u_n giving us an $(n+1)$-frame. It follows that there is a unique element $v \in \{e_0, \dots, e_{n-1}\}^\perp$ such that $gv = u_n$. We define

$$\beta_{n+1}(u_0, \dots, u_n)) = (\beta_n(u_0, \dots, u_{n-1}), L^{n-1}v) \in \left(\prod_{k<n} S^\infty\right) \times S^\infty = \prod_{k<n+1} S^\infty.$$

It is not hard to check that this is continuous and inverse to α_n. □

Corollary 1.23. *There are commutative diagrams as follows, in which the vertical maps are homeomorphisms (and therefore the top horizontal maps are trivialisable fibre bundles).*

$$\begin{tikzcd}
\mathcal{L}(1) \arrow[r, two heads] \arrow[d, "\beta"'] & \mathcal{L}^+(\mathbb{R}^n, \mathbb{R}^\infty) \arrow[r, two heads] \arrow[d, "\beta_n"'] & \mathcal{L}^+(\mathbb{R}^{n-1}, \mathbb{R}^\infty) \arrow[d, "\beta_{n-1}"] \\
\prod_k S^\infty \arrow[r, two heads, "\pi"'] & \prod_{k<n} S^\infty \arrow[r, two heads, "\pi"'] & \prod_{k<n-1} S^\infty
\end{tikzcd}$$

Proof. This follows immediately from the lemma and the fact that $\mathcal{L}(1) = \varprojlim_n \mathcal{L}^+(\mathbb{R}^n, \mathbb{R}^\infty)$. ☐

Corollary 1.24. *Given universes $\mathcal{U}$ and $\mathcal{V}$ (where $\mathcal{V}$ is infinite) and a subspace $U \ll \mathcal{U}$, the restriction map*

$$\mathcal{L}^+(\mathcal{U}, \mathcal{V}) \to \mathcal{L}^+(U, \mathcal{V})$$

is a trivialisable fibre bundle.

Proof. As all universes of the same dimension are isomorphic, we may assume that $\mathcal{V} = \mathbb{R}^\infty$ and $U = \mathbb{R}^n$ for some $n \in \mathbb{N}$ and $\mathcal{U} \ominus U = \mathbb{R}^m$ for some $m \in \mathbb{N} \cup \{\infty\}$. In this case the claim is clear from the previous corollary. ☐

We next define a map $\lambda \colon \mathcal{L}(1) \to \mathcal{L}(1)$ as follows. Observe that $g = \sigma(fe_0)^{-1} \circ f \in \mathcal{L}(1)$ and $g(e_0) = e_0$, so that $g = Th$ for a unique $h \in \mathcal{L}(1)$; we set $\lambda(f) = h$. We also define $\mu \colon \prod_k S^\infty \to \prod_k S^\infty$ by $\mu(v_0, v_1, \ldots) = (v_1, v_2, \ldots)$. The following lemma follows by inspection of definitions:

Lemma 1.25. *There is a commutative diagram*

$$\begin{tikzcd}
\prod_k S^\infty \arrow[r, "\alpha", "\sim"'] \arrow[d, two heads, "\mu"'] & \mathcal{L}(1) \arrow[d, two heads, "\lambda"] \\
\prod_k S^\infty \arrow[r, "\sim", "\alpha"'] & \mathcal{L}(1)
\end{tikzcd}$$

Moreover, if $f = \alpha(\underline{v})$ then $f = \sigma_n(\alpha) \circ T^n \lambda^n f$, so f lies in the same orbit as $T^n \lambda^n f$ under the left action of O. ☐

The next result would be easy if we were using the ordinary product topology, but requires a little more work in the compactly generated category.

Lemma 1.26. *Let Y be a nonempty subset of $\prod_k S^\infty$, with the property that for any $n \geq 0$, $\mu^n \underline{x} \in \mu^n Y$ implies $\underline{x} \in Y$. Then Y is dense in $\prod_k S^\infty$.*

Proof. Choose $\underline{y} \in Y$. Consider an arbitrary element $\underline{x} \in \prod_k S^\infty$, and set $K = \prod_k \{x_k, y_k\}$. This is compact in the ordinary product topology, so it is compact in the CG topology and its topology as a subspace of the CG product is the ordinary one. Define $\underline{x}_N \in K$ by $x_{N,k} = x_k$ for $k < n$ and $x_{N,k} = y_k$ for $k \geq n$. Clearly $\mu^n \underline{x}_N = \mu^n \underline{y} \in \mu^n Y$, so $\underline{x}_N \in Y \cap K \subseteq \overline{Y} \cap K$. Clearly $\underline{x}_N$ converges to $\underline{x}$, and $\overline{Y} \cap K$ is closed, so $\underline{x} \in \overline{Y}$ as required. ☐

Proposition 1.27. *Every O-orbit Of in $\mathcal{L}(1)$ is dense. It follows that $\mathcal{L}(1)f$ is always dense, and that*

$$\mathcal{L}^0(1) = \{g \in \mathcal{L}(1) \mid \dim g(\mathbb{R}^\infty)^\perp = \infty\}$$

is dense in $\mathcal{L}(1)$.

Proof. Write $Y = \alpha^{-1}(Of) \subseteq \prod_k S^\infty$. We claim that Lemma 1.26 applies to this Y. Clearly $Y \neq \emptyset$, so suppose that $\mu^n \underline{x} = \mu^n \underline{y}$ with $\underline{y} \in Y$. It follows easily from Lemma 1.25 that $\lambda^n \alpha(\underline{x}) = \lambda^n \alpha(\underline{y}) = \lambda^n(gf)$ for some $g \in O$, and thus that $\alpha(\underline{x})$ lies in the same orbit under O as gf or f, so that $\underline{x} \in Y$ as required. Thus Lemma 1.26 shows that Y is dense in $\prod_k S^\infty$; as α is a homeomorphism we conclude that Of is dense in $\mathcal{L}(1)$. It follows immediately that $\mathcal{L}(1)f$ is dense. Now consider the special case where $f(e_i) = e_{2i}$ for all i. It is easy to see that $f \in \mathcal{L}^0(1)$ and thus $\mathcal{L}(1)f \subseteq \mathcal{L}^0(1)$, so $\mathcal{L}^0(1)$ is dense. ☐

1.4. **The monoid structure of $\mathcal{L}(1)$.** We now prove various facts about the structure of $\mathcal{L}(1)$ as a monoid.

Lemma 1.28. *Let $\mathcal{L}'$ be obtained from the category $\mathcal{L}$ of infinite universes by inverting all morphisms. Then $\mathcal{L}'(\mathcal{U},\mathcal{V})$ has precisely one element for all $\mathcal{U}$ and $\mathcal{V}$.*

Proof. We know that $\mathcal{L}(\mathcal{U},\mathcal{V}) \neq \emptyset$ so it suffices to show that any two maps $f, g\colon \mathcal{U} \to \mathcal{V}$ become equal in $\mathcal{L}'$. We have a map $i_0\colon \mathcal{U} \to \mathcal{U} \times \mathcal{U}$ with $(1 \times f)i_0 = (1 \times g)i_0$ and i_0 is invertible in $\mathcal{L}'$ so $1 \times f = 1 \times g$ there. We next have a commutative diagram in $\mathcal{L}'$ as follows.

$$\begin{tikzcd}
\mathcal{U} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "i_1"'] & \mathcal{V} \arrow[d, "i_1"] \\
\mathcal{U} \times \mathcal{U} \arrow[r, shift left, "1\times f"] \arrow[r, shift right, "1\times g"'] & \mathcal{U} \times \mathcal{V}
\end{tikzcd}$$

As $1 \times f = 1 \times g$ and the maps i_1 are invertible we have $f = g$. ☐

Corollary 1.29. *The group completion of the monoid $\mathcal{L}(1)$ is trivial.*

Proof. This is immediate from the lemma and the fact that $\mathcal{L}$ is equivalent to the category with one object $\mathbb{R}^\infty$ whose endomorphisms are $\mathcal{L}(1)$. ☐

Definition 1.30. Let M be a topological monoid and let X and Y be spaces with a right and a left action of M, respectively. We then write $X \times_M Y$ for the coequaliser of the maps $d_0, d_1\colon X \times M \times Y \to X \times Y$, where

$$\begin{aligned} d_0(x, m, y) &= (xm, y) \\ d_1(x, m, y) &= (x, my). \end{aligned}$$

We now give a proof of a result of Hopkins, which is the key point in the proof that the smash product of $\mathbb{L}$-spectra is associative. While it is not significantly different from that given in [1], it is a nice illustration of the point of view we will take in a number of proofs later in this paper.

Proposition 1.31. *Let $\mathcal{U}, \mathcal{V}, \mathcal{W}, \mathcal{X}$ and $\mathcal{Y}$ be infinite universes, so that $\mathcal{L}(\mathcal{V}) \times \mathcal{L}(\mathcal{X})$ acts on the right on $\mathcal{L}(\mathcal{V} \oplus \mathcal{X}, \mathcal{Y})$ and on the left on $\mathcal{L}(\mathcal{U},\mathcal{V}) \times \mathcal{L}(\mathcal{W},\mathcal{X})$. Then*

$$\mathcal{L}(\mathcal{V} \oplus \mathcal{X}, \mathcal{Y}) \times_{\mathcal{L}(\mathcal{V}) \times \mathcal{L}(\mathcal{X})} (\mathcal{L}(\mathcal{U},\mathcal{V}) \times \mathcal{L}(\mathcal{W},\mathcal{X})) = \mathcal{L}(\mathcal{U} \oplus \mathcal{W}, \mathcal{Y}).$$

Proof. As all infinite universes are isomorphic, we may assume that $\mathcal{U} = \mathcal{V}$ and $\mathcal{W} = \mathcal{X}$. The proposition then becomes an instance of the easy fact that $X \times_M M = X$. ☐

Corollary 1.32 (Hopkins)**.** *If $k, l > 0$ we have $\mathcal{L}(2) \times_{\mathcal{L}(1)^2} (\mathcal{L}(k) \times \mathcal{L}(l)) = \mathcal{L}(k + l)$.*

Proof. Take $\mathcal{U} = (\mathbb{R}^\infty)^k$, $\mathcal{W} = (\mathbb{R}^\infty)^l$, $\mathcal{V} = \mathcal{X} = \mathcal{Y} = \mathbb{R}^\infty$. ☐

Definition 1.33. If X is a space with an action of a monoid M, we write X/M for the coequaliser of the action map and the projection map $M \times X \to X$. This is the quotient of X by the smallest closed equivalence relation such that $x \sim mx$ for all $x \in X$ and $m \in M$. We say that a space X is *quasi-transitive* if X/M is a one-element set.

Observe that if G is a group then

$$G \simeq (G \times G)/G \not\simeq (G/G) \times (G/G) = 1.$$

Thus, the functor $X \mapsto X/G$ fails badly to preserve products. This makes the following result rather curious.

Proposition 1.34. *If X and Y are spaces with a right action of $\mathcal{L}(\mathcal{U})$ (where $\mathcal{U}$ is an infinite universe) then*

$$(X \times Y)/\mathcal{L}(\mathcal{U}) = (X/\mathcal{L}(\mathcal{U})) \times (Y/\mathcal{L}(\mathcal{U})).$$

The proof will follow after two lemmas.

Definition 1.35. If $\mathcal{U}$ is a subuniverse of $\mathcal{V}$, we write $\mathcal{U}^\perp = \{v \in \mathcal{V} \mid \langle v, \mathcal{U}\rangle = \{0\}\}$. We say that $\mathcal{U}$ is *complemented* if $\mathcal{V} = \mathcal{U} \oplus \mathcal{U}^\perp$. It is not hard to see that any finite subuniverse is complemented (choose an orthonormal basis $\{u_i\}$ and observe that $v - \sum_i \langle v, u_i\rangle u_i$ lies in $\mathcal{U}^\perp$).

Lemma 1.36. *Given two maps $f, g\colon \mathcal{U} \to \mathcal{V}$ of infinite universes, there is an infinite subuniverse $\mathcal{W} \leq \mathcal{U}$ such that $f\mathcal{W}$, $g\mathcal{W}$ and $f\mathcal{W} + g\mathcal{W}$ are complemented in $\mathcal{V}$ and the universe $(f\mathcal{W}+g\mathcal{W})^\perp = (f\mathcal{W})^\perp \cap (g\mathcal{W})^\perp$ is infinite.*

Proof. We may assume that $\mathcal{V} = \mathbb{R}^\infty$, and let $\{e_i \mid i \geq 0\}$ be the standard basis. We choose an orthonormal sequence $w_0, w_1, \dots$ in $\mathcal{U}$ as follows. Let w_0 be any unit vector. Given $w_0, \dots, w_{n-1}$, let $d = d_{n-1}$ be the least integer such that $f(w_j)$ and $g(w_j)$ both lie in $\mathbb{R}^{d_{n-1}} < \mathbb{R}^\infty$ for all $j < n$ and $d_{n-1} > d_{n-2}$. Let $\mathcal{U}_n$ be the set of vectors $u \in \mathcal{U}$ such that

(a) $\langle u, w_i \rangle = 0$ for $i < n$.
(b) $\langle f(u), e_i \rangle = \langle g(u), e_i \rangle = 0$ for $i \leq d$.

This has finite codimension in $\mathcal{U}$ and thus is nonzero. Let w_n be any unit vector in $\mathcal{U}_n$.

We can now set $\mathcal{W} = \operatorname{span}\{w_i \mid i \geq 0\}$. If we write $\mathcal{V}_i = \operatorname{span}\{e_j \mid d_{i-1} \leq j < d_i\}$ then $\mathcal{V}_i$ is finite and $\mathcal{V} = \bigoplus_i \mathcal{V}_i$. It follows easily that if a subuniverse $\mathcal{X} \leq \mathcal{V}$ has the form $\mathcal{X} = \bigoplus_i \mathcal{X}_i$ with $\mathcal{X}_i \leq \mathcal{V}_i$, then $\mathcal{X}$ is complemented. Given that $fw_i \in \mathcal{V}_i$ and $f\mathcal{W} = \operatorname{span}\{fw_i \mid i \geq 0\}$, it follows that $f\mathcal{W}$ is complemented. Similarly, $g\mathcal{W}$ and $f\mathcal{W}+g\mathcal{W}$ are complemented. We have $e_{d_i} \in (f\mathcal{W})^\perp \cap (g\mathcal{W})^\perp$ for all i, so $(f\mathcal{W})^\perp \cap (g\mathcal{W})^\perp$ is infinite. ☐

Lemma 1.37. *If we let $\mathcal{L}(\mathcal{U})$ act on the right on $\mathcal{L}(\mathcal{U})^2$ by $(f, g)h = (fh, gh)$ then the action is quasi-transitive.*

Proof. Suppose that $f, g\colon \mathcal{U} \to \mathcal{U}$. Let $\sim$ be the equivalence relation on $\mathcal{L}(\mathcal{U})^2$ generated by the action of $\mathcal{L}(\mathcal{U})$, so we need to prove that $(f, g) \sim (1, 1)$. Choose a subuniverse $\mathcal{W} \leq \mathcal{U}$ as in Lemma 1.36, and let $j\colon \mathcal{W} \to \mathcal{U}$ be the inclusion. As all infinite universes are isomorphic, we can choose an isomorphism $h\colon \mathcal{U} \to \mathcal{W}$. We then have $(f, g) \sim (fjh, gjh)$. After replacing f by fjh and g by gjh we may assume that $f\mathcal{U}$, $g\mathcal{U}$ and $f\mathcal{U} + g\mathcal{U}$ are complemented and that $\mathcal{V} = (f\mathcal{U} + g\mathcal{U})^\perp < \mathcal{U}$ is an infinite universe. We define a map $u\colon \mathcal{U} \to \mathcal{U}$ as follows. We may assume that $\dim((f\mathcal{U} + g\mathcal{U}) \ominus f\mathcal{U}) \leq \dim((f\mathcal{U} + g\mathcal{U}) \ominus g\mathcal{U})$ (otherwise exchange f and g). We start with the map $gf^{-1}\colon f\mathcal{U} \to g\mathcal{U}$. We take the direct sum with an arbitrary isometric embedding $(f\mathcal{U}+g\mathcal{U}) \ominus f\mathcal{U} \to (f\mathcal{U}+g\mathcal{U}) \ominus g\mathcal{U}$ to get an endomorphism of $f\mathcal{U}+g\mathcal{U}$. We then take the direct sum with the identity map of $(f\mathcal{U} + g\mathcal{U})^\perp$ to get an endomorphism u of $\mathcal{U}$. Note that $uf = g$. Thus $(f, g) = (1f, uf) \sim (1, u)$. Choose an isomorphism $k\colon \mathcal{U} \to (f\mathcal{U} + g\mathcal{U})^\perp$ and let $i\colon (f\mathcal{U} + g\mathcal{U})^\perp \to \mathcal{U}$ be the inclusion. Then $ui = i$ and thus $(1, u) \sim (ik, uik) = (ik, ik) \sim (1, 1)$. ☐

Proof of Proposition 1.34. Write $\sim$ for the equivalence relation on $X \times Y$ generated by the action of $\mathcal{L}(\mathcal{U})$. Given $(x, y) \in X \times Y$ we define $S = \{(u, v) \in \mathcal{L}(\mathcal{U})^2 \mid (xu, yv) \sim (x, y)\}$. Clearly, for any $w \in \mathcal{L}(\mathcal{U})$ we have $(xuw, yvw) \sim (xu, yv)$, so $(uw, vw) \in S$ if and only if $(u, v) \in S$. It follows from Lemma 1.37 that $S = \mathcal{L}(\mathcal{U})^2$. This implies that $(X \times Y)/\mathcal{L}(\mathcal{U}) = (X \times Y)/\mathcal{L}(\mathcal{U})^2 = (X/\mathcal{L}(\mathcal{U})) \times (Y/\mathcal{L}(\mathcal{U}))$, as required. ☐

Proposition 1.38. *If $\mathcal{U}$ is an infinite complex universe and $\mathcal{V}$ is an infinite real universe then $\mathcal{L}_\mathbb{C}(\mathcal{U})$ acts quasi-transitively on $\mathcal{L}_\mathbb{R}(\mathcal{U}, \mathcal{V})$.*

Proof. As all complex universes are isomorphic, it is not hard to identify $\mathcal{L}_\mathbb{R}(\mathcal{U}, \mathcal{V})/\mathcal{L}_\mathbb{C}(\mathcal{U})$ with $\varinjlim_{\mathcal{W}} \mathcal{L}_\mathbb{R}(\mathcal{W}, \mathcal{V})$, where $\mathcal{W}$ runs over the category of complex universes. This is the set of pairs $(\mathcal{W}, g)$ where $\mathcal{W}$ is a complex universe, $g\colon \mathcal{W} \to \mathcal{V}$, and $(\mathcal{W}, g)$ is identified with $(\mathcal{U}, gh)$ whenever $h\colon \mathcal{U} \to \mathcal{W}$ is a complex linear isometry.

Consider two maps $\mathcal{U} \xrightarrow{f} \mathcal{V} \xleftarrow{g} \mathcal{W}$, where $\mathcal{U}$ and $\mathcal{W}$ are complex. We need to show that $(\mathcal{U}, f) \sim (\mathcal{W}, g)$. We first claim that there are infinite complex subuniverses $\mathcal{U}'$ and $\mathcal{W}'$ such that $f\mathcal{U}'$ is orthogonal to $g\mathcal{W}'$. To prove this, we choose orthonormal sequences $u_0, u_1, \dots$ and $w_0, w_1, \dots$ as follows. Given $u_0, \dots, u_{n-1}$ and $w_0, \dots, w_{n-1}$ we define

$$\mathcal{U}_n = \operatorname{span}_\mathbb{C}\{u_j \mid j < n\}^\perp \cap f^{-1}((g \operatorname{span}_\mathbb{C}\{w_j \mid j < n\})^\perp) \cap if^{-1}((g \operatorname{span}_\mathbb{C}\{w_j \mid j < n\})^\perp).$$

This is a complex subspace of finite codimension in $\mathcal{U}$, so it is nonzero. We take u_n to be any unit vector in $\mathcal{U}_n$. We then define

$$\mathcal{W}_n = \operatorname{span}_\mathbb{C}\{w_j \mid j < n\}^\perp \cap g^{-1}((f \operatorname{span}_\mathbb{C}\{u_j \mid j \leq n\})^\perp) \cap ig^{-1}((f \operatorname{span}_\mathbb{C}\{u_j \mid j \leq n\})^\perp),$$

and let w_n be any unit vector in $\mathcal{W}_n$. It is not hard to check that $\mathcal{U}' = \operatorname{span}_\mathbb{C}\{u_j \mid j \geq 0\}$ and $\mathcal{W}' = \operatorname{span}_\mathbb{C}\{w_j \mid j \geq 0\}$ are as advertised. Let $j\colon \mathcal{U}' \to \mathcal{U}$ and $k\colon \mathcal{W}' \to \mathcal{W}$ be the inclusions. As the images of

fj and gk are orthogonal, we have a linear isometry $(fj, gk)\colon \mathcal{U}' \oplus \mathcal{W}' \to \mathcal{V}$. This fits into the following diagram:

$$\begin{tikzcd}
\mathcal{U}' \arrow[r, tail, "j"] \arrow[dd, tail] & \mathcal{U} \arrow[r, tail, "f"] & \mathcal{V} \\
 & & \mathcal{W} \arrow[u, tail, "g"] \\
\mathcal{U}' \oplus \mathcal{V}' \arrow[uurr, tail, "(fi,gj)"] & & \mathcal{W}' \arrow[u, tail, "k"] \arrow[ll, tail]
\end{tikzcd}$$

This shows that

$$(\mathcal{U}, f) \sim (\mathcal{U}', fi) \sim (\mathcal{U}' \oplus \mathcal{V}', (fi, gj)) \sim (\mathcal{W}', gj) \sim (\mathcal{W}, g).$$

□

1.5. **Complements of subuniverses.** We next explore a little further the question of when a subuniverse $\mathcal{U} \leq \mathcal{V}$ is complemented. Amongst other things, we will argue that this is a very rare phenomenon; all most all universes are uncomplemented. It will be convenient to expand the terminology slightly: we say that a linear isometry $f\colon \mathcal{U} \to \mathcal{V}$ is *complemented* if $f\mathcal{U}$ is a complemented subuniverse of $\mathcal{V}$.

Proposition 1.39. *Let $\mathcal{U}$ be a subuniverse of $\mathcal{V}$, and let $\{u_k \mid k \in \mathbb{N}\}$ be an orthonormal basis of $\mathcal{U}$. Then $\mathcal{U}$ is complemented iff for all $v \in \mathcal{V}$ we have $\langle v, u_i \rangle = 0$ for $i \gg 0$.*

Proof. If $\mathcal{U}$ is complemented we can write $v = w + u$ for some $w \in \mathcal{U}^{\perp}$ and $u \in \mathcal{U}$, say $u = \sum_i x_i u_i$ with $x_i = 0$ for $i \gg 0$. Clearly $\langle v, u_i \rangle = x_i$ and the claim follows.

Conversely, suppose that $\langle v, u_i \rangle = 0$ for $i \gg 0$ for all $v \in \mathcal{V}$. We can then define $\pi(v) = \sum_i \langle v, u_i \rangle u_i$. It is easy to check that $\pi(v) \in \mathcal{U}$ and $v - \pi(v) \in \mathcal{U}^{\perp}$, and thus that $\mathcal{V} = \mathcal{U} \oplus \mathcal{U}^{\perp}$. □

Concrete examples?

For $k, l \geq 0$ we define $a_k, b_l \in \mathbb{R}^{\infty}$ by

$$a_k = e_{2k} + \sum_{j<k} e_{2j+1}$$
$$b_l = e_{2l+3} + e_{2l+2} - e_{2l+1}.$$

By looking at the leading terms we see that the a_k's and b_l's are linearly independent, and that e_1 does not lie in their span. We put $\mathcal{A} = \mathbb{R}\{a_k \mid k \geq 0\}$ and $\mathcal{B} = \mathbb{R}\{b_l \mid l \geq 0\}$.

Proposition 1.40. *We have $\mathcal{A} = \mathcal{B}^{\perp}$ and $\mathcal{B} = \mathcal{A}^{\perp}$ but $\mathcal{A} \oplus \mathcal{B} \neq \mathbb{R}^{\infty}$.*

Proof. First, we have

$$\begin{aligned}
\langle a_k, b_l \rangle &= \langle e_{2k}, b_l \rangle + \sum_{j=0}^{k-1} \langle e_{2j+1}, b_l \rangle \\
&= I_{l+1=k} + \sum_{j=0}^{k-1} (I_{j=l+1} - I_{j=l}) \\
&= I_{l+1=k} + I_{l+1<k} - I_{l<k} = 0.
\end{aligned}$$

Thus $\langle \mathcal{A}, \mathcal{B} \rangle = 0$. Next, suppose that $x \in \mathcal{A}^{\perp}$. Put $x' = \sum_l x_{2l+2} b_l \in \mathcal{B} \leq \mathcal{A}^{\perp}$, so the element $x'' := x' - x$ also lies in $A^{\perp}$. By construction we have $x''_{2l+2} = 0$ for $l \geq 0$, and also $x''_0 = 0$ because x'' is orthogonal to $a_0 = e_0$, so x'' is orthogonal to e_{2k} for all k. As $e_{2k+1} = a_{k+1} - a_k - e_{2k+2} + e_{2k}$ we find that $x''_{2k+1} = 0$ for all k, so $x'' = 0$, so $x \in \mathcal{B}$ as required.

Now suppose instead that $y \in \mathcal{B}^{\perp}$. Put $y' = \sum_k y_{2k} a_k \in \mathcal{A}$ and $y'' = y - y' \in \mathcal{B}^{\perp}$ so $y''_{2k} = 0$. Given this, the equation $\langle y'', b_l \rangle = 0$ reduces to $y''_{2l+1} = y''_{2l+3}$, so y''_{2j+1} is independent of j. As $y''_{2j+1} = 0$ for $j \gg 0$, this implies that $y'' = 0$. Thus $y \in \mathcal{A}$, as required.

We have already observed that $e_1 \notin \mathcal{A} \oplus \mathcal{B}$, so $\mathcal{A} \oplus \mathcal{B} \neq \mathbb{R}^{\infty}$. □

2. Orthogonal diagrams

Let $\mathcal{L}_0$ be the category of finite universes, and let $\mathcal{C}$ be the category of those functors from $\mathcal{L}_0$ to the category of sets that preserve orthogonal pullbacks. We call the objects of $\mathcal{C}$ *orthogonal diagrams.*

A *basic diagram* is a functor of the form $A \mapsto \mathcal{L}_0(\mathbb{R}^n, A)/G$, where G is a subgroup of $O(n)$.

Theorem 2.1. *Any orthogonal diagram is a disjoint union of basic diagrams.*

Corollary 2.2. *An orthogonal diagram preserves* all *pullbacks in $\mathcal{L}_0$.*

The rest of this document constitutes the proof.

Lemma 2.3. *Any functor $X \in \mathcal{C}$ sends all morphisms to monomorphisms.*

Proof. Any morphism is isomorphic to one of the form $A \to A \oplus B$, and by using the orthogonal pullback square $\{A, A\oplus B, A\oplus B, A\oplus B\oplus B\}$ we see that the map $X(A) \to X(A\oplus B)$ is injective. ☐

Suppose we have $X \in \mathcal{C}$ and $a \in X(A)$ and $b \in X(B)$. Put

$$E(a,b)(C) = \{(\alpha,\beta) \in \mathcal{L}_0(A,C) \times \mathcal{L}_0(B,C) \mid \alpha_*(a) = \beta_*(b)\}.$$

This gives an object $E(a,b) \in \mathcal{C}$.

Definition 2.4. Let $Q(A,B)$ be the set of positive semidefinite quadratic forms q on $A \oplus B$ such that $q(u) = \|u\|_A^2$ for $u \in A$ and $q(v) = \|v\|_B^2$ for $v \in B$.

Definition 2.5. Let $Q'(A,B)$ be the set of linear maps $\gamma\colon A \to B$ such $\|\gamma(u)\| \leq \|u\|$ for all $u \in A$. Given $\gamma \in Q'(A,B)$, we can define $q_\gamma \in Q(A,B)$ by

$$q_\gamma(u,v) = \|u\|_A^2 + \|v\|_B^2 + 2\langle\gamma(u)\rangle v_B.$$

One checks that this construction gives a bijection $Q(A,B) \simeq Q'(A,B)$.

Definition 2.6. Given any pair of maps $(\alpha,\beta) \in \mathcal{L}_0(A,C)\times\mathcal{L}_0(B,C)$, we define $q_{\alpha,\beta}(u,v) = \|\alpha(u)+\beta(v)\|^2$. One checks easily that $q_{\alpha,\beta} \in Q(A,B)$ and $q_{\alpha,\beta} = q_\gamma$ where $\gamma = \beta^*\alpha\colon A \to B$.

For any $q \in Q(A,B)$ we define $Z_q = \{(u,v) \mid q(u,v) = 0\}$ and $C_q = (A\oplus B)/Z_q$. The form q gives an inner product on C_q, the evident maps $\alpha_q\colon A \to C_q$ and $\beta_q\colon B \to C_q$ are isometries, and clearly $q_{\alpha_q,\beta_q} = q$.

Lemma 2.7. *There is a set $Q(a,b) \subseteq Q(A,B)$ (independent of C) such that $(\alpha,\beta) \in E(a,b)(C)$ iff $q_{\alpha,\beta} \in Q(a,b)$.*

Proof. We put

$$Q(a,b) = \{q \mid (\alpha_q,\beta_q) \in E(a,b)(C_q)\}.$$

If $\alpha\colon A \to C$ and $\beta\colon B \to C$ and $q = q_{\alpha,\beta}$, then $\alpha(u)+\beta(v) = 0$ for $(u,v) \in Z_q$ so there is a unique map $\phi \in \mathcal{L}_0(C_q, C)$ with $\phi\alpha_q = \alpha$ and $\phi\beta_q = \beta$. As $\phi_*\colon X(C_q) \to X(C)$ is injective, we see that $(\alpha,\beta) \in E(a,b)(C)$ if and only if $(\alpha_q,\beta_q) \in E(a,b)(C_q)$, if and only if $q \in Q(a,b)$. ☐

Definition 2.8. We write $Q'(a,b) = \{\gamma \in Q'(A,B) \mid q_\gamma \in Q(a,b)\}$.

Lemma 2.9. *Suppose that $\gamma\colon A \to B$ has $\|\gamma\| < 1$. Define isometric embeddings $\alpha_\theta\colon A \to A^2$ by $\alpha_\theta(u) = (\cos(2\theta)u, \sin(2\theta)u)$, and let $j\colon A^2 \to A^3 \oplus B$ be the evident inclusion. Then for sufficiently small θ there exists an isometric embedding $\beta_\theta\colon B \to A^2 \oplus B$ such that $\beta_\theta^* j\alpha_0 = \beta_\theta^* j\alpha_\theta = \gamma\colon A \to B$.*

Proof. Suppose that $\|\gamma\| \leq \cos(\theta)$; as $\|\gamma\| < 1$, this will hold for sufficiently small θ.

Let $e_1, \dots, e_n$ be an orthonormal basis of eigenvectors for the positive self-adjoint operator $\gamma^*\gamma$. Let t_i be the eigenvalue of e_i, and order the basis so that $t_1 \geq \ldots \geq t_n \geq 0$. In particular, if r is the rank of γ then this means that $t_1, \dots, t_r > 0$ and $t_i = 0$ for $i > r$. Next note that $t_i = \|\gamma(e_i)\|^2 \leq \|\gamma\|^2 \leq \cos^2(\theta)$, so we can write $t_i = \cos^2(\theta)\cos^2(\phi_i)$ for some $\phi_i \in [0, \pi/2]$. For $i \leq r$ we define $f_i = \gamma(e_i)/\sqrt{t_i}$; these vectors form an an orthonormal basis for $(\gamma) = B \ominus \ker(\gamma^*)$, which can be extended to an orthonormal basis $\{f_1, \dots, f_m\}$ for B. Now define

$$g_i = \begin{cases} (\cos(\phi_i)\cos(\theta)e_i, \cos(\phi_i)\sin(\theta)e_i, \sin(\phi_i)e_i, 0) & \text{if } 1 \leq i \leq r \\ (0,0,0,f_i) & \text{if } r < i \leq m. \end{cases}$$

It is clear that the g_i are orthonormal, so we can define an isometric embedding $\beta_\theta\colon B \to A^3 \oplus B$ by $\beta_\theta(f_i) = g_i$. It is also clear that $\langle g_i\rangle\alpha_\psi(e_j) = 0$ for all ψ if $i \neq j$. Also, for $i \leq r$ we have

$$\langle g_i\rangle j\alpha_0(e_i) = \langle g_i\rangle(e_i, 0, 0, 0) = \cos(\phi_i)\cos(\theta) = \langle f_i\rangle\gamma(e_i).$$

Using this and easy arguments for the cases $i > r$ or $k > r$, we see that

$$\langle f_i\rangle\beta_\theta^* j\alpha_0(e_k) = \langle g_i\rangle j\alpha_0(e_k) = \langle f_i\rangle\gamma(e_k)$$

for $i = 1, \ldots, n$ and $k = 1, \ldots, m$, so $\beta_\theta^* j\alpha_0 = \gamma$.

With a little more work, one checks that

$$\begin{aligned}\langle g_i\rangle j\alpha_\theta(e_i) &= \langle(\cos(2\theta), \sin(2\theta), 0)\rangle(\cos(\theta)\cos(\phi_i), \sin(\theta)\cos(\phi_i), \sin(\phi_i))\\ &= \cos(\phi_i)(\cos(2\theta)\cos(\theta) + \sin(2\theta)\sin(\theta))\\ &= \cos(\phi_i)\cos(\theta)\\ &= \langle f_i\rangle\gamma(e_i),\end{aligned}$$

and it follows that $\beta_\theta^* j\alpha_\theta = \gamma$ also. □

Proposition 2.10. *Suppose that $a \in X(A)$, $b \in X(B)$ and there exists $\gamma \in Q'(a, b)$ such that $\|\gamma\| < 1$. Then a lies in the image of the map $X(0) \to X(A)$.*

Proof. We let $\rho_\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation through angle 2ϕ, and we also write ρ_ϕ for the map $\rho_\phi \otimes 1\colon \mathbb{R}^2 \otimes A = A^2 \to A^2$. Let $\alpha_\theta\colon A \to A^2$ be as in the previous proof, so that $\rho_\phi\alpha_\theta = \alpha_{\phi+\theta}$. If θ is sufficently small then we can apply the lemma and we find that

$$j_*\alpha_{0*}(a) = \beta_{\theta*}(b) = j_*\alpha_{\theta*}(a) = j_*\rho_{\theta*}\alpha_{0*}(a).$$

As j_* is injective, we see that $\alpha_{0*}(a)$ is invariant under the group generated by ρ_θ. By taking $\theta = \pi/4N$ for sufficiently large N, we see that $(\alpha_{\pi/4})_*(a) = \alpha_{0*}(a)$. The claim now follows from the following orthogonal pullback square.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & A \arrow[d, "\alpha_0"] \\
A \arrow[r, "\alpha_{\pi/4}"'] & A^2
\end{tikzcd}$$

□

Definition 2.11. For any $q \in Q(A, B)$, we let A_1 and B_1 be the images of $Z := \{(a, b) \in A \oplus B \mid q(a, b) = 0\}$ under the projection maps from $A \oplus B$ to A and B. We also put $A_0 = A \ominus A_1$ and $B_0 = B \ominus B_1$.

Lemma 2.12. *Suppose that $q = q_\gamma$ for some $\gamma\colon A \to B$. Then $A_1 = \ker(\gamma^*\gamma - 1)$ and $B_1 = \ker(\gamma\gamma^* - 1)$. Moreover, we have $\gamma = \gamma_0 \oplus \gamma_1$, where $\gamma_1\colon A_1 \to B_1$ is an isomorphism and $\gamma_0\colon A_0 \to B_0$ has $\|\gamma_0\| < 1$.*

Proof. From the definitions, we have

$$q(u, v) = (\|u\|^2 - \|\gamma(u)\|^2) + \|\gamma(u) + v\|^2.$$

Both terms on the right hand side are nonnegative, so we can only have $q(u, v) = 0$ if $\|u\| = \|\gamma(u)\|$ and $\gamma(u) + v = 0$, or in other words $v = -\gamma(u)$ and $\|\gamma(u)\| = \|u\|$. Thus, $A_1 = \{u \in A \mid \|\gamma(u)\| = \|u\|\}$, and by a similar argument, $B_1 = \{v \in B \mid \|\gamma(v)\| = \|v\|\}$. Clearly, if $\gamma^*\gamma(u) = u$ then

$$\|\gamma(u)\|^2 = \langle\gamma(u)\rangle\gamma(u) = \langle u\rangle\gamma^*\gamma(u) = \|u\|^2$$

so $\|\gamma(u)\| = \|u\|$. For the converse, we first recall the well-known fact that $\|\gamma^*\| = \|\gamma\| \leq 1$, so $\|\gamma^*\gamma(u)\| \leq \|\gamma(u)\| = \|u\|$. Thus, we have

$$\begin{aligned}0 &\geq \|\gamma^*\gamma(u)\|^2 - \|u\|^2\\ &= \|\gamma^*\gamma(u)\|^2 - 2\|\gamma(u)\|^2 + \|u\|^2\\ &= \|\gamma^*\gamma(u)\|^2 - 2\langle u\rangle\gamma^*\gamma(u) + \|u\|^2\\ &= \|u - \gamma^*\gamma(u)\| \geq 0.\end{aligned}$$

This means that $\|u-\gamma^*\gamma(u)\|^2=0$, so $\gamma^*\gamma(u)=u$. Thus $A_1=\ker(\gamma^*\gamma-1)$, and $B_1=\ker(\gamma\gamma^*-1)$ by a dual argument. If $u\in A_1$ then $\gamma^*\gamma(u)=u$ so $\gamma\gamma^*\gamma(u)=\gamma(u)$, so $\gamma(u)\in B_1$. Similarly, we have $\gamma^*B_1\leq A_1$, so $\langle\gamma(A_0)\rangle B_1=\langle A_0\rangle\gamma^*B_1=\{0\}$, so $\gamma(A_0)\leq B_1^\perp=B_0$. It is easy to see that $\gamma\colon A_1\to B_1$ is an isometry with inverse γ^*. The norm of the map $\gamma\colon A_0\to A_0$ is the supremum of the values $\|\gamma(u)\|$ as u runs over the unit sphere of A_0. For such u we have $u\not\in A_1$ so $\|\gamma(u)\|<\|u\|=1$, and the unit sphere is compact so the supremum is the maximum and thus is strictly less than one. ☐

Corollary 2.13. *Suppose that $a\in X(A)$, $b\in X(B)$ and $q\in Q(a,b)$. Then a lies in the image of the map $X(A_1)\to X(A)$ and b lies in the image of the map $X(B_1)\to X(B)$.*

Proof. Define $Y(C)=X(A_1\oplus C)$, so $a\in Y(A_0)=X(A)$. Define $b'=(\gamma_1^*\oplus 1)_*(b)\in X(A_1\oplus B_0)=Y(B_0)$. One checks that $\gamma_0\in Q(a,b')$ and $\|\gamma_0\|<1$, so we see from Proposition 2.10 that a lies in the image of the map $Y(0)\to Y(A_0)$, or in other words the map $X(A_1)\to X(A)$. The argument for b is similar. ☐

Definition 2.14. An element $a\in X(A)$ is *minimal* if there is no proper subspace $A'<A$ such that a lies in the image of $X(A')$.

If $a\in X(A)$ is arbitrary, then it is clear that there exists $A'\leq A$ and a minimal element $a'\in X(A')$ that maps to a in $X(A)$. As $A'\simeq\mathbb{R}^n$ for some n, it follows in turn that there is a minimal element $a''\in X(\mathbb{R}^n)$ and a map $\alpha\colon\mathbb{R}^n\to A$ such that $a=\alpha_*(a'')$.

Proposition 2.15. *If $a\in X(A)$ and $b\in X(B)$ are minimal and $\alpha_*(a)=\beta_*(b)$ then the map $\gamma:=\beta^*\alpha\colon A\to B$ is an isometric isomorphism, $\alpha=\beta\gamma\colon A\to C$ and $b=\gamma_*(a)$.*

Proof. The map γ lies in $Q'(a,b)$, so Corollary 2.13 tells us that a lies in the image of $X(\ker(\gamma^*\gamma-1))$. As a is minimal we must have $\ker(\gamma^*\gamma-1)=A$, so $\gamma^*\gamma=1$. A similar argument gives $\gamma\gamma^*=1$, so γ is an isometric isomorphism. Now put $\delta=\alpha-\beta\gamma\colon A\to C$; we claim that $\delta^*\delta=0$. As α, β and γ are isometries, we have $\alpha^*\alpha=\beta^*\beta=1$ and $\alpha^*\beta\beta^*\alpha=\gamma^*\gamma=1$. It follows that

$$\delta^*\delta=(\alpha^*-\alpha^*\beta\beta^*)(\alpha-\beta\beta^*\alpha)=\alpha^*\alpha-2\alpha^*\beta\beta^*\alpha+\alpha^*\beta\beta^*\beta\beta^*\alpha=1-2+1=0,$$

as required. This means that $\|\delta(a)\|^2=\langle a\rangle\delta^*\delta(a)=0$ for all a, so $\delta=0$, so $\alpha=\beta\gamma$. This means that $\beta_*\gamma_*(a)=\alpha_*(a)=\beta_*(b)$ and β_* is injective, so $\gamma_*(a)=b$. ☐

Corollary 2.16. *If we let X'_n be the set of minimal elements in $X(\mathbb{R}^n)$, then there is a natural isomorphism*

$$X(C)=\coprod_n X'_n\times_{O(n)}\mathcal{L}^+(\mathbb{R}^n,C).\quad\square$$

The main theorem now follows, by choosing representatives for the orbits of $O(n)$ on X'_n.

3. The infinite case

We next give examples to show that these results do not extend to infinite universes.

First note that if $\mathcal{U}\leq\mathcal{V}\leq\mathcal{W}$ and $\mathcal{U}$ is complemented in $\mathcal{W}$ then any $w\in\mathcal{W}$ can be written as $u+x$ with $u\in\mathcal{U}$ and $x\perp\mathcal{U}$. If $w\in\mathcal{V}$ then $x=v-u\in\mathcal{V}$ also. Using this, we see that $\mathcal{U}$ is complemented in $\mathcal{V}$. To say this the other way around, if $\mathcal{U}$ is uncomplemented in $\mathcal{V}$ then it is uncomplemented in $\mathcal{W}$.

Now define

$$X(\mathcal{U})=\{f\in\mathcal{L}(\mathbb{R}^\infty,\mathcal{U})\mid f(\mathbb{R}^\infty)\text{ is uncomplemented in }\mathcal{U}\}.$$

One can also identify $X(\mathcal{U})$ with the space of orthonormal sequences (u_k) in $\mathcal{U}$ such that there exists $u\in\mathcal{U}$ with $\langle u\rangle u_k\neq 0$ for infinitely many k.

This gives a continuous functor from universes to unbased spaces **check for topological pathology here**. It is easy to see that X converts orthogonal pullback squares to pullbacks. However, if $\mathcal{U}$ is uncomplemented in $\mathcal{V}$, I claim that X does not preserve the following pullback:

$$\begin{tikzcd}
\mathcal{U}\oplus\mathcal{U} \arrow[r] \arrow[d] & \mathcal{U}\oplus\mathcal{V} \arrow[d] \\
\mathcal{V}\oplus\mathcal{U} \arrow[r] & \mathcal{V}\oplus\mathcal{V}
\end{tikzcd}$$

Indeed, if $f\colon\mathbb{R}^\infty\to\mathcal{U}\oplus\mathcal{U}$ is any isomorphism then $f\in X(\mathcal{U}\oplus\mathcal{V})\cap X(\mathcal{V}\oplus\mathcal{U})$ but $f\not\in X(\mathcal{U}\oplus\mathcal{U})$.

Next, I claim that if $\mathcal{U}_0 > \mathcal{U}_1 > \mathcal{U}_2 > \dots$, then we need not have $X(\bigcap_n \mathcal{U}_n) = \bigcap_n X(\mathcal{U}_n)$. To see this, put $\mathcal{U}_0 = \mathbb{R}^\infty \oplus \mathbb{R}^\infty$, and define $\phi\colon \mathbb{R}^\infty \to \mathbb{R}$ by $\phi(x) = \sum_i x_i/2^i$. Define a quadratic form on $\mathcal{U}_0$ by $q(x,y) = \|x\|^2 + \|y\|^2 + \phi(x)\phi(y)$. The Cauchy-Schwartz inequality implies that $|\phi(x)|^2 \leq 4\|x\|^2/3$ and thus that $|\phi(x)\phi(y)| \leq 2\|x\|\|y\|$, with equality only when $x = 0$ or $y = 0$. From this it is not hard to deduce that q is positive definite, so it makes $\mathcal{U}_0$ into a universe. Let $\mathcal{U}_k$ be the subspace of pairs (x, y) where $y_i = 0$ for $i < k$. The inclusion of the first copy of $\mathbb{R}^\infty$ in $\mathcal{U}_0$ gives an element of $\bigcap_k X(\mathcal{U}_k)$ that does not lie in $X(\bigcap_k \mathcal{U}_k)$.

4. The Whitehead product

Given finite universes U and V, we define a Whitehead map

$$w = w_{U,V}\colon S(U \oplus V) \to S(U \oplus \mathbb{R}) \vee S(V \oplus \mathbb{R})$$

by

$$w(u,v) = \begin{cases} v^{-2}(2u\sqrt{v^2-u^2}, 2u^2 - v^2) \in S(U \oplus \mathbb{R}) & \text{if } \|u\| \leq \|v\| \\ u^{-2}(2v\sqrt{u^2-v^2}, 2v^2 - u^2) \in S(V \oplus \mathbb{R}) & \text{if } \|v\| \leq \|u\|. \end{cases}$$

Note that when $\|u\| = \|v\|$, both clauses give $w(u,v) = (0,1)$, which is the basepoint in $S(U \oplus \mathbb{R}) \vee S(V \oplus \mathbb{R})$, so the definition is self-consistent and gives a continuous map.

If $(x,t) \in S(U \oplus \mathbb{R})$ is not the basepoint, we define $f = f_{(x,t)}\colon S(V) \to S(U \oplus V)$ by

$$f(y) = \left(\frac{x}{\sqrt{(1-t)(3+t)}}, \sqrt{\frac{2}{3+t}}\,y \right).$$

One can check that f gives a homeomorphism $S(V) \to w^{-1}\{(x,t)\}$.

I think that w fits in a diagram as follows:

$$\begin{tikzcd}
S(U \oplus V) \arrow[r, "w"] \arrow[d, tail] & S(U \oplus \mathbb{R}) \vee S(V \oplus \mathbb{R}) \arrow[r, "\simeq"] \arrow[d, tail, "\text{inc}"] & S^U \vee S^V \arrow[d, tail, "\text{inc}"] \\
B(U \oplus V) \arrow[r] \arrow[d, two heads] & S(U \oplus \mathbb{R}) \times S(V \oplus \mathbb{R}) \arrow[r, "\simeq"] \arrow[d, two heads] & S^U \times S^V \arrow[d, two heads] \\
B(U \oplus V)/S(U \oplus V) \arrow[r, "\simeq"'] & S(U \oplus V \oplus \mathbb{R}) \arrow[r, "\simeq"'] & S^{U \oplus V}
\end{tikzcd}$$

5. Embedding products

Define $j(V)\colon I \times S^V \to V \times I$ by

$$j(V)(s,t,v) = \tfrac{1}{2}(1+s)(v,t) + \tfrac{1}{2}(1-s)(0,\tfrac{1}{2}).$$

This is easily seen to be an embedding. Given universes $V_0, \dots, V_n$, define

$$j(\underline{V})\colon I \times \prod_{k \leq n} S^{V_k} \to \left(\prod_{k \leq n} V_k \right) \times I$$

to be the composite

$$I \times \prod_{k \leq n} S^{V_k} \xrightarrow{j(V_0,\dots,V_{n-1}) \times 1} \prod_{k<n} V_k \times I \times S^{V_n} \xrightarrow{1 \times j(V_n)} \prod_{k \leq n} V_k \times I.$$

This is again an embedding. In particular, this embeds $\prod_k S^{V_k}$ as a hypersurface in $(\prod_k V_k) \times \mathbb{R}$, by a polynomial map between real algebraic varieties. The map is natural in the V_k's but is not symmetric.

If we instead use the unit sphere U^V in $\mathbb{R} \oplus V$ and use $(-1,1)$ instead of I then I think the formulae are

$$i(s,a,u) = \tfrac{1}{2}(1+s)(u,a)$$

$$i((s,(a_0,u_0),\dots,(a_{n-1},u_{n-1})) = (b_0u_0/2, \dots, b_{n-1}u_{n-1}/2^n, b_{n-1}a_{n-1}/2^n)$$

where

$$b_k = \sum_{l=0}^{k-1} 2^l \prod_{j=l}^{k-1} a_j + s \prod_{j=0}^{k-1} a_j.$$

6. The clutching map

Here I just record some formulae for the map $\Sigma U(V) \to G_d(V^2)$. We regard V^2 as $V \otimes_{\mathbb{R}} \mathbb{C}$. We define $\kappa\colon \mathbb{C} \to \mathbb{C}$ by $\kappa(z) = \overline{z}$, and $\gamma\colon U(1) \to \mathbb{R} \cup \{\infty\}$ by $\gamma(z) = i(1+z)/(1-z)$, so $\gamma^{-1}(t) = (it+1)/(it-1)$. We define $\beta(t)$ to be the square root of $\gamma^{-1}(t)$, using the branch such that $\beta(t) \to \mp 1$ as $t \to \pm\infty$. We then define

$$\phi(t,g) = ((1+g) \otimes \kappa + (1-g) \otimes \beta(t))/2 \in U(V \otimes \mathbb{C}),$$

and we let $\psi(t,g) = W_{(t,g)}$ be the image of $V \otimes 1$ under $\phi(t,g)$. We find that

$$\begin{aligned} \phi(t,g) &\to g \oplus (-1) && \text{as } t \to \infty \\ \phi(t,g) &\to 1 \oplus (-g) && \text{as } t \to -\infty. \end{aligned}$$

It follows that ψ extends to a based map $\Sigma U(V) \to G_d(V^2)$, if we use $V \otimes 1$ as a basepoint in $G_d(V^2)$. Now define I_+ to be the closure of the image of $[0,\infty)$ in S^1, and let I_- be the closure of the image of $(-\infty, 0]$, so $I_+ \cap I_- = \{0, \infty\} = S^0$. Put $C_\pm = I_\pm \wedge U(V)$, and define trivialisations $\theta_\pm\colon C_\pm \times V \to W|_{C_\pm}$ by

$$\begin{aligned} \theta_+(t,g)(v) &= \phi(t,g)(g^{-1}v) \\ \theta_-(t,g)(v) &= \phi(t,g)(v). \end{aligned}$$

Note that $\theta_\pm(\infty, g) = v$, and that $\theta_+(0,g)^{-1}\theta_-(0,g) = g\colon V \to V$. This means that the identity map of $U(V)$ is the clutching function for W.

7. More rotations and reflections

Given vectors $u, v \in S(V)$ with $u+v \neq 0$, we observe that $\langle u, v\rangle > -1$ so we can define $\operatorname{rot}(u,v) \in \operatorname{End}(V)$ by

$$\operatorname{rot}(u,v)(x) = x - \frac{\langle u+v, x\rangle(u+v)}{1+\langle u,v\rangle} + 2\langle u,x\rangle v.$$

One checks that this actually lies in $SO(V)$ and that it sends u to v and acts as the identity on $\mathbb{R}\{u,v\}^\perp$. This defines a continuous map

$$\operatorname{rot}\colon \{(u,v) \in S(V)^2 \mid u+v \neq 0\} \to SO(V).$$

We also define $\operatorname{ref}(u)(x) = x - 2\langle u,x\rangle u$, so $\operatorname{ref}\colon S(V) \to O(V)$ and $\operatorname{ref}(u)(u) = -u$ and $\operatorname{ref}(u)$ acts as the identity on $\mathbb{R}\{u\}^\perp$.

Lemma 7.1. *If $V = \mathbb{H}$ or $V = \mathbb{O}$ and $u \in S(V)$ then $\operatorname{ref}(u)(x) = -u\overline{x}u$. (In the octonionic case, we recall that $\mathbb{O}$ is alternative, which means that $(u\overline{x})u = u(\overline{x}u)$, so brackets can be omitted.)*

Proof. Put $g(x) = -u\overline{x}u$; it will suffice to show that $g(u) = -u$ and $g(x) = -x$ for $x \in u^\perp$. The first of these is clear (becaues $u\overline{u} = 1$ for $u \in S(V)$). For the second recall that $\langle x, u\rangle = \operatorname{Re}(\overline{u}x)$, so $x \in u^\perp$ iff $x = uz$ for some z with $\overline{z} = -z$. The alternative property implies that the subalgebra of V generated by z and $\operatorname{Im}(u)$ is associative, so $\overline{x}u = (\overline{z}\overline{u})u = \overline{z}(\overline{u}u) = \overline{z} = -z$. This gives $g(x) = -u((\overline{z}\overline{u})u) = uz = x$, as required. □

We define a stereographic projection map $\phi_V\colon S(V \oplus \mathbb{R}) \to S^V$ by $\phi(v,t) = v/(1-t)$, which has inverse

$$\phi^{-1}(w) = (2w, \|w\|^2 - 1)/(\|w\|^2 + 1).$$

Note that for $v \in S(V)$ we have $\phi(v, 0) = v$ and so $\phi^{-1}(v) = (v, 0)$.

Now suppose that $V = \mathbb{K}_0 = \{a \in \mathbb{K} \mid a + \overline{a} = 0\}$ for $\mathbb{K} = \mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$. We identify $V \oplus \mathbb{R}$ with $\mathbb{K}$ in the obvious way. Then for $w \in \mathbb{K}_0$ we have $\|w\|^2 = -w^2$ and so

$$\phi^{-1}(w) = \frac{2w - w^2 - 1}{1 - w^2} = \frac{(w-1)^2}{w^2 - 1} = \frac{w-1}{w+1}.$$

It follows that for $a \in S(\mathbb{K})$ we have

$$\phi(a) = (1+a)/(1-a) = (1+a)^2/(1+\|a\|^2).$$

(Note that all the above takes place in a subalgebra of $\mathbb{K}$ generated by a single element, and all such subalgebras are commutative and associative even when $\mathbb{K} = \mathbb{O}$.)

8. THE J-HOMOMORPHISM

We define an unbased map $J\colon O(V) \to F(S^V, S^V)$ as follows. Given $v \in S(V)$ and $r \in [0,\infty]$ we put

$$J(g)(rv) = \begin{cases} v\sqrt{r^{-2}-1} & \text{if } r \leq 1 \\ g(v)\sqrt{1-r^{-2}} & \text{if } r \geq 1. \end{cases}$$

We are using some obvious conventions about infinity, so $J(g)(0) = J(g)(\infty) = \infty$ and $J(g)(v) = 0$ for $v \in S(V)$. Note that the formula is independent of v when $r = 0$ or $r = \infty$, which implies that we have a well-defined, continuous map.

We next show that $J(1)$ is joined to the basepoint in $F(S^V, S^V)$ by a canonical path. Indeed, we can just define

$$H_t(rv) = (t + |r^{-2} - 1|^{1/2})v$$

for $0 \leq t, r \leq \infty$ and $v \in S(V)$. We then have $H_t(0) = H_t(\infty) = \infty$ and $H_0(v) = J(1)(v)$ and $H_\infty(v) = \infty$, as required.

Note that the function $r \mapsto \sqrt{1-r^{-2}}$ gives an increasing homeomorphism $(1,\infty) \to (0,\infty)$, whereas the function $r \mapsto \sqrt{r^{-2}-1}$ gives a decreasing homeomorphism $(0,1) \to (0,\infty)$. This means that $J = \alpha - \beta$, where $\alpha, \beta\colon O(V) \to F(S^V, S^V)$ are given by $\alpha(g) = S^g$ and $\beta(g) = 1_{S^V}$, and subtraction uses the co-H-structure of the first S^V. There is a wrinkle here because the co-H-structure depends on a choice of embedding $\mathbb{R} \to V$, whereas our definition uses the collapse $S^V \to S^V/S(V)$ which is not a comultiplication because the basepoint does not lie in $S(V)$. However, I think that everything is OK up to a non-canonical homotopy.

Using our homotopy H we can extend J to a based map defined on the whiskered space $O(V)' = (O(V) \amalg [0,\infty])/(1_V \sim 0)$ (where ∞ is taken as the basepoint). We can also define a canonical homotopy equivalence $O(V) \to O(V)'$ by functional calculus. Define $u\colon O(V) \to [-1,1]$ by $u(g) = \min(\mathrm{Re}(\mathrm{spec}(g)))$. Define $p\colon S^1 \to S^1$ by

$$p(-e^{i\theta}) = \begin{cases} -e^{2i\theta} & \text{if } 0 \leq |\theta| \leq \pi/2 \\ 1 & \text{if } \pi/2 \leq |\theta| \leq \pi, \end{cases}$$

so p collapses the right-hand semicircle to a point and stretches out the left-hand semicircle to fill the whole circle. By functional calculus we get a map $p\colon U(\mathbb{C} \otimes V) \to U(\mathbb{C} \otimes V)$ satisfying $\mathrm{spec}(p(g)) = p(\mathrm{spec}(g))$. This is homotopic to the identity, and I think it preserves $O(V)$. Now choose an increasing homeomorphism $f\colon [0,1] \to [0,\infty]$ and define $p'\colon O(V) \to O(V)'$ by

$$p'(g) = \begin{cases} p(g) \in O(V) & \text{if } u(g) \leq 0 \\ f(u(g)) \in [0,\infty] & \text{if } u(g) \geq 0. \end{cases}$$

This is the required equivalence.

9. COMPLEX SPHERES

Given $z, w \in \mathbb{C}^n$ we put $z.w = \sum_a z_a w_a \in \mathbb{C}$, so $\|z\|^2 = z.\overline{z}$. We then put $X^n = \{z \in \mathbb{C}^{n+1} \mid z.z = 1\}$, so $X^n \cap \mathbb{R}^{n+1} = S^n$.

Proposition 9.1. *The inclusion $i_n\colon S^n \to X^n$ is a homotopy equivalence.*

We treat the odd and even cases separately.

Lemma 9.2. *The inclusion $i = i_{2m-1}\colon S^{2m-1} \to X^{2m-1}$ is a homotopy equivalence.*

Proof. Put $Y = \{(u,v) \in \mathbb{C}^m \times \mathbb{C}^m \mid u.v = 1\}$, and define $f\colon X \to Y$ by $f(z) = (u,v)$, where $u_b = z_{2b} + iz_{2b+1}$ and $v_b = z_{2b} - iz_{2b+1}$. It is clear that this is a homeomorphism. If $(u,v) \in Y$ then $u \in \mathbb{C}^m \setminus \{0\}$ so we can define $p\colon Y \to S^{2m-1}$ by $p(u,v) = x$, where $x_{2b} = \mathrm{Re}(u_b/\|u\|)$ and $x_{2b+1} = \mathrm{Im}(u_b/\|u\|)$. One checks that $pfi = 1\colon S^{2m-1} \to S^{2m-1}$. We also have $fip(u,v) = (u/\|u\|, \overline{u}/\|u\|)$. By considering the maps $h_t(u,v) = (u\|u\|^{-t}, \overline{u}\|u\|^{t-2})$ and $k_t(u,v) = (u, (1-t)v + t\overline{u}/\|u\|^2)$ we see that $fip \simeq 1_Y$. As f is a homeomorphism it follows that $ipf \simeq 1_X$, so pf is homotopy inverse to i. $\square$

Lemma 9.3. *The inclusion* $i = i_{2m} \colon S^{2m} \to X^{2m}$ *is a homotopy equivalence.*

Proof. Here it is more convenient to fatten things up a litle. Put

$$\begin{aligned}\widetilde{X} &= \{z \in \mathbb{C}^{2m+1} \mid z.z \in (0,\infty)\} \\ \widetilde{S} &= \widetilde{X} \cap \mathbb{R}^{2m+1} = \mathbb{R}^{2m+1} \setminus \{0\} \\ \widetilde{Y} &= \{(u,v,w) \in \mathbb{C}^m \times \mathbb{C}^m \times \mathbb{C} \mid u.v + w^2 \in (0,\infty)\}.\end{aligned}$$

There are evident homeomorphisms $\widetilde{X} = (0,\infty) \times X^{2m}$ and $\widetilde{S} = (0,\infty) \times S^{2m}$, so it will suffice to show that the inclusion $i \colon \widetilde{S} \to \widetilde{X}$ is a homotopy equivalence. Define $f \colon \widetilde{X} \to \widetilde{Y}$ by $f(z) = (u,v,w)$, where $u_b = z_{2b} + iz_{2b+1}$ and $v_b = z_{2b} - iz_{2b+1}$ and $w = z_{2m}$. It is clear that this is a homeomorphism. If $(u,v,w) \in Y$ and $u = 0$ then $w^2 = 1$ so $\mathrm{Re}(w) \neq 0$. It follows that we can define $p \colon \widetilde{Y} \to \widetilde{S}$ by

$$p(u,v,w) = (\mathrm{Re}(u_0), (u_0), \dots, \mathrm{Re}(u_{m-1}), (u_{m-1}), \mathrm{Re}(w)).$$

One checks that $pfi = 1 \colon \widetilde{S} \to \widetilde{S}$. Now consider a point $(u, v, x+iy) \in \widetilde{Y}$, so the number $r = u.v + x^2 - y^2 + 2ixy$ lies in $(0,\infty)$. One checks that $fip(u,v,x+iy) = (u, \overline{u}, x)$. Given $t \in [0,1]$, consider the point

$$h_t(u,v,x) = (u, (1-t)\overline{u} + tv, x + ity).$$

We have

$$\begin{aligned}u.((1-t)\overline{u} + tv) + (x+ity)^2 &= (1-t)\|u\|^2 + tu.v + x^2 - t^2y^2 + 2itxy \\ &= (1-t)\|u\|^2 + t(r - x^2 + y^2 - 2ixy) + x^2 - t^2y^2 + 2itxy \\ &= (1-t)(\|u\|^2 + x^2 + ty^2) + tr.\end{aligned}$$

As $r > 0$ this is positive unless $t = 0$ and $u = 0$ and $x = 0$, which would contradict the positivity of r. It follows that $h_t(u,v,x) \in \widetilde{Y}$, so we have a homotopy between $h_0 = fip$ and $h_1 = 1_{\widetilde{Y}}$. As $fip \simeq 1$ and f is a homeomorphism we have $ipf \simeq 1$, so pf is a homotopy inverse for i. ☐

Corollary 9.4. *The inclusion* $O_n(\mathbb{R}) \to O_n(\mathbb{C})$ *is a homotopy equivalence.*

Proof. We have a commutative diagram

$$\begin{tikzcd} O_{n-1}(\mathbb{R}) \arrow[r] \arrow[d] & O_n(\mathbb{R}) \arrow[r] \arrow[d] & S^n \arrow[d] \\ O_{n-1}(\mathbb{C}) \arrow[r, "j"'] & O_n(\mathbb{C}) \arrow[r, "q"'] & X^n \end{tikzcd}$$

The right hand vertical is a homotopy equivalence, and we may assume inductively that the same is true of the left hand vertical. Both rows are fibre bundles so it follows that the middle vertical is also a homotopy equivalence. ☐

References

[1] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules and algebras in stable homotopy theory*, Amer. Math. Soc. Surveys and Monographs, vol. 47, American Mathematical Society, 1996.

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