The known part of the Bousfield semiring

Neil Strickland

November 1, 2016

Outline of the talk

- Fix a prime p, and let $\mathcal{L}$ denote the semiring of p-local Bousfield classes.
- The literature contains many results about the structure of $\mathcal{L}$. We seek a consolidated statement that incorporates as much of this information as possible.
- The Telescope Conjecture is a key open question about $\mathcal{L}$. It is widely expected to be false, but this remains unproven. We will work with a quotient semiring $\overline{\mathcal{L}}$ in which TC is true.
- We will give a complete description of a subsemiring $\mathcal{A} \leq \overline{\mathcal{L}}$ which contains almost all classes that have previously been named and studied.

Outline of the talk

- Fix a prime p, and let $\mathcal{L}$ denote the semiring of p-local Bousfield classes.
- The literature contains many results about the structure of $\mathcal{L}$. We seek a consolidated statement that incorporates as much of this information as possible.
- The Telescope Conjecture is a key open question about $\mathcal{L}$. It is widely expected to be false, but this remains unproven. We will work with a quotient semiring $\overline{\mathcal{L}}$ in which TC is true.
- We will give a complete description of a subsemiring $\mathcal{A} \leq \overline{\mathcal{L}}$ which contains almost all classes that have previously been named and studied.

Outline of the talk

- Fix a prime p, and let $\mathcal{L}$ denote the semiring of p-local Bousfield classes.
- The literature contains many results about the structure of $\mathcal{L}$. We seek a consolidated statement that incorporates as much of this information as possible.
- The Telescope Conjecture is a key open question about $\mathcal{L}$. It is widely expected to be false, but this remains unproven. We will work with a quotient semiring $\overline{\mathcal{L}}$ in which TC is true.
- We will give a complete description of a subsemiring $\mathcal{A} \leq \overline{\mathcal{L}}$ which contains almost all classes that have previously been named and studied.

Outline of the talk

- Fix a prime p, and let $\mathcal{L}$ denote the semiring of p-local Bousfield classes.
- The literature contains many results about the structure of $\mathcal{L}$. We seek a consolidated statement that incorporates as much of this information as possible.
- The Telescope Conjecture is a key open question about $\mathcal{L}$. It is widely expected to be false, but this remains unproven. We will work with a quotient semiring $\overline{\mathcal{L}}$ in which TC is true.
- We will give a complete description of a subsemiring $\mathcal{A} \leq \overline{\mathcal{L}}$ which contains almost all classes that have previously been named and studied.

Outline of the talk

- Fix a prime p, and let $\mathcal{L}$ denote the semiring of p-local Bousfield classes.
- The literature contains many results about the structure of $\mathcal{L}$. We seek a consolidated statement that incorporates as much of this information as possible.
- The Telescope Conjecture is a key open question about $\mathcal{L}$. It is widely expected to be false, but this remains unproven. We will work with a quotient semiring $\overline{\mathcal{L}}$ in which TC is true.
- We will give a complete description of a subsemiring $\mathcal{A} \leq \overline{\mathcal{L}}$ which contains almost all classes that have previously been named and studied.

Basic definitions

- $\mathcal{B} = \{p - \text{local spectra}\}$.
- This is a triangulated category, and in particular is additive.
- There is a binary coproduct written $X \vee Y$, and more generally an indexed coproduct written $\bigvee_i X_i$.
- There is a bilinear symmetric monoidal smash product written $X \wedge Y$, with unit object S.
- All this is similar to the derived category $D(R)$ of a ring R, with $\vee$ like $\oplus$ and $\wedge$ like $\otimes$.
- $\langle X \rangle = \{T \mid X \wedge T = 0\}$ and $\mathcal{L} = \{\langle X \rangle \mid X \in \mathcal{B}\}$.
- Theorem of Ohkawa: $\mathcal{L}$ is a set, not a proper class.
- There are well-defined operations $\langle X \rangle \vee \langle Y \rangle = \langle X \vee Y \rangle$ and $\langle X \rangle \wedge \langle Y \rangle = \langle X \wedge Y \rangle$. We put $0 = \langle 0 \rangle$ and $1 = \langle S \rangle$.
- We order Bousfield classes by reverse inclusion, so $\langle X \rangle \leq \langle Y \rangle$ means $\langle X \rangle \supseteq \langle Y \rangle$.

Basic definitions

- $\mathcal{B} = \{p - \text{local spectra}\}$.
- This is a triangulated category, and in particular is additive.
- There is a binary coproduct written $X \vee Y$, and more generally an indexed coproduct written $\bigvee_i X_i$.
- There is a bilinear symmetric monoidal smash product written $X \wedge Y$, with unit object S.
- All this is similar to the derived category $D(R)$ of a ring R, with $\vee$ like $\oplus$ and $\wedge$ like $\otimes$.
- $\langle X \rangle = \{T \mid X \wedge T = 0\}$ and $\mathcal{L} = \{\langle X \rangle \mid X \in \mathcal{B}\}$.
- Theorem of Ohkawa: $\mathcal{L}$ is a set, not a proper class.
- There are well-defined operations $\langle X \rangle \vee \langle Y \rangle = \langle X \vee Y \rangle$ and $\langle X \rangle \wedge \langle Y \rangle = \langle X \wedge Y \rangle$. We put $0 = \langle 0 \rangle$ and $1 = \langle S \rangle$.
- We order Bousfield classes by reverse inclusion, so $\langle X \rangle \leq \langle Y \rangle$ means $\langle X \rangle \supseteq \langle Y \rangle$.

Basic definitions

- $\mathcal{B} = \{p - \text{local spectra}\}$.
- This is a triangulated category, and in particular is additive.
- There is a binary coproduct written $X \vee Y$, and more generally an indexed coproduct written $\bigvee_i X_i$.
- There is a bilinear symmetric monoidal smash product written $X \wedge Y$, with unit object S.
- All this is similar to the derived category $D(R)$ of a ring R, with $\vee$ like $\oplus$ and $\wedge$ like $\otimes$.
- $\langle X \rangle = \{T \mid X \wedge T = 0\}$ and $\mathcal{L} = \{\langle X \rangle \mid X \in \mathcal{B}\}$.
- Theorem of Ohkawa: $\mathcal{L}$ is a set, not a proper class.
- There are well-defined operations $\langle X \rangle \vee \langle Y \rangle = \langle X \vee Y \rangle$ and $\langle X \rangle \wedge \langle Y \rangle = \langle X \wedge Y \rangle$. We put $0 = \langle 0 \rangle$ and $1 = \langle S \rangle$.
- We order Bousfield classes by reverse inclusion, so $\langle X \rangle \leq \langle Y \rangle$ means $\langle X \rangle \supseteq \langle Y \rangle$.

Basic definitions

- $\mathcal{B} = \{p - \text{local spectra}\}$.
- This is a triangulated category, and in particular is additive.
- There is a binary coproduct written $X \vee Y$, and more generally an indexed coproduct written $\bigvee_i X_i$.
- There is a bilinear symmetric monoidal smash product written $X \wedge Y$, with unit object S.
- All this is similar to the derived category $D(R)$ of a ring R, with $\vee$ like $\oplus$ and $\wedge$ like $\otimes$.
- $\langle X \rangle = \{T \mid X \wedge T = 0\}$ and $\mathcal{L} = \{\langle X \rangle \mid X \in \mathcal{B}\}$.
- Theorem of Ohkawa: $\mathcal{L}$ is a set, not a proper class.
- There are well-defined operations $\langle X \rangle \vee \langle Y \rangle = \langle X \vee Y \rangle$ and $\langle X \rangle \wedge \langle Y \rangle = \langle X \wedge Y \rangle$. We put $0 = \langle 0 \rangle$ and $1 = \langle S \rangle$.
- We order Bousfield classes by reverse inclusion, so $\langle X \rangle \leq \langle Y \rangle$ means $\langle X \rangle \supseteq \langle Y \rangle$.

Basic definitions

- $\mathcal{B} = \{p - \text{local spectra}\}$.
- This is a triangulated category, and in particular is additive.
- There is a binary coproduct written $X \vee Y$, and more generally an indexed coproduct written $\bigvee_i X_i$.
- There is a bilinear symmetric monoidal smash product written $X \wedge Y$, with unit object S.
- All this is similar to the derived category $D(R)$ of a ring R, with $\vee$ like $\oplus$ and $\wedge$ like $\otimes$.
- $\langle X \rangle = \{T \mid X \wedge T = 0\}$ and $\mathcal{L} = \{\langle X \rangle \mid X \in \mathcal{B}\}$.
- Theorem of Ohkawa: $\mathcal{L}$ is a set, not a proper class.
- There are well-defined operations $\langle X \rangle \vee \langle Y \rangle = \langle X \vee Y \rangle$ and $\langle X \rangle \wedge \langle Y \rangle = \langle X \wedge Y \rangle$. We put $0 = \langle 0 \rangle$ and $1 = \langle S \rangle$.
- We order Bousfield classes by reverse inclusion, so $\langle X \rangle \leq \langle Y \rangle$ means $\langle X \rangle \supseteq \langle Y \rangle$.

Basic definitions

- $\mathcal{B} = \{p - \text{local spectra}\}$.
- This is a triangulated category, and in particular is additive.
- There is a binary coproduct written $X \vee Y$, and more generally an indexed coproduct written $\bigvee_i X_i$.
- There is a bilinear symmetric monoidal smash product written $X \wedge Y$, with unit object S.
- All this is similar to the derived category $D(R)$ of a ring R, with $\vee$ like $\oplus$ and $\wedge$ like $\otimes$.
- $\langle X \rangle = \{T \mid X \wedge T = 0\}$ and $\mathcal{L} = \{\langle X \rangle \mid X \in \mathcal{B}\}$.
- Theorem of Ohkawa: $\mathcal{L}$ is a set, not a proper class.
- There are well-defined operations $\langle X \rangle \vee \langle Y \rangle = \langle X \vee Y \rangle$ and $\langle X \rangle \wedge \langle Y \rangle = \langle X \wedge Y \rangle$. We put $0 = \langle 0 \rangle$ and $1 = \langle S \rangle$.
- We order Bousfield classes by reverse inclusion, so $\langle X \rangle \leq \langle Y \rangle$ means $\langle X \rangle \supseteq \langle Y \rangle$.

Basic definitions

- $\mathcal{B} = \{p - \text{local spectra}\}$.
- This is a triangulated category, and in particular is additive.
- There is a binary coproduct written $X \vee Y$, and more generally an indexed coproduct written $\bigvee_i X_i$.
- There is a bilinear symmetric monoidal smash product written $X \wedge Y$, with unit object S.
- All this is similar to the derived category $D(R)$ of a ring R, with $\vee$ like $\oplus$ and $\wedge$ like $\otimes$.
- $\langle X \rangle = \{T \mid X \wedge T = 0\}$ and $\mathcal{L} = \{\langle X \rangle \mid X \in \mathcal{B}\}$.
- Theorem of Ohkawa: $\mathcal{L}$ is a set, not a proper class.
- There are well-defined operations $\langle X \rangle \vee \langle Y \rangle = \langle X \vee Y \rangle$ and $\langle X \rangle \wedge \langle Y \rangle = \langle X \wedge Y \rangle$. We put $0 = \langle 0 \rangle$ and $1 = \langle S \rangle$.
- We order Bousfield classes by reverse inclusion, so $\langle X \rangle \leq \langle Y \rangle$ means $\langle X \rangle \supseteq \langle Y \rangle$.

- $\mathcal{B} = \{p-\text{local spectra}\}$.
- This is a triangulated category, and in particular is additive.
- There is a binary coproduct written $X \vee Y$, and more generally an indexed coproduct written $\bigvee_i X_i$.
- There is a bilinear symmetric monoidal smash product written $X \wedge Y$, with unit object S.
- All this is similar to the derived category $D(R)$ of a ring R, with $\vee$ like $\oplus$ and $\wedge$ like $\otimes$.
- $\langle X \rangle = \{T \mid X \wedge T = 0\}$ and $\mathcal{L} = \{\langle X \rangle \mid X \in \mathcal{B}\}$.
- Theorem of Ohkawa: $\mathcal{L}$ is a set, not a proper class.
- There are well-defined operations $\langle X \rangle \vee \langle Y \rangle = \langle X \vee Y \rangle$ and $\langle X \rangle \wedge \langle Y \rangle = \langle X \wedge Y \rangle$. We put $0 = \langle 0 \rangle$ and $1 = \langle S \rangle$.
- We order Bousfield classes by reverse inclusion, so $\langle X \rangle \leq \langle Y \rangle$ means $\langle X \rangle \supseteq \langle Y \rangle$.

Basic definitions

- $\mathcal{B} = \{p - \text{local spectra}\}$.
- This is a triangulated category, and in particular is additive.
- There is a binary coproduct written $X \vee Y$, and more generally an indexed coproduct written $\bigvee_i X_i$.
- There is a bilinear symmetric monoidal smash product written $X \wedge Y$, with unit object S.
- All this is similar to the derived category $D(R)$ of a ring R, with $\vee$ like $\oplus$ and $\wedge$ like $\otimes$.
- $\langle X \rangle = \{T \mid X \wedge T = 0\}$ and $\mathcal{L} = \{\langle X \rangle \mid X \in \mathcal{B}\}$.
- Theorem of Ohkawa: $\mathcal{L}$ is a set, not a proper class.
- There are well-defined operations $\langle X \rangle \vee \langle Y \rangle = \langle X \vee Y \rangle$ and $\langle X \rangle \wedge \langle Y \rangle = \langle X \wedge Y \rangle$. We put $0 = \langle 0 \rangle$ and $1 = \langle S \rangle$.
- We order Bousfield classes by reverse inclusion, so $\langle X \rangle \leq \langle Y \rangle$ means $\langle X \rangle \supseteq \langle Y \rangle$.

Basic definitions

- $\mathcal{B} = \{p - \text{local spectra}\}$.
- This is a triangulated category, and in particular is additive.
- There is a binary coproduct written $X \vee Y$, and more generally an indexed coproduct written $\bigvee_i X_i$.
- There is a bilinear symmetric monoidal smash product written $X \wedge Y$, with unit object S.
- All this is similar to the derived category $D(R)$ of a ring R, with $\vee$ like $\oplus$ and $\wedge$ like $\otimes$.
- $\langle X \rangle = \{T \mid X \wedge T = 0\}$ and $\mathcal{L} = \{\langle X \rangle \mid X \in \mathcal{B}\}$.
- Theorem of Ohkawa: $\mathcal{L}$ is a set, not a proper class.
- There are well-defined operations $\langle X \rangle \vee \langle Y \rangle = \langle X \vee Y \rangle$ and $\langle X \rangle \wedge \langle Y \rangle = \langle X \wedge Y \rangle$. We put $0 = \langle 0 \rangle$ and $1 = \langle S \rangle$.
- We order Bousfield classes by reverse inclusion, so $\langle X \rangle \leq \langle Y \rangle$ means $\langle X \rangle \supseteq \langle Y \rangle$.

Ordered semirings

An *ordered semiring* is a set $\mathcal{R}$ with $0, 1 \in \mathcal{R}$ and operations $\vee$ and $\wedge$ such that:

(a) $\vee$ is commutative and associative, with 0 as an identity element.

(b) $\wedge$ is commutative and associative, with 1 as an identity element.

(c) $\wedge$ distributes over $\vee$.

(d) For all $u \in \mathcal{R}$ we have $0 \wedge u = 0$ and $1 \vee u = 1$ and $u \vee u = u$.

- This gives a partial order by the rule $u \leq v$ iff $u \vee v = v$.
- The binary operations preserve this order, and 0 and 1 are the smallest and largest elements.
- $u \vee v$ is the smallest element satisfying $w \geq u$ and $w \geq v$.
- There is no similar statement for $u \wedge v$ in general.
- We say that $\mathcal{R}$ is *complete* if every family of elements $(u_i)_{i \in I}$ has least upper bound $\bigvee_i u_i$.
- We say that $\mathcal{R}$ is *completely distributive* if, in addition, $x \wedge \bigvee_i u_i = \bigvee_i (x \wedge u_i)$.
- The set $\mathcal{L}$ is naturally a completely distributive ordered semiring.

Ordered semirings

An *ordered semiring* is a set $\mathcal{R}$ with $0, 1 \in \mathcal{R}$ and operations $\vee$ and $\wedge$ such that:

(a) $\vee$ is commutative and associative, with 0 as an identity element.

(b) $\wedge$ is commutative and associative, with 1 as an identity element.

(c) $\wedge$ distributes over $\vee$.

(d) For all $u \in \mathcal{R}$ we have $0 \wedge u = 0$ and $1 \vee u = 1$ and $u \vee u = u$.

- This gives a partial order by the rule $u \leq v$ iff $u \vee v = v$.
- The binary operations preserve this order, and 0 and 1 are the smallest and largest elements.
- $u \vee v$ is the smallest element satisfying $w \geq u$ and $w \geq v$.
- There is no similar statement for $u \wedge v$ in general.
- We say that $\mathcal{R}$ is *complete* if every family of elements $(u_i)_{i \in I}$ has least upper bound $\bigvee_i u_i$.
- We say that $\mathcal{R}$ is *completely distributive* if, in addition, $x \wedge \bigvee_i u_i = \bigvee_i (x \wedge u_i)$.
- The set $\mathcal{L}$ is naturally a completely distributive ordered semiring.

Ordered semirings

An *ordered semiring* is a set $\mathcal{R}$ with $0, 1 \in \mathcal{R}$ and operations $\vee$ and $\wedge$ such that:

(a) $\vee$ is commutative and associative, with 0 as an identity element.

(b) $\wedge$ is commutative and associative, with 1 as an identity element.

(c) $\wedge$ distributes over $\vee$.

(d) For all $u \in \mathcal{R}$ we have $0 \wedge u = 0$ and $1 \vee u = 1$ and $u \vee u = u$.

- This gives a partial order by the rule $u \leq v$ iff $u \vee v = v$.
- The binary operations preserve this order, and 0 and 1 are the smallest and largest elements.
- $u \vee v$ is the smallest element satisfying $w \geq u$ and $w \geq v$.
- There is no similar statement for $u \wedge v$ in general.
- We say that $\mathcal{R}$ is *complete* if every family of elements $(u_i)_{i \in I}$ has least upper bound $\bigvee_i u_i$.
- We say that $\mathcal{R}$ is *completely distributive* if, in addition, $x \wedge \bigvee_i u_i = \bigvee_i (x \wedge u_i)$.
- The set $\mathcal{L}$ is naturally a completely distributive ordered semiring.

Ordered semirings

An *ordered semiring* is a set $\mathcal{R}$ with $0, 1 \in \mathcal{R}$ and operations $\vee$ and $\wedge$ such that:

(a) $\vee$ is commutative and associative, with 0 as an identity element.

(b) $\wedge$ is commutative and associative, with 1 as an identity element.

(c) $\wedge$ distributes over $\vee$.

(d) For all $u \in \mathcal{R}$ we have $0 \wedge u = 0$ and $1 \vee u = 1$ and $u \vee u = u$.

- This gives a partial order by the rule $u \leq v$ iff $u \vee v = v$.
- The binary operations preserve this order, and 0 and 1 are the smallest and largest elements.
- $u \vee v$ is the smallest element satisfying $w \geq u$ and $w \geq v$.
- There is no similar statement for $u \wedge v$ in general.
- We say that $\mathcal{R}$ is *complete* if every family of elements $(u_i)_{i \in I}$ has least upper bound $\bigvee_i u_i$.
- We say that $\mathcal{R}$ is *completely distributive* if, in addition, $x \wedge \bigvee_i u_i = \bigvee_i (x \wedge u_i)$.
- The set $\mathcal{L}$ is naturally a completely distributive ordered semiring.

Ordered semirings

An *ordered semiring* is a set $\mathcal{R}$ with $0, 1 \in \mathcal{R}$ and operations $\vee$ and $\wedge$ such that:

(a) $\vee$ is commutative and associative, with 0 as an identity element.

(b) $\wedge$ is commutative and associative, with 1 as an identity element.

(c) $\wedge$ distributes over $\vee$.

(d) For all $u \in \mathcal{R}$ we have $0 \wedge u = 0$ and $1 \vee u = 1$ and $u \vee u = u$.

- This gives a partial order by the rule $u \leq v$ iff $u \vee v = v$.
- The binary operations preserve this order, and 0 and 1 are the smallest and largest elements.
- $u \vee v$ is the smallest element satisfying $w \geq u$ and $w \geq v$.
- There is no similar statement for $u \wedge v$ in general.
- We say that $\mathcal{R}$ is *complete* if every family of elements $(u_i)_{i \in I}$ has least upper bound $\bigvee_i u_i$.
- We say that $\mathcal{R}$ is *completely distributive* if, in addition, $x \wedge \bigvee_i u_i = \bigvee_i (x \wedge u_i)$.
- The set $\mathcal{L}$ is naturally a completely distributive ordered semiring.

Ordered semirings

An *ordered semiring* is a set $\mathcal{R}$ with $0, 1 \in \mathcal{R}$ and operations $\vee$ and $\wedge$ such that:

(a) $\vee$ is commutative and associative, with 0 as an identity element.

(b) $\wedge$ is commutative and associative, with 1 as an identity element.

(c) $\wedge$ distributes over $\vee$.

(d) For all $u \in \mathcal{R}$ we have $0 \wedge u = 0$ and $1 \vee u = 1$ and $u \vee u = u$.

- This gives a partial order by the rule $u \leq v$ iff $u \vee v = v$.
- The binary operations preserve this order, and 0 and 1 are the smallest and largest elements.
- $u \vee v$ is the smallest element satisfying $w \geq u$ and $w \geq v$.
- There is no similar statement for $u \wedge v$ in general.
- We say that $\mathcal{R}$ is *complete* if every family of elements $(u_i)_{i \in I}$ has least upper bound $\bigvee_i u_i$.
- We say that $\mathcal{R}$ is *completely distributive* if, in addition, $x \wedge \bigvee_i u_i = \bigvee_i (x \wedge u_i)$.
- The set $\mathcal{L}$ is naturally a completely distributive ordered semiring.

Ordered semirings

An *ordered semiring* is a set $\mathcal{R}$ with $0, 1 \in \mathcal{R}$ and operations $\vee$ and $\wedge$ such that:

(a) $\vee$ is commutative and associative, with 0 as an identity element.

(b) $\wedge$ is commutative and associative, with 1 as an identity element.

(c) $\wedge$ distributes over $\vee$.

(d) For all $u \in \mathcal{R}$ we have $0 \wedge u = 0$ and $1 \vee u = 1$ and $u \vee u = u$.

- This gives a partial order by the rule $u \leq v$ iff $u \vee v = v$.
- The binary operations preserve this order, and 0 and 1 are the smallest and largest elements.
- $u \vee v$ is the smallest element satisfying $w \geq u$ and $w \geq v$.
- There is no similar statement for $u \wedge v$ in general.
- We say that $\mathcal{R}$ is *complete* if every family of elements $(u_i)_{i \in I}$ has least upper bound $\bigvee_i u_i$.
- We say that $\mathcal{R}$ is *completely distributive* if, in addition, $x \wedge \bigvee_i u_i = \bigvee_i (x \wedge u_i)$.
- The set $\mathcal{L}$ is naturally a completely distributive ordered semiring.

Ordered semirings

An *ordered semiring* is a set $\mathcal{R}$ with $0, 1 \in \mathcal{R}$ and operations $\vee$ and $\wedge$ such that:

(a) $\vee$ is commutative and associative, with 0 as an identity element.

(b) $\wedge$ is commutative and associative, with 1 as an identity element.

(c) $\wedge$ distributes over $\vee$.

(d) For all $u \in \mathcal{R}$ we have $0 \wedge u = 0$ and $1 \vee u = 1$ and $u \vee u = u$.

- This gives a partial order by the rule $u \leq v$ iff $u \vee v = v$.
- The binary operations preserve this order, and 0 and 1 are the smallest and largest elements.
- $u \vee v$ is the smallest element satisfying $w \geq u$ and $w \geq v$.
- There is no similar statement for $u \wedge v$ in general.
- We say that $\mathcal{R}$ is *complete* if every family of elements $(u_i)_{i \in I}$ has least upper bound $\bigvee_i u_i$.
- We say that $\mathcal{R}$ is *completely distributive* if, in addition, $x \wedge \bigvee_i u_i = \bigvee_i (x \wedge u_i)$.
- The set $\mathcal{L}$ is naturally a completely distributive ordered semiring.

Let $\mathcal{R}$ be an ordered semiring.

- We say that $u \in \mathcal{R}$ is *idempotent* if $u \wedge u = u$.
- We write $\mathcal{R}_{\text{latt}}$ for the set of idempotent elements. This is a subsemiring of $\mathcal{L}$ and is a distributive lattice.
- We say that $u \in \mathcal{R}$ is *complemented* if there is a (necessarily unique) element $\neg u$ with $u \vee \neg u = 1$ and $u \wedge \neg u = 0$.
- We write $\mathcal{R}_{\text{bool}}$ for the set of complemented elements. This is a sublattice of $\mathcal{R}_{\text{latt}}$ and is a Boolean algebra.
- If $e \in \mathcal{R}$ is idempotent then there is a semiring $\mathcal{R}/e$ and a homomorphism $\pi\colon \mathcal{R} \to \mathcal{R}/e$ that is initial among homomorphisms sending e to zero.
- In fact, we can take $\mathcal{R}/e = \{x \in \mathcal{R} \mid x \geq e\}$ and $\pi(x) = x \vee e$ and define operations on $\mathcal{R}/e$ so as to make π a homomorphism.
- $\overline{\mathcal{L}}$ will be a colimit of quotients $\mathcal{L}/\epsilon(n)$ for some idempotents $\epsilon(n)$ to be described later.

Let $\mathcal{R}$ be an ordered semiring.

- We say that $u \in \mathcal{R}$ is *idempotent* if $u \wedge u = u$.
- We write $\mathcal{R}_{\text{latt}}$ for the set of idempotent elements. This is a subsemiring of $\mathcal{L}$ and is a distributive lattice.
- We say that $u \in \mathcal{R}$ is *complemented* if there is a (necessarily unique) element $\neg u$ with $u \vee \neg u = 1$ and $u \wedge \neg u = 0$.
- We write $\mathcal{R}_{\text{bool}}$ for the set of complemented elements. This is a sublattice of $\mathcal{R}_{\text{latt}}$ and is a Boolean algebra.
- If $e \in \mathcal{R}$ is idempotent then there is a semiring $\mathcal{R}/e$ and a homomorphism $\pi\colon \mathcal{R} \to \mathcal{R}/e$ that is initial among homomorphisms sending e to zero.
- In fact, we can take $\mathcal{R}/e = \{x \in \mathcal{R} \mid x \geq e\}$ and $\pi(x) = x \vee e$ and define operations on $\mathcal{R}/e$ so as to make π a homomorphism.
- $\overline{\mathcal{L}}$ will be a colimit of quotients $\mathcal{L}/\epsilon(n)$ for some idempotents $\epsilon(n)$ to be described later.

Lattices and Boolean algebras

Let $\mathcal{R}$ be an ordered semiring.

- We say that $u \in \mathcal{R}$ is *idempotent* if $u \wedge u = u$.
- We write $\mathcal{R}_{\text{latt}}$ for the set of idempotent elements. This is a subsemiring of $\mathcal{L}$ and is a distributive lattice.
- We say that $u \in \mathcal{R}$ is *complemented* if there is a (necessarily unique) element $\neg u$ with $u \vee \neg u = 1$ and $u \wedge \neg u = 0$.
- We write $\mathcal{R}_{\text{bool}}$ for the set of complemented elements. This is a sublattice of $\mathcal{R}_{\text{latt}}$ and is a Boolean algebra.
- If $e \in \mathcal{R}$ is idempotent then there is a semiring $\mathcal{R}/e$ and a homomorphism $\pi \colon \mathcal{R} \to \mathcal{R}/e$ that is initial among homomorphisms sending e to zero.
- In fact, we can take $\mathcal{R}/e = \{x \in \mathcal{R} \mid x \geq e\}$ and $\pi(x) = x \vee e$ and define operations on $\mathcal{R}/e$ so as to make π a homomorphism.
- $\overline{\mathcal{L}}$ will be a colimit of quotients $\mathcal{L}/\epsilon(n)$ for some idempotents $\epsilon(n)$ to be described later.

Lattices and Boolean algebras

Let $\mathcal{R}$ be an ordered semiring.

- We say that $u \in \mathcal{R}$ is *idempotent* if $u \wedge u = u$.
- We write $\mathcal{R}_{\text{latt}}$ for the set of idempotent elements. This is a subsemiring of $\mathcal{L}$ and is a distributive lattice.
- We say that $u \in \mathcal{R}$ is *complemented* if there is a (necessarily unique) element $\neg u$ with $u \vee \neg u = 1$ and $u \wedge \neg u = 0$.
- We write $\mathcal{R}_{\text{bool}}$ for the set of complemented elements. This is a sublattice of $\mathcal{R}_{\text{latt}}$ and is a Boolean algebra.
- If $e \in \mathcal{R}$ is idempotent then there is a semiring $\mathcal{R}/e$ and a homomorphism $\pi \colon \mathcal{R} \to \mathcal{R}/e$ that is initial among homomorphisms sending e to zero.
- In fact, we can take $\mathcal{R}/e = \{x \in \mathcal{R} \mid x \geq e\}$ and $\pi(x) = x \vee e$ and define operations on $\mathcal{R}/e$ so as to make π a homomorphism.
- $\overline{\mathcal{L}}$ will be a colimit of quotients $\mathcal{L}/\epsilon(n)$ for some idempotents $\epsilon(n)$ to be described later.

Lattices and Boolean algebras

Let $\mathcal{R}$ be an ordered semiring.

- We say that $u \in \mathcal{R}$ is *idempotent* if $u \wedge u = u$.
- We write $\mathcal{R}_{\text{latt}}$ for the set of idempotent elements. This is a subsemiring of $\mathcal{L}$ and is a distributive lattice.
- We say that $u \in \mathcal{R}$ is *complemented* if there is a (necessarily unique) element $\neg u$ with $u \vee \neg u = 1$ and $u \wedge \neg u = 0$.
- We write $\mathcal{R}_{\text{bool}}$ for the set of complemented elements. This is a sublattice of $\mathcal{R}_{\text{latt}}$ and is a Boolean algebra.
- If $e \in \mathcal{R}$ is idempotent then there is a semiring $\mathcal{R}/e$ and a homomorphism $\pi \colon \mathcal{R} \to \mathcal{R}/e$ that is initial among homomorphisms sending e to zero.
- In fact, we can take $\mathcal{R}/e = \{x \in \mathcal{R} \mid x \geq e\}$ and $\pi(x) = x \vee e$ and define operations on $\mathcal{R}/e$ so as to make π a homomorphism.
- $\overline{\mathcal{L}}$ will be a colimit of quotients $\mathcal{L}/\epsilon(n)$ for some idempotents $\epsilon(n)$ to be described later.

Lattices and Boolean algebras

Let $\mathcal{R}$ be an ordered semiring.

- We say that $u \in \mathcal{R}$ is *idempotent* if $u \wedge u = u$.
- We write $\mathcal{R}_{\text{latt}}$ for the set of idempotent elements. This is a subsemiring of $\mathcal{L}$ and is a distributive lattice.
- We say that $u \in \mathcal{R}$ is *complemented* if there is a (necessarily unique) element $\neg u$ with $u \vee \neg u = 1$ and $u \wedge \neg u = 0$.
- We write $\mathcal{R}_{\text{bool}}$ for the set of complemented elements. This is a sublattice of $\mathcal{R}_{\text{latt}}$ and is a Boolean algebra.
- If $e \in \mathcal{R}$ is idempotent then there is a semiring $\mathcal{R}/e$ and a homomorphism $\pi \colon \mathcal{R} \to \mathcal{R}/e$ that is initial among homomorphisms sending e to zero.
- In fact, we can take $\mathcal{R}/e = \{x \in \mathcal{R} \mid x \geq e\}$ and $\pi(x) = x \vee e$ and define operations on $\mathcal{R}/e$ so as to make π a homomorphism.
- $\overline{\mathcal{L}}$ will be a colimit of quotients $\mathcal{L}/\epsilon(n)$ for some idempotents $\epsilon(n)$ to be described later.

Lattices and Boolean algebras

Let $\mathcal{R}$ be an ordered semiring.

- We say that $u \in \mathcal{R}$ is *idempotent* if $u \wedge u = u$.
- We write $\mathcal{R}_{\text{latt}}$ for the set of idempotent elements. This is a subsemiring of $\mathcal{L}$ and is a distributive lattice.
- We say that $u \in \mathcal{R}$ is *complemented* if there is a (necessarily unique) element $\neg u$ with $u \vee \neg u = 1$ and $u \wedge \neg u = 0$.
- We write $\mathcal{R}_{\text{bool}}$ for the set of complemented elements. This is a sublattice of $\mathcal{R}_{\text{latt}}$ and is a Boolean algebra.
- If $e \in \mathcal{R}$ is idempotent then there is a semiring $\mathcal{R}/e$ and a homomorphism $\pi\colon \mathcal{R} \to \mathcal{R}/e$ that is initial among homomorphisms sending e to zero.
- In fact, we can take $\mathcal{R}/e = \{x \in \mathcal{R} \mid x \geq e\}$ and $\pi(x) = x \vee e$ and define operations on $\mathcal{R}/e$ so as to make π a homomorphism.
- $\overline{\mathcal{L}}$ will be a colimit of quotients $\mathcal{L}/\epsilon(n)$ for some idempotents $\epsilon(n)$ to be described later.

Let $\mathcal{R}$ be an ordered semiring.

- We say that $u \in \mathcal{R}$ is *idempotent* if $u \wedge u = u$.
- We write $\mathcal{R}_{\text{latt}}$ for the set of idempotent elements. This is a subsemiring of $\mathcal{L}$ and is a distributive lattice.
- We say that $u \in \mathcal{R}$ is *complemented* if there is a (necessarily unique) element $\neg u$ with $u \vee \neg u = 1$ and $u \wedge \neg u = 0$.
- We write $\mathcal{R}_{\text{bool}}$ for the set of complemented elements. This is a sublattice of $\mathcal{R}_{\text{latt}}$ and is a Boolean algebra.
- If $e \in \mathcal{R}$ is idempotent then there is a semiring $\mathcal{R}/e$ and a homomorphism $\pi \colon \mathcal{R} \to \mathcal{R}/e$ that is initial among homomorphisms sending e to zero.
- In fact, we can take $\mathcal{R}/e = \{x \in \mathcal{R} \mid x \geq e\}$ and $\pi(x) = x \vee e$ and define operations on $\mathcal{R}/e$ so as to make π a homomorphism.
- $\overline{\mathcal{L}}$ will be a colimit of quotients $\mathcal{L}/\epsilon(n)$ for some idempotents $\epsilon(n)$ to be described later.

The combinatorial model

- $\mathbb{N}_\infty = \{0, 1, 2, 3, 4, \ldots, \infty\}$ $\mathbb{N}_\omega = \{0, 1, 2, 3, 4, \ldots, \omega, \infty\}$
- A set $S \subset \mathbb{N}_\infty$ is *small* if $S \subseteq [0, n)$ for some $n < \infty$, otherwise *large*.
- We say that $S \subseteq \mathbb{N}_\infty$ is *cosmall* if $\mathbb{N}_\infty \setminus S$ is small, or equivalently $S \supseteq [n, \infty]$ for some finite n.
- The set $\mathcal{A}$ has elements as follows:
  - $t(q, T)$ for $q \in \mathbb{N}_\infty$ and $T \subseteq \mathbb{N}_\infty$ cosmall.
  - $j(m, S)$ for $m \in \mathbb{N}_\omega$ and $S \subset \mathbb{N}_\infty$ small.
  - $k(U)$ for $U \subseteq \mathbb{N}_\infty$ arbitrary.
- The top and bottom elements are $1 = t(0, \mathbb{N}_\infty)$ and $0 = k(\emptyset)$.
- The set $\mathcal{P} = \{U \mid U \subseteq \mathbb{N}_\infty\}$ is an ordered semiring under $\cup$ and $\cap$.
- We define $\text{tail}: \mathcal{A} \to \mathcal{P}$ by $\text{tail}(t(q, T)) = T$ and $\text{tail}(j(m, S)) = S$ and $\text{tail}(k(U)) = U$. This will be a homomorphism.

The combinatorial model

- $\mathbb{N}_\infty = \{0, 1, 2, 3, 4, \dots, \infty\}$ $\quad$ $\mathbb{N}_\omega = \{0, 1, 2, 3, 4, \dots, \omega, \infty\}$
- A set $S \subset \mathbb{N}_\infty$ is *small* if $S \subseteq [0, n)$ for some $n < \infty$, otherwise *large*.
- We say that $S \subseteq \mathbb{N}_\infty$ is *cosmall* if $\mathbb{N}_\infty \setminus S$ is small, or equivalently $S \supseteq [n, \infty]$ for some finite n.
- The set $\mathcal{A}$ has elements as follows:
  - $t(q, T)$ for $q \in \mathbb{N}_\infty$ and $T \subseteq \mathbb{N}_\infty$ cosmall.
  - $j(m, S)$ for $m \in \mathbb{N}_\omega$ and $S \subset \mathbb{N}_\infty$ small.
  - $k(U)$ for $U \subseteq \mathbb{N}_\infty$ arbitrary.
- The top and bottom elements are $1 = t(0, \mathbb{N}_\infty)$ and $0 = k(\emptyset)$.
- The set $\mathcal{P} = \{U \mid U \subseteq \mathbb{N}_\infty\}$ is an ordered semiring under $\cup$ and $\cap$.
- We define $\text{tail}\colon \mathcal{A} \to \mathcal{P}$ by $\text{tail}(t(q, T)) = T$ and $\text{tail}(j(m, S)) = S$ and $\text{tail}(k(U)) = U$. This will be a homomorphism.

The combinatorial model

- $\mathbb{N}_\infty = \{0, 1, 2, 3, 4, \dots, \infty\}$ $\mathbb{N}_\omega = \{0, 1, 2, 3, 4, \dots, \omega, \infty\}$
- A set $S \subset \mathbb{N}_\infty$ is *small* if $S \subseteq [0, n)$ for some $n < \infty$, otherwise *large*.
- We say that $S \subseteq \mathbb{N}_\infty$ is *cosmall* if $\mathbb{N}_\infty \setminus S$ is small, or equivalently $S \supseteq [n, \infty]$ for some finite n.
- The set $\mathcal{A}$ has elements as follows:
  - $t(q, T)$ for $q \in \mathbb{N}_\infty$ and $T \subseteq \mathbb{N}_\infty$ cosmall.
  - $j(m, S)$ for $m \in \mathbb{N}_\omega$ and $S \subset \mathbb{N}_\infty$ small.
  - $k(U)$ for $U \subseteq \mathbb{N}_\infty$ arbitrary.
- The top and bottom elements are $1 = t(0, \mathbb{N}_\infty)$ and $0 = k(\emptyset)$.
- The set $\mathcal{P} = \{U \mid U \subseteq \mathbb{N}_\infty\}$ is an ordered semiring under $\cup$ and $\cap$.
- We define $\mathrm{tail}\colon \mathcal{A} \to \mathcal{P}$ by $\mathrm{tail}(t(q, T)) = T$ and $\mathrm{tail}(j(m, S)) = S$ and $\mathrm{tail}(k(U)) = U$. This will be a homomorphism.

The combinatorial model

- $\mathbb{N}_\infty = \{0, 1, 2, 3, 4, \ldots, \infty\}$ $\qquad$ $\mathbb{N}_\omega = \{0, 1, 2, 3, 4, \ldots, \omega, \infty\}$
- A set $S \subset \mathbb{N}_\infty$ is *small* if $S \subseteq [0, n)$ for some $n < \infty$, otherwise *large*.
- We say that $S \subseteq \mathbb{N}_\infty$ is *cosmall* if $\mathbb{N}_\infty \setminus S$ is small, or equivalently $S \supseteq [n, \infty]$ for some finite n.
- The set $\mathcal{A}$ has elements as follows:
  - $t(q, T)$ for $q \in \mathbb{N}_\infty$ and $T \subseteq \mathbb{N}_\infty$ cosmall.
  - $j(m, S)$ for $m \in \mathbb{N}_\omega$ and $S \subset \mathbb{N}_\infty$ small.
  - $k(U)$ for $U \subseteq \mathbb{N}_\infty$ arbitrary.
- The top and bottom elements are $1 = t(0, \mathbb{N}_\infty)$ and $0 = k(\emptyset)$.
- The set $\mathcal{P} = \{U \mid U \subseteq \mathbb{N}_\infty\}$ is an ordered semiring under $\cup$ and $\cap$.
- We define tail: $\mathcal{A} \to \mathcal{P}$ by $\text{tail}(t(q, T)) = T$ and $\text{tail}(j(m, S)) = S$ and $\text{tail}(k(U)) = U$. This will be a homomorphism.

The combinatorial model

- $\mathbb{N}_\infty = \{0, 1, 2, 3, 4, \dots, \infty\}$ $\qquad$ $\mathbb{N}_\omega = \{0, 1, 2, 3, 4, \dots, \omega, \infty\}$
- A set $S \subset \mathbb{N}_\infty$ is *small* if $S \subseteq [0, n)$ for some $n < \infty$, otherwise *large*.
- We say that $S \subseteq \mathbb{N}_\infty$ is *cosmall* if $\mathbb{N}_\infty \setminus S$ is small, or equivalently $S \supseteq [n, \infty]$ for some finite n.
- The set $\mathcal{A}$ has elements as follows:
  - $t(q, T)$ for $q \in \mathbb{N}_\infty$ and $T \subseteq \mathbb{N}_\infty$ cosmall.
  - $j(m, S)$ for $m \in \mathbb{N}_\omega$ and $S \subset \mathbb{N}_\infty$ small.
  - $k(U)$ for $U \subseteq \mathbb{N}_\infty$ arbitrary.
- The top and bottom elements are $1 = t(0, \mathbb{N}_\infty)$ and $0 = k(\emptyset)$.
- The set $\mathcal{P} = \{U \mid U \subseteq \mathbb{N}_\infty\}$ is an ordered semiring under $\cup$ and $\cap$.
- We define $\text{tail}\colon \mathcal{A} \to \mathcal{P}$ by $\text{tail}(t(q, T)) = T$ and $\text{tail}(j(m, S)) = S$ and $\text{tail}(k(U)) = U$. This will be a homomorphism.

The combinatorial model

- $\mathbb{N}_\infty = \{0, 1, 2, 3, 4, \dots, \infty\}$ $\qquad$ $\mathbb{N}_\omega = \{0, 1, 2, 3, 4, \dots, \omega, \infty\}$
- A set $S \subset \mathbb{N}_\infty$ is *small* if $S \subseteq [0, n)$ for some $n < \infty$, otherwise *large*.
- We say that $S \subseteq \mathbb{N}_\infty$ is *cosmall* if $\mathbb{N}_\infty \setminus S$ is small, or equivalently $S \supseteq [n, \infty]$ for some finite n.
- The set $\mathcal{A}$ has elements as follows:
  - $t(q, T)$ for $q \in \mathbb{N}_\infty$ and $T \subseteq \mathbb{N}_\infty$ cosmall.
  - $j(m, S)$ for $m \in \mathbb{N}_\omega$ and $S \subset \mathbb{N}_\infty$ small.
  - $k(U)$ for $U \subseteq \mathbb{N}_\infty$ arbitrary.
- The top and bottom elements are $1 = t(0, \mathbb{N}_\infty)$ and $0 = k(\emptyset)$.
- The set $\mathcal{P} = \{U \mid U \subseteq \mathbb{N}_\infty\}$ is an ordered semiring under $\cup$ and $\cap$.
- We define $\mathrm{tail}\colon \mathcal{A} \to \mathcal{P}$ by $\mathrm{tail}(t(q, T)) = T$ and $\mathrm{tail}(j(m, S)) = S$ and $\mathrm{tail}(k(U)) = U$. This will be a homomorphism.

The combinatorial model

- $\mathbb{N}_\infty = \{0, 1, 2, 3, 4, \ldots, \infty\}$ $\qquad$ $\mathbb{N}_\omega = \{0, 1, 2, 3, 4, \ldots, \omega, \infty\}$
- A set $S \subset \mathbb{N}_\infty$ is *small* if $S \subseteq [0, n)$ for some $n < \infty$, otherwise *large*.
- We say that $S \subseteq \mathbb{N}_\infty$ is *cosmall* if $\mathbb{N}_\infty \setminus S$ is small, or equivalently $S \supseteq [n, \infty]$ for some finite n.
- The set $\mathcal{A}$ has elements as follows:
  - $t(q, T)$ for $q \in \mathbb{N}_\infty$ and $T \subseteq \mathbb{N}_\infty$ cosmall.
  - $j(m, S)$ for $m \in \mathbb{N}_\omega$ and $S \subset \mathbb{N}_\infty$ small.
  - $k(U)$ for $U \subseteq \mathbb{N}_\infty$ arbitrary.
- The top and bottom elements are $1 = t(0, \mathbb{N}_\infty)$ and $0 = k(\emptyset)$.
- The set $\mathcal{P} = \{U \mid U \subseteq \mathbb{N}_\infty\}$ is an ordered semiring under $\cup$ and $\cap$.
- We define $\mathrm{tail}\colon \mathcal{A} \to \mathcal{P}$ by $\mathrm{tail}(t(q, T)) = T$ and $\mathrm{tail}(j(m, S)) = S$ and $\mathrm{tail}(k(U)) = U$. This will be a homomorphism.

The combinatorial model

- $\mathbb{N}_\infty = \{0, 1, 2, 3, 4, \ldots, \infty\}$ $\qquad$ $\mathbb{N}_\omega = \{0, 1, 2, 3, 4, \ldots, \omega, \infty\}$
- A set $S \subset \mathbb{N}_\infty$ is *small* if $S \subseteq [0, n)$ for some $n < \infty$, otherwise *large*.
- We say that $S \subseteq \mathbb{N}_\infty$ is *cosmall* if $\mathbb{N}_\infty \setminus S$ is small, or equivalently $S \supseteq [n, \infty]$ for some finite n.
- The set $\mathcal{A}$ has elements as follows:
  - $t(q, T)$ for $q \in \mathbb{N}_\infty$ and $T \subseteq \mathbb{N}_\infty$ cosmall.
  - $j(m, S)$ for $m \in \mathbb{N}_\omega$ and $S \subset \mathbb{N}_\infty$ small.
  - $k(U)$ for $U \subseteq \mathbb{N}_\infty$ arbitrary.
- The top and bottom elements are $1 = t(0, \mathbb{N}_\infty)$ and $0 = k(\emptyset)$.
- The set $\mathcal{P} = \{U \mid U \subseteq \mathbb{N}_\infty\}$ is an ordered semiring under $\cup$ and $\cap$.
- We define $\text{tail}\colon \mathcal{A} \to \mathcal{P}$ by $\text{tail}(t(q, T)) = T$ and $\text{tail}(j(m, S)) = S$ and $\text{tail}(k(U)) = U$. This will be a homomorphism.

The combinatorial model: addition

- $t(q, T)$ for $q \in \mathbb{N}_\infty$ and $T \subseteq \mathbb{N}_\infty$ cosmall.
- $j(m, S)$ for $m \in \mathbb{N}_\omega$ and $S \subset \mathbb{N}_\infty$ small.
- $k(U)$ for $U \subseteq \mathbb{N}_\infty$ arbitrary.

$$\begin{aligned}
t(q, T) \vee t(q', T') &= t(\min(q, q'), T \cup T') \\
t(q, T) \vee j(m', S') &= t(q, T \cup S') \\
t(q, T) \vee k(U') &= t(q, T \cup U') \\
j(m, S) \vee j(m', S') &= j(\max(m, m'), S \cup S') \\
j(m, S) \vee k(U') &= \begin{cases} j(m, S \cup U') & \text{if } U' \text{ is small} \\ k(S \cup U') & \text{if } U' \text{ is big} \end{cases} \\
k(U) \vee k(U') &= k(U \cup U').
\end{aligned}$$

Note that $\mathrm{tail}(a \vee b) = \mathrm{tail}(a) \cup \mathrm{tail}(b)$.

The combinatorial model: multiplication

- $t(q, T)$ for $q \in \mathbb{N}_\infty$ and $T \subseteq \mathbb{N}_\infty$ cosmall.
- $j(m, S)$ for $m \in \mathbb{N}_\omega$ and $S \subset \mathbb{N}_\infty$ small.
- $k(U)$ for $U \subseteq \mathbb{N}_\infty$ arbitrary.

$$\begin{aligned}
t(q, T) \wedge t(q', T') &= t(\max(q, q'), T \cap T') \\
t(q, T) \wedge j(m', S') &= \begin{cases} j(m', T \cap S') & \text{if } q \leq m' \\ k(T \cap S') & \text{if } q > m' \end{cases} \\
t(q, T) \wedge k(U') &= k(T \cap U') \\
j(m, S) \wedge j(m', S') &= k(S \cap S') \\
j(m, S) \wedge k(U') &= k(S \cap U') \\
k(U) \wedge k(U') &= k(U \cap U').
\end{aligned}$$

Note that $\text{tail}(a \wedge b) = \text{tail}(a) \cap \text{tail}(b)$.

$\mathcal{A}$ is a semiring

Theorem: these operations make $\mathcal{A}$ a completely distributive ordered semiring.

$$t(q, T) \vee t(q', T') = t(\min(q, q'), T \cup T') \qquad t(q, T) \wedge t(q', T') = t(\max(q, q'), T \cap T')$$

$$t(q, T) \vee j(m', S') = t(q, T \cup S') \qquad t(q, T) \wedge j(m', S') = \begin{cases} j(m', T \cap S') & \text{if } q \leq m' \\ k(T \cap S') & \text{if } q > m' \end{cases}$$

$$t(q, T) \vee k(U') = t(q, T \cup U') \qquad t(q, T) \wedge k(U') = k(T \cap U')$$

$$j(m, S) \vee j(m', S') = j(\max(m, m'), S \cup S') \qquad j(m, S) \wedge j(m', S') = k(S \cap S')$$

$$j(m, S) \vee k(U') = \begin{cases} j(m, S \cup U') & \text{if } U' \text{ is small} \\ k(S \cup U') & \text{if } U' \text{ is big} \end{cases} \qquad j(m, S) \wedge k(U') = k(S \cap U')$$

$$k(U) \vee k(U') = k(U \cup U') \qquad k(U) \wedge k(U') = k(U \cap U').$$

Outline of proof:

- It is long but straightforward to check that the operations satisfy all axioms for an ordered semiring.
- A lemma shows that complete distributivity reduces to a statement about least upper bounds for ideals.
- Ideals in $\mathcal{A}$ have a fairly simple structure. In many cases, they have a largest element, which makes other questions trivial.

$\mathcal{A}$ is a semiring

Theorem: these operations make $\mathcal{A}$ a completely distributive ordered semiring.

$$t(q,T) \vee t(q',T') = t(\min(q,q'), T \cup T')$$

$$t(q,T) \vee j(m',S') = t(q, T \cup S')$$

$$t(q,T) \vee k(U') = t(q, T \cup U')$$

$$j(m,S) \vee j(m',S') = j(\max(m,m'), S \cup S')$$

$$j(m,S) \vee k(U') = \begin{cases} j(m, S \cup U') & \text{if } U' \text{ is small} \\ k(S \cup U') & \text{if } U' \text{ is big} \end{cases}$$

$$k(U) \vee k(U') = k(U \cup U')$$

$$t(q,T) \wedge t(q',T') = t(\max(q,q'), T \cap T')$$

$$t(q,T) \wedge j(m',S') = \begin{cases} j(m', T \cap S') & \text{if } q \le m' \\ k(T \cap S') & \text{if } q > m' \end{cases}$$

$$t(q,T) \wedge k(U') = k(T \cap U')$$

$$j(m,S) \wedge j(m',S') = k(S \cap S')$$

$$j(m,S) \wedge k(U') = k(S \cap U')$$

$$k(U) \wedge k(U') = k(U \cap U').$$

Outline of proof:

- It is long but straightforward to check that the operations satisfy all axioms for an ordered semiring.
- A lemma shows that complete distributivity reduces to a statement about least upper bounds for ideals.
- Ideals in $\mathcal{A}$ have a fairly simple structure. In many cases, they have a largest element, which makes other questions trivial.

$\mathcal{A}$ is a semiring

Theorem: these operations make $\mathcal{A}$ a completely distributive ordered semiring.

$$t(q,T) \vee t(q',T') = t(\min(q,q'), T \cup T')$$

$$t(q,T) \vee j(m',S') = t(q, T \cup S')$$

$$t(q,T) \vee k(U') = t(q, T \cup U')$$

$$j(m,S) \vee j(m',S') = j(\max(m,m'), S \cup S')$$

$$j(m,S) \vee k(U') = \begin{cases} j(m, S \cup U') & \text{if } U' \text{ is small} \\ k(S \cup U') & \text{if } U' \text{ is big} \end{cases}$$

$$k(U) \vee k(U') = k(U \cup U')$$

$$t(q,T) \wedge t(q',T') = t(\max(q,q'), T \cap T')$$

$$t(q,T) \wedge j(m',S') = \begin{cases} j(m', T \cap S') & \text{if } q \leq m' \\ k(T \cap S') & \text{if } q > m' \end{cases}$$

$$t(q,T) \wedge k(U') = k(T \cap U')$$

$$j(m,S) \wedge j(m',S') = k(S \cap S')$$

$$j(m,S) \wedge k(U') = k(S \cap U')$$

$$k(U) \wedge k(U') = k(U \cap U').$$

Outline of proof:

- It is long but straightforward to check that the operations satisfy all axioms for an ordered semiring.
- A lemma shows that complete distributivity reduces to a statement about least upper bounds for ideals.
- Ideals in $\mathcal{A}$ have a fairly simple structure. In many cases, they have a largest element, which makes other questions trivial.

$\mathcal{A}$ is a semiring

Theorem: these operations make $\mathcal{A}$ a completely distributive ordered semiring.

$$t(q,T) \vee t(q',T') = t(\min(q,q'), T \cup T')$$

$$t(q,T) \vee j(m',S') = t(q, T \cup S')$$

$$t(q,T) \vee k(U') = t(q, T \cup U')$$

$$j(m,S) \vee j(m',S') = j(\max(m,m'), S \cup S')$$

$$j(m,S) \vee k(U') = \begin{cases} j(m, S \cup U') & \text{if } U' \text{ is small} \\ k(S \cup U') & \text{if } U' \text{ is big} \end{cases}$$

$$k(U) \vee k(U') = k(U \cup U')$$

$$t(q,T) \wedge t(q',T') = t(\max(q,q'), T \cap T')$$

$$t(q,T) \wedge j(m',S') = \begin{cases} j(m', T \cap S') & \text{if } q \leq m' \\ k(T \cap S') & \text{if } q > m' \end{cases}$$

$$t(q,T) \wedge k(U') = k(T \cap U')$$

$$j(m,S) \wedge j(m',S') = k(S \cap S')$$

$$j(m,S) \wedge k(U') = k(S \cap U')$$

$$k(U) \wedge k(U') = k(U \cap U').$$

Outline of proof:

- It is long but straightforward to check that the operations satisfy all axioms for an ordered semiring.
- A lemma shows that complete distributivity reduces to a statement about least upper bounds for ideals.
- Ideals in $\mathcal{A}$ have a fairly simple structure. In many cases, they have a largest element, which makes other questions trivial.

The order on $\mathcal{A}$

The order on $\mathcal{A}$ can be made more explicit as follows:

- We have $t(q, T) \leq t(q', T')$ iff $T \subseteq T'$ and $q \geq q'$.
- We never have $t(q, T) \leq j(m, S)$ or $t(q, T) \leq k(U)$.
- We have $j(m, S) \leq t(q, T)$ iff $S \subseteq T$.
- We have $j(m, S) \leq j(m', S')$ iff $S \subseteq S'$ and $m \leq m'$.
- We have $j(m, S) \leq k(U)$ iff $S \subseteq U$ and U is big.
- We have $k(U) \leq t(q, T)$ iff $U \subseteq T$.
- We have $k(U) \leq j(m, S)$ iff $U \subseteq S$.
- We have $k(U) \leq k(U')$ iff $U \subseteq U'$.

- $\mathcal{A}_{\text{latt}}$ consists of all elements of the form $t(q, T)$ or $k(U)$.
- $\mathcal{A}_{\text{bool}}$ consists of all elements of the form $k(S)$ or $\neg k(S) = t(0, \mathbb{N}_\infty \setminus S)$ with S small.

The order on $\mathcal{A}$

The order on $\mathcal{A}$ can be made more explicit as follows:

- We have $t(q, T) \leq t(q', T')$ iff $T \subseteq T'$ and $q \geq q'$.
- We never have $t(q, T) \leq j(m, S)$ or $t(q, T) \leq k(U)$.
- We have $j(m, S) \leq t(q, T)$ iff $S \subseteq T$.
- We have $j(m, S) \leq j(m', S')$ iff $S \subseteq S'$ and $m \leq m'$.
- We have $j(m, S) \leq k(U)$ iff $S \subseteq U$ and U is big.
- We have $k(U) \leq t(q, T)$ iff $U \subseteq T$.
- We have $k(U) \leq j(m, S)$ iff $U \subseteq S$.
- We have $k(U) \leq k(U')$ iff $U \subseteq U'$.

- $\mathcal{A}_{\text{latt}}$ consists of all elements of the form $t(q, T)$ or $k(U)$.
- $\mathcal{A}_{\text{bool}}$ consists of all elements of the form $k(S)$ or $\neg k(S) = t(0, \mathbb{N}_\infty \setminus S)$ with S small.

The order on $\mathcal{A}$

The order on $\mathcal{A}$ can be made more explicit as follows:

- We have $t(q, T) \leq t(q', T')$ iff $T \subseteq T'$ and $q \geq q'$.
- We never have $t(q, T) \leq j(m, S)$ or $t(q, T) \leq k(U)$.
- We have $j(m, S) \leq t(q, T)$ iff $S \subseteq T$.
- We have $j(m, S) \leq j(m', S')$ iff $S \subseteq S'$ and $m \leq m'$.
- We have $j(m, S) \leq k(U)$ iff $S \subseteq U$ and U is big.
- We have $k(U) \leq t(q, T)$ iff $U \subseteq T$.
- We have $k(U) \leq j(m, S)$ iff $U \subseteq S$.
- We have $k(U) \leq k(U')$ iff $U \subseteq U'$.

- $\mathcal{A}_{\text{latt}}$ consists of all elements of the form $t(q, T)$ or $k(U)$.
- $\mathcal{A}_{\text{bool}}$ consists of all elements of the form $k(S)$ or $\neg k(S) = t(0, \mathbb{N}_\infty \setminus S)$ with S small.

The order on $\mathcal{A}$

The order on $\mathcal{A}$ can be made more explicit as follows:

- We have $t(q,T) \leq t(q',T')$ iff $T \subseteq T'$ and $q \geq q'$.
- We never have $t(q,T) \leq j(m,S)$ or $t(q,T) \leq k(U)$.
- We have $j(m,S) \leq t(q,T)$ iff $S \subseteq T$.
- We have $j(m,S) \leq j(m',S')$ iff $S \subseteq S'$ and $m \leq m'$.
- We have $j(m,S) \leq k(U)$ iff $S \subseteq U$ and U is big.
- We have $k(U) \leq t(q,T)$ iff $U \subseteq T$.
- We have $k(U) \leq j(m,S)$ iff $U \subseteq S$.
- We have $k(U) \leq k(U')$ iff $U \subseteq U'$.

- $\mathcal{A}_{\text{latt}}$ consists of all elements of the form $t(q,T)$ or $k(U)$.
- $\mathcal{A}_{\text{bool}}$ consists of all elements of the form $k(S)$ or $\neg k(S) = t(0, \mathbb{N}_\infty \setminus S)$ with S small.

The order on $\mathcal{A}$

The order on $\mathcal{A}$ can be made more explicit as follows:

- We have $t(q, T) \leq t(q', T')$ iff $T \subseteq T'$ and $q \geq q'$.
- We never have $t(q, T) \leq j(m, S)$ or $t(q, T) \leq k(U)$.
- We have $j(m, S) \leq t(q, T)$ iff $S \subseteq T$.
- We have $j(m, S) \leq j(m', S')$ iff $S \subseteq S'$ and $m \leq m'$.
- We have $j(m, S) \leq k(U)$ iff $S \subseteq U$ and U is big.
- We have $k(U) \leq t(q, T)$ iff $U \subseteq T$.
- We have $k(U) \leq j(m, S)$ iff $U \subseteq S$.
- We have $k(U) \leq k(U')$ iff $U \subseteq U'$.

- $\mathcal{A}_{\text{latt}}$ consists of all elements of the form $t(q, T)$ or $k(U)$.
- $\mathcal{A}_{\text{bool}}$ consists of all elements of the form $k(S)$ or $\neg k(S) = t(0, \mathbb{N}_\infty \setminus S)$ with S small.

The order on $\mathcal{A}$

The order on $\mathcal{A}$ can be made more explicit as follows:

- We have $t(q, T) \leq t(q', T')$ iff $T \subseteq T'$ and $q \geq q'$.
- We never have $t(q, T) \leq j(m, S)$ or $t(q, T) \leq k(U)$.
- We have $j(m, S) \leq t(q, T)$ iff $S \subseteq T$.
- We have $j(m, S) \leq j(m', S')$ iff $S \subseteq S'$ and $m \leq m'$.
- We have $j(m, S) \leq k(U)$ iff $S \subseteq U$ and U is big.
- We have $k(U) \leq t(q, T)$ iff $U \subseteq T$.
- We have $k(U) \leq j(m, S)$ iff $U \subseteq S$.
- We have $k(U) \leq k(U')$ iff $U \subseteq U'$.

- $\mathcal{A}_{\text{latt}}$ consists of all elements of the form $t(q, T)$ or $k(U)$.
- $\mathcal{A}_{\text{bool}}$ consists of all elements of the form $k(S)$ or $\neg k(S) = t(0, \mathbb{N}_\infty \setminus S)$ with S small.

The order on $\mathcal{A}$

The order on $\mathcal{A}$ can be made more explicit as follows:

- We have $t(q,T) \leq t(q',T')$ iff $T \subseteq T'$ and $q \geq q'$.
- We never have $t(q,T) \leq j(m,S)$ or $t(q,T) \leq k(U)$.
- We have $j(m,S) \leq t(q,T)$ iff $S \subseteq T$.
- We have $j(m,S) \leq j(m',S')$ iff $S \subseteq S'$ and $m \leq m'$.
- We have $j(m,S) \leq k(U)$ iff $S \subseteq U$ and U is big.
- We have $k(U) \leq t(q,T)$ iff $U \subseteq T$.
- We have $k(U) \leq j(m,S)$ iff $U \subseteq S$.
- We have $k(U) \leq k(U')$ iff $U \subseteq U'$.

- $\mathcal{A}_{\text{latt}}$ consists of all elements of the form $t(q,T)$ or $k(U)$.
- $\mathcal{A}_{\text{bool}}$ consists of all elements of the form $k(S)$ or $\neg k(S) = t(0, \mathbb{N}_\infty \setminus S)$ with S small.

The order on $\mathcal{A}$

The order on $\mathcal{A}$ can be made more explicit as follows:

- We have $t(q, T) \leq t(q', T')$ iff $T \subseteq T'$ and $q \geq q'$.
- We never have $t(q, T) \leq j(m, S)$ or $t(q, T) \leq k(U)$.
- We have $j(m, S) \leq t(q, T)$ iff $S \subseteq T$.
- We have $j(m, S) \leq j(m', S')$ iff $S \subseteq S'$ and $m \leq m'$.
- We have $j(m, S) \leq k(U)$ iff $S \subseteq U$ and U is big.
- We have $k(U) \leq t(q, T)$ iff $U \subseteq T$.
- We have $k(U) \leq j(m, S)$ iff $U \subseteq S$.
- We have $k(U) \leq k(U')$ iff $U \subseteq U'$.

- $\mathcal{A}_{\text{latt}}$ consists of all elements of the form $t(q, T)$ or $k(U)$.
- $\mathcal{A}_{\text{bool}}$ consists of all elements of the form $k(S)$ or $\neg k(S) = t(0, \mathbb{N}_\infty \setminus S)$ with S small.

The order on $\mathcal{A}$

The order on $\mathcal{A}$ can be made more explicit as follows:

- We have $t(q, T) \leq t(q', T')$ iff $T \subseteq T'$ and $q \geq q'$.
- We never have $t(q, T) \leq j(m, S)$ or $t(q, T) \leq k(U)$.
- We have $j(m, S) \leq t(q, T)$ iff $S \subseteq T$.
- We have $j(m, S) \leq j(m', S')$ iff $S \subseteq S'$ and $m \leq m'$.
- We have $j(m, S) \leq k(U)$ iff $S \subseteq U$ and U is big.
- We have $k(U) \leq t(q, T)$ iff $U \subseteq T$.
- We have $k(U) \leq j(m, S)$ iff $U \subseteq S$.
- We have $k(U) \leq k(U')$ iff $U \subseteq U'$.

- $\mathcal{A}_{\text{latt}}$ consists of all elements of the form $t(q, T)$ or $k(U)$.
- $\mathcal{A}_{\text{bool}}$ consists of all elements of the form $k(S)$ or $\neg k(S) = t(0, \mathbb{N}_\infty \setminus S)$ with S small.

The order on $\mathcal{A}$

The order on $\mathcal{A}$ can be made more explicit as follows:

- We have $t(q,T) \leq t(q',T')$ iff $T \subseteq T'$ and $q \geq q'$.
- We never have $t(q,T) \leq j(m,S)$ or $t(q,T) \leq k(U)$.
- We have $j(m,S) \leq t(q,T)$ iff $S \subseteq T$.
- We have $j(m,S) \leq j(m',S')$ iff $S \subseteq S'$ and $m \leq m'$.
- We have $j(m,S) \leq k(U)$ iff $S \subseteq U$ and U is big.
- We have $k(U) \leq t(q,T)$ iff $U \subseteq T$.
- We have $k(U) \leq j(m,S)$ iff $U \subseteq S$.
- We have $k(U) \leq k(U')$ iff $U \subseteq U'$.

- $\mathcal{A}_{\text{latt}}$ consists of all elements of the form $t(q,T)$ or $k(U)$.
- $\mathcal{A}_{\text{bool}}$ consists of all elements of the form $k(S)$ or $\neg k(S) = t(0, \mathbb{N}_\infty \setminus S)$ with S small.

The order on $\mathcal{A}$

The order on $\mathcal{A}$ can be made more explicit as follows:

- We have $t(q, T) \leq t(q', T')$ iff $T \subseteq T'$ and $q \geq q'$.
- We never have $t(q, T) \leq j(m, S)$ or $t(q, T) \leq k(U)$.
- We have $j(m, S) \leq t(q, T)$ iff $S \subseteq T$.
- We have $j(m, S) \leq j(m', S')$ iff $S \subseteq S'$ and $m \leq m'$.
- We have $j(m, S) \leq k(U)$ iff $S \subseteq U$ and U is big.
- We have $k(U) \leq t(q, T)$ iff $U \subseteq T$.
- We have $k(U) \leq j(m, S)$ iff $U \subseteq S$.
- We have $k(U) \leq k(U')$ iff $U \subseteq U'$.

- $\mathcal{A}_{\text{latt}}$ consists of all elements of the form $t(q, T)$ or $k(U)$.
- $\mathcal{A}_{\text{bool}}$ consists of all elements of the form $k(S)$ or $\neg k(S) = t(0, \mathbb{N}_\infty \setminus S)$ with S small.

The main theorem

- Theorem: There is an injective semiring homomorphism $\phi\colon \mathcal{A} \to \overline{\mathcal{L}}$ which preserves all joins.
- This is defined as a composite $\mathcal{A} \xrightarrow{\phi_0} \mathcal{L} \xrightarrow{\pi} \overline{\mathcal{L}}$, but ϕ_0 is not a homomorphism of semirings unless TC holds.
- For each element x in $\mathcal{A}$, we will define an element in $\mathcal{L}$ with the same name, which will be the image of x under ϕ_0.

The main theorem

- Theorem: There is an injective semiring homomorphism $\phi\colon \mathcal{A} \to \overline{\mathcal{L}}$ which preserves all joins.
- This is defined as a composite $\mathcal{A} \xrightarrow{\phi_0} \mathcal{L} \xrightarrow{\pi} \overline{\mathcal{L}}$, but ϕ_0 is not a homomorphism of semirings unless TC holds.
- For each element x in $\mathcal{A}$, we will define an element in $\mathcal{L}$ with the same name, which will be the image of x under ϕ_0.

The main theorem

- Theorem: There is an injective semiring homomorphism $\phi\colon \mathcal{A} \to \overline{\mathcal{L}}$ which preserves all joins.
- This is defined as a composite $\mathcal{A} \xrightarrow{\phi_0} \mathcal{L} \xrightarrow{\pi} \overline{\mathcal{L}}$, but ϕ_0 is not a homomorphism of semirings unless TC holds.
- For each element x in $\mathcal{A}$, we will define an element in $\mathcal{L}$ with the same name, which will be the image of x under ϕ_0.

The main theorem

- Theorem: There is an injective semiring homomorphism $\phi\colon \mathcal{A} \to \overline{\mathcal{L}}$ which preserves all joins.
- This is defined as a composite $\mathcal{A} \xrightarrow{\phi_0} \mathcal{L} \xrightarrow{\pi} \overline{\mathcal{L}}$, but ϕ_0 is not a homomorphism of semirings unless TC holds.
- For each element x in $\mathcal{A}$, we will define an element in $\mathcal{L}$ with the same name, which will be the image of x under ϕ_0.

Basic Bousfield classes

- $K(n)$ = Morava K-theory ($K(0) = H\mathbb{Q}$, $K(\infty) = H/p$); $k(n) = \langle K(n) \rangle$.
- For $U \subseteq \mathbb{N}_\infty$ we put $K(U) = \bigvee_{i \in U} K(i)$ and $k(U) = \langle K(U) \rangle$.
- $F(n)$ = a finite spectrum of type n, so $K(i)_* F(n) = 0$ iff $i < n$. This can be chosen so $F(n)$ is a self-dual ring spectrum and $F(0) = S$. Put $f(n) = \langle F(n) \rangle$.
- For $q \in \mathbb{N}$ we recall that the Bott periodicity isomorphism $\Omega SU = BU$ gives a natural virtual vector bundle over $\Omega SU(p^q)$, and the associated Thom spectrum $X(p^q)$ has a natural ring structure. The p-localisation of this has a p-typical summand called $T(q)$. We have $T(0) = S$ and $T(\infty) = BP$. In all cases we put $t(q) = \langle T(q) \rangle$ and $t(q; n) = t(q) \wedge f(n)$.
- Suppose $q \in \mathbb{N}_\infty$ and $T \subseteq \mathbb{N}_\infty$ is cosmall.
  If $[n, \infty] \subseteq T$, we put $t(q, T; n) = t(q; n) \vee k(T)$.
  Put $t(q, T) = t(q, T; n_0)$, where n_0 is smallest such that $[n_0, \infty] \subseteq T$.
- For $m \in \mathbb{N}_\infty$ we let $J(m)$ denote the Brown-Comenetz dual of $T(m)$, so there is a natural isomorphism

  $$[X, J(m)] \simeq \mathrm{Hom}(\pi_0(T(m) \wedge X), \mathbb{Q}/\mathbb{Z}).$$

  Put $J(\omega) = \bigvee_{m \in \mathbb{N}} J(m)$, and $j(m) = \langle J(m) \rangle$ for all $m \in \mathbb{N}_\omega$.
  Given a small set S, put $j(m, S) = j(m) \vee k(S)$.

Basic Bousfield classes

- $K(n)$ = Morava K-theory ($K(0) = H\mathbb{Q}$, $K(\infty) = H/p$); $k(n) = \langle K(n) \rangle$.
- For $U \subseteq \mathbb{N}_\infty$ we put $K(U) = \bigvee_{i \in U} K(i)$ and $k(U) = \langle K(U) \rangle$.
- $F(n)$ = a finite spectrum of type n, so $K(i)_* F(n) = 0$ iff $i < n$. This can be chosen so $F(n)$ is a self-dual ring spectrum and $F(0) = S$. Put $f(n) = \langle F(n) \rangle$.
- For $q \in \mathbb{N}$ we recall that the Bott periodicity isomorphism $\Omega SU = BU$ gives a natural virtual vector bundle over $\Omega SU(p^q)$, and the associated Thom spectrum $X(p^q)$ has a natural ring structure. The p-localisation of this has a p-typical summand called $T(q)$. We have $T(0) = S$ and $T(\infty) = BP$. In all cases we put $t(q) = \langle T(q) \rangle$ and $t(q; n) = t(q) \wedge f(n)$.
- Suppose $q \in \mathbb{N}_\infty$ and $T \subseteq \mathbb{N}_\infty$ is cosmall.
  If $[n, \infty] \subseteq T$, we put $t(q, T; n) = t(q; n) \vee k(T)$.
  Put $t(q, T) = t(q, T; n_0)$, where n_0 is smallest such that $[n_0, \infty] \subseteq T$.
- For $m \in \mathbb{N}_\infty$ we let $J(m)$ denote the Brown-Comenetz dual of $T(m)$, so there is a natural isomorphism

$$[X, J(m)] \simeq \mathrm{Hom}(\pi_0(T(m) \wedge X), \mathbb{Q}/\mathbb{Z}).$$

  Put $J(\omega) = \bigvee_{m \in \mathbb{N}} J(m)$, and $j(m) = \langle J(m) \rangle$ for all $m \in \mathbb{N}_\omega$.
  Given a small set S, put $j(m, S) = j(m) \vee k(S)$.

Basic Bousfield classes

- $K(n)$ = Morava K-theory ($K(0) = H\mathbb{Q}$, $K(\infty) = H/p$); $k(n) = \langle K(n) \rangle$.
- For $U \subseteq \mathbb{N}_\infty$ we put $K(U) = \bigvee_{i \in U} K(i)$ and $k(U) = \langle K(U) \rangle$.
- $F(n)$ = a finite spectrum of type n, so $K(i)_* F(n) = 0$ iff $i < n$. This can be chosen so $F(n)$ is a self-dual ring spectrum and $F(0) = S$. Put $f(n) = \langle F(n) \rangle$.
- For $q \in \mathbb{N}$ we recall that the Bott periodicity isomorphism $\Omega SU = BU$ gives a natural virtual vector bundle over $\Omega SU(p^q)$, and the associated Thom spectrum $X(p^q)$ has a natural ring structure. The p-localisation of this has a p-typical summand called $T(q)$. We have $T(0) = S$ and $T(\infty) = BP$. In all cases we put $t(q) = \langle T(q) \rangle$ and $t(q; n) = t(q) \wedge f(n)$.
- Suppose $q \in \mathbb{N}_\infty$ and $T \subseteq \mathbb{N}_\infty$ is cosmall.
  If $[n, \infty] \subseteq T$, we put $t(q, T; n) = t(q; n) \vee k(T)$.
  Put $t(q, T) = t(q, T; n_0)$, where n_0 is smallest such that $[n_0, \infty] \subseteq T$.
- For $m \in \mathbb{N}_\infty$ we let $J(m)$ denote the Brown-Comenetz dual of $T(m)$, so there is a natural isomorphism

$$[X, J(m)] \simeq \mathrm{Hom}(\pi_0(T(m) \wedge X), \mathbb{Q}/\mathbb{Z}).$$

  Put $J(\omega) = \bigvee_{m \in \mathbb{N}} J(m)$, and $j(m) = \langle J(m) \rangle$ for all $m \in \mathbb{N}_\omega$.
  Given a small set S, put $j(m, S) = j(m) \vee k(S)$.

Basic Bousfield classes

- $K(n)$ = Morava K-theory ($K(0) = H\mathbb{Q}$, $K(\infty) = H/p$); $k(n) = \langle K(n) \rangle$.
- For $U \subseteq \mathbb{N}_\infty$ we put $K(U) = \bigvee_{i \in U} K(i)$ and $k(U) = \langle K(U) \rangle$.
- $F(n)$ = a finite spectrum of type n, so $K(i)_* F(n) = 0$ iff $i < n$. This can be chosen so $F(n)$ is a self-dual ring spectrum and $F(0) = S$. Put $f(n) = \langle F(n) \rangle$.
- For $q \in \mathbb{N}$ we recall that the Bott periodicity isomorphism $\Omega SU = BU$ gives a natural virtual vector bundle over $\Omega SU(p^q)$, and the associated Thom spectrum $X(p^q)$ has a natural ring structure. The p-localisation of this has a p-typical summand called $T(q)$. We have $T(0) = S$ and $T(\infty) = BP$. In all cases we put $t(q) = \langle T(q) \rangle$ and $t(q; n) = t(q) \wedge f(n)$.
- Suppose $q \in \mathbb{N}_\infty$ and $T \subseteq \mathbb{N}_\infty$ is cosmall.
If $[n, \infty] \subseteq T$, we put $t(q, T; n) = t(q; n) \vee k(T)$.
Put $t(q, T) = t(q, T; n_0)$, where n_0 is smallest such that $[n_0, \infty] \subseteq T$.
- For $m \in \mathbb{N}_\infty$ we let $J(m)$ denote the Brown-Comenetz dual of $T(m)$, so there is a natural isomorphism

$$[X, J(m)] \simeq \mathrm{Hom}(\pi_0(T(m) \wedge X), \mathbb{Q}/\mathbb{Z}).$$

Put $J(\omega) = \bigvee_{m \in \mathbb{N}} J(m)$, and $j(m) = \langle J(m) \rangle$ for all $m \in \mathbb{N}_\omega$.
Given a small set S, put $j(m, S) = j(m) \vee k(S)$.

Basic Bousfield classes

- $K(n)$ = Morava K-theory ($K(0) = H\mathbb{Q}$, $K(\infty) = H/p$); $k(n) = \langle K(n) \rangle$.
- For $U \subseteq \mathbb{N}_\infty$ we put $K(U) = \bigvee_{i \in U} K(i)$ and $k(U) = \langle K(U) \rangle$.
- $F(n)$ = a finite spectrum of type n, so $K(i)_* F(n) = 0$ iff $i < n$. This can be chosen so $F(n)$ is a self-dual ring spectrum and $F(0) = S$. Put $f(n) = \langle F(n) \rangle$.
- For $q \in \mathbb{N}$ we recall that the Bott periodicity isomorphism $\Omega SU = BU$ gives a natural virtual vector bundle over $\Omega SU(p^q)$, and the associated Thom spectrum $X(p^q)$ has a natural ring structure. The p-localisation of this has a p-typical summand called $T(q)$. We have $T(0) = S$ and $T(\infty) = BP$. In all cases we put $t(q) = \langle T(q) \rangle$ and $t(q; n) = t(q) \wedge f(n)$.
- Suppose $q \in \mathbb{N}_\infty$ and $T \subseteq \mathbb{N}_\infty$ is cosmall.
  If $[n, \infty] \subseteq T$, we put $t(q, T; n) = t(q; n) \vee k(T)$.
  Put $t(q, T) = t(q, T; n_0)$, where n_0 is smallest such that $[n_0, \infty] \subseteq T$.
- For $m \in \mathbb{N}_\infty$ we let $J(m)$ denote the Brown-Comenetz dual of $T(m)$, so there is a natural isomorphism

$$[X, J(m)] \simeq \mathrm{Hom}(\pi_0(T(m) \wedge X), \mathbb{Q}/\mathbb{Z}).$$

  Put $J(\omega) = \bigvee_{m \in \mathbb{N}} J(m)$, and $j(m) = \langle J(m) \rangle$ for all $m \in \mathbb{N}_\omega$.
  Given a small set S, put $j(m, S) = j(m) \vee k(S)$.

- $K(n)$ = Morava K-theory ($K(0) = H\mathbb{Q}$, $K(\infty) = H/p$); $k(n) = \langle K(n) \rangle$.
- For $U \subseteq \mathbb{N}_\infty$ we put $K(U) = \bigvee_{i \in U} K(i)$ and $k(U) = \langle K(U) \rangle$.
- $F(n)$ = a finite spectrum of type n, so $K(i)_* F(n) = 0$ iff $i < n$. This can be chosen so $F(n)$ is a self-dual ring spectrum and $F(0) = S$. Put $f(n) = \langle F(n) \rangle$.
- For $q \in \mathbb{N}$ we recall that the Bott periodicity isomorphism $\Omega SU = BU$ gives a natural virtual vector bundle over $\Omega SU(p^q)$, and the associated Thom spectrum $X(p^q)$ has a natural ring structure. The p-localisation of this has a p-typical summand called $T(q)$. We have $T(0) = S$ and $T(\infty) = BP$. In all cases we put $t(q) = \langle T(q) \rangle$ and $t(q; n) = t(q) \wedge f(n)$.
- Suppose $q \in \mathbb{N}_\infty$ and $T \subseteq \mathbb{N}_\infty$ is cosmall.
  If $[n, \infty] \subseteq T$, we put $t(q, T; n) = t(q; n) \vee k(T)$.
  Put $t(q, T) = t(q, T; n_0)$, where n_0 is smallest such that $[n_0, \infty] \subseteq T$.
- For $m \in \mathbb{N}_\infty$ we let $J(m)$ denote the Brown-Comenetz dual of $T(m)$, so there is a natural isomorphism

$$[X, J(m)] \simeq \mathrm{Hom}(\pi_0(T(m) \wedge X), \mathbb{Q}/\mathbb{Z}).$$

  Put $J(\omega) = \bigvee_{m \in \mathbb{N}} J(m)$, and $j(m) = \langle J(m) \rangle$ for all $m \in \mathbb{N}_\omega$.
  Given a small set S, put $j(m, S) = j(m) \vee k(S)$.

- $K(n)$ = Morava K-theory ($K(0) = H\mathbb{Q}$, $K(\infty) = H/p$); $k(n) = \langle K(n) \rangle$.
- For $U \subseteq \mathbb{N}_\infty$ we put $K(U) = \bigvee_{i \in U} K(i)$ and $k(U) = \langle K(U) \rangle$.
- $F(n)$ = a finite spectrum of type n, so $K(i)_* F(n) = 0$ iff $i < n$. This can be chosen so $F(n)$ is a self-dual ring spectrum and $F(0) = S$. Put $f(n) = \langle F(n) \rangle$.
- For $q \in \mathbb{N}$ we recall that the Bott periodicity isomorphism $\Omega SU = BU$ gives a natural virtual vector bundle over $\Omega SU(p^q)$, and the associated Thom spectrum $X(p^q)$ has a natural ring structure. The p-localisation of this has a p-typical summand called $T(q)$. We have $T(0) = S$ and $T(\infty) = BP$. In all cases we put $t(q) = \langle T(q) \rangle$ and $t(q; n) = t(q) \wedge f(n)$.
- Suppose $q \in \mathbb{N}_\infty$ and $T \subseteq \mathbb{N}_\infty$ is cosmall.
  If $[n, \infty] \subseteq T$, we put $t(q, T; n) = t(q; n) \vee k(T)$.
  Put $t(q, T) = t(q, T; n_0)$, where n_0 is smallest such that $[n_0, \infty] \subseteq T$.
- For $m \in \mathbb{N}_\infty$ we let $J(m)$ denote the Brown-Comenetz dual of $T(m)$, so there is a natural isomorphism

  $$[X, J(m)] \simeq \mathrm{Hom}(\pi_0(T(m) \wedge X), \mathbb{Q}/\mathbb{Z}).$$

  Put $J(\omega) = \bigvee_{m \in \mathbb{N}} J(m)$, and $j(m) = \langle J(m) \rangle$ for all $m \in \mathbb{N}_\omega$.
  Given a small set S, put $j(m, S) = j(m) \vee k(S)$.

Basic Bousfield classes

- For $n \in \mathbb{N}$ we choose a good v_n element $w_n \in \pi_*(F(n))$. This means that $w_n = v_n^{p^{d_n}}$ in $BP_*F(n)$ for some d_n, plus some additional properties. Put $K'(n) = F(n)[w_n^{-1}]$ and $k'(n) = \langle K'(n) \rangle$.
- Now fix $n \in \mathbb{N}$. Let L_n denote the Bousfield localisation functor with respect to the Johnson-Wilson spectrum $E(n)$, and let C_nX denote the fibre of the natural map $X \to L_nX$. We also put $A(n) = C_nK'(n)$ and $a(n) = \langle A(n) \rangle$. We also put $\epsilon(n) = \bigvee_{i<n} a(i)$ for all $n \in \mathbb{N}_\infty$.
- Proposition: the elements $a(i)$ and $\epsilon(n)$ are idempotent.
- The Telescope Conjecture is equivalent to $a(i) = 0$ for all i, or $\epsilon(n) = 0$ for all $n \in \mathbb{N}$, or $\epsilon(\infty) = 0$.
- We work with $\overline{\mathcal{L}} = \varinjlim_n \mathcal{L}/\epsilon(n)$. One can also consider $\widehat{\mathcal{L}} = \mathcal{L}/\epsilon(\infty)$, which is a quotient of $\overline{\mathcal{L}}$, but that gives weaker results.
- We have $k'(n) = k(n) \vee a(n)$, so $k'(n) = k(n)$ in $\overline{\mathcal{L}}$.

Basic Bousfield classes

- For $n \in \mathbb{N}$ we choose a good v_n element $w_n \in \pi_*(F(n))$. This means that $w_n = v_n^{p^{d_n}}$ in $BP_*F(n)$ for some d_n, plus some additional properties. Put $K'(n) = F(n)[w_n^{-1}]$ and $k'(n) = \langle K'(n) \rangle$.
- Now fix $n \in \mathbb{N}$. Let L_n denote the Bousfield localisation functor with respect to the Johnson-Wilson spectrum $E(n)$, and let C_nX denote the fibre of the natural map $X \to L_nX$. We also put $A(n) = C_nK'(n)$ and $a(n) = \langle A(n) \rangle$. We also put $\epsilon(n) = \bigvee_{i<n} a(i)$ for all $n \in \mathbb{N}_\infty$.
- Proposition: the elements $a(i)$ and $\epsilon(n)$ are idempotent.
- The Telescope Conjecture is equivalent to $a(i) = 0$ for all i, or $\epsilon(n) = 0$ for all $n \in \mathbb{N}$, or $\epsilon(\infty) = 0$.
- We work with $\overline{\mathcal{L}} = \varinjlim_n \mathcal{L}/\epsilon(n)$. One can also consider $\widehat{\mathcal{L}} = \mathcal{L}/\epsilon(\infty)$, which is a quotient of $\overline{\mathcal{L}}$, but that gives weaker results.
- We have $k'(n) = k(n) \vee a(n)$, so $k'(n) = k(n)$ in $\overline{\mathcal{L}}$.

Basic Bousfield classes

- For $n \in \mathbb{N}$ we choose a good v_n element $w_n \in \pi_*(F(n))$. This means that $w_n = v_n^{p^{d_n}}$ in $BP_*F(n)$ for some d_n, plus some additional properties. Put $K'(n) = F(n)[w_n^{-1}]$ and $k'(n) = \langle K'(n) \rangle$.
- Now fix $n \in \mathbb{N}$. Let L_n denote the Bousfield localisation functor with respect to the Johnson-Wilson spectrum $E(n)$, and let C_nX denote the fibre of the natural map $X \to L_nX$. We also put $A(n) = C_nK'(n)$ and $a(n) = \langle A(n) \rangle$. We also put $\epsilon(n) = \bigvee_{i<n} a(i)$ for all $n \in \mathbb{N}_\infty$.
- Proposition: the elements $a(i)$ and $\epsilon(n)$ are idempotent.
- The Telescope Conjecture is equivalent to $a(i) = 0$ for all i, or $\epsilon(n) = 0$ for all $n \in \mathbb{N}$, or $\epsilon(\infty) = 0$.
- We work with $\overline{\mathcal{L}} = \varinjlim_n \mathcal{L}/\epsilon(n)$. One can also consider $\widehat{\mathcal{L}} = \mathcal{L}/\epsilon(\infty)$, which is a quotient of $\overline{\mathcal{L}}$, but that gives weaker results.
- We have $k'(n) = k(n) \vee a(n)$, so $k'(n) = k(n)$ in $\overline{\mathcal{L}}$.

Basic Bousfield classes

- For $n \in \mathbb{N}$ we choose a good v_n element $w_n \in \pi_*(F(n))$. This means that $w_n = v_n^{p^{d_n}}$ in $BP_*F(n)$ for some d_n, plus some additional properties. Put $K'(n) = F(n)[w_n^{-1}]$ and $k'(n) = \langle K'(n) \rangle$.
- Now fix $n \in \mathbb{N}$. Let L_n denote the Bousfield localisation functor with respect to the Johnson-Wilson spectrum $E(n)$, and let C_nX denote the fibre of the natural map $X \to L_nX$. We also put $A(n) = C_nK'(n)$ and $a(n) = \langle A(n) \rangle$. We also put $\epsilon(n) = \bigvee_{i<n} a(i)$ for all $n \in \mathbb{N}_\infty$.
- Proposition: the elements $a(i)$ and $\epsilon(n)$ are idempotent.
- The Telescope Conjecture is equivalent to $a(i) = 0$ for all i, or $\epsilon(n) = 0$ for all $n \in \mathbb{N}$, or $\epsilon(\infty) = 0$.
- We work with $\overline{\mathcal{L}} = \varinjlim_n \mathcal{L}/\epsilon(n)$. One can also consider $\widehat{\mathcal{L}} = \mathcal{L}/\epsilon(\infty)$, which is a quotient of $\overline{\mathcal{L}}$, but that gives weaker results.
- We have $k'(n) = k(n) \vee a(n)$, so $k'(n) = k(n)$ in $\overline{\mathcal{L}}$.

Basic Bousfield classes

- For $n \in \mathbb{N}$ we choose a good v_n element $w_n \in \pi_*(F(n))$. This means that $w_n = v_n^{p^{d_n}}$ in $BP_*F(n)$ for some d_n, plus some additional properties. Put $K'(n) = F(n)[w_n^{-1}]$ and $k'(n) = \langle K'(n) \rangle$.
- Now fix $n \in \mathbb{N}$. Let L_n denote the Bousfield localisation functor with respect to the Johnson-Wilson spectrum $E(n)$, and let C_nX denote the fibre of the natural map $X \to L_nX$. We also put $A(n) = C_nK'(n)$ and $a(n) = \langle A(n) \rangle$. We also put $\epsilon(n) = \bigvee_{i<n} a(i)$ for all $n \in \mathbb{N}_\infty$.
- Proposition: the elements $a(i)$ and $\epsilon(n)$ are idempotent.
- The Telescope Conjecture is equivalent to $a(i) = 0$ for all i, or $\epsilon(n) = 0$ for all $n \in \mathbb{N}$, or $\epsilon(\infty) = 0$.
- We work with $\overline{\mathcal{L}} = \varinjlim_n \mathcal{L}/\epsilon(n)$. One can also consider $\widehat{\mathcal{L}} = \mathcal{L}/\epsilon(\infty)$, which is a quotient of $\overline{\mathcal{L}}$, but that gives weaker results.
- We have $k'(n) = k(n) \vee a(n)$, so $k'(n) = k(n)$ in $\overline{\mathcal{L}}$.

Basic Bousfield classes

- For $n \in \mathbb{N}$ we choose a good v_n element $w_n \in \pi_*(F(n))$. This means that $w_n = v_n^{p^{d_n}}$ in $BP_*F(n)$ for some d_n, plus some additional properties. Put $K'(n) = F(n)[w_n^{-1}]$ and $k'(n) = \langle K'(n) \rangle$.
- Now fix $n \in \mathbb{N}$. Let L_n denote the Bousfield localisation functor with respect to the Johnson-Wilson spectrum $E(n)$, and let C_nX denote the fibre of the natural map $X \to L_nX$. We also put $A(n) = C_nK'(n)$ and $a(n) = \langle A(n) \rangle$. We also put $\epsilon(n) = \bigvee_{i<n} a(i)$ for all $n \in \mathbb{N}_\infty$.
- Proposition: the elements $a(i)$ and $\epsilon(n)$ are idempotent.
- The Telescope Conjecture is equivalent to $a(i) = 0$ for all i, or $\epsilon(n) = 0$ for all $n \in \mathbb{N}$, or $\epsilon(\infty) = 0$.
- We work with $\overline{\mathcal{L}} = \varinjlim_n \mathcal{L}/\epsilon(n)$. One can also consider $\widehat{\mathcal{L}} = \mathcal{L}/\epsilon(\infty)$, which is a quotient of $\overline{\mathcal{L}}$, but that gives weaker results.
- We have $k'(n) = k(n) \vee a(n)$, so $k'(n) = k(n)$ in $\overline{\mathcal{L}}$.

Basic Bousfield classes

- For $n \in \mathbb{N}$ we choose a good v_n element $w_n \in \pi_*(F(n))$. This means that $w_n = v_n^{p^{d_n}}$ in $BP_*F(n)$ for some d_n, plus some additional properties. Put $K'(n) = F(n)[w_n^{-1}]$ and $k'(n) = \langle K'(n) \rangle$.
- Now fix $n \in \mathbb{N}$. Let L_n denote the Bousfield localisation functor with respect to the Johnson-Wilson spectrum $E(n)$, and let C_nX denote the fibre of the natural map $X \to L_nX$. We also put $A(n) = C_nK'(n)$ and $a(n) = \langle A(n) \rangle$. We also put $\epsilon(n) = \bigvee_{i<n} a(i)$ for all $n \in \mathbb{N}_\infty$.
- Proposition: the elements $a(i)$ and $\epsilon(n)$ are idempotent.
- The Telescope Conjecture is equivalent to $a(i) = 0$ for all i, or $\epsilon(n) = 0$ for all $n \in \mathbb{N}$, or $\epsilon(\infty) = 0$.
- We work with $\overline{\mathcal{L}} = \varinjlim_n \mathcal{L}/\epsilon(n)$. One can also consider $\widehat{\mathcal{L}} = \mathcal{L}/\epsilon(\infty)$, which is a quotient of $\overline{\mathcal{L}}$, but that gives weaker results.
- We have $k'(n) = k(n) \vee a(n)$, so $k'(n) = k(n)$ in $\overline{\mathcal{L}}$.

General facts about Bousfield classes

(a) If R is a ring spectrum then $\langle R\rangle \wedge \langle R\rangle = \langle R\rangle$. Moreover, if M is any R-module spectrum then $\langle M\rangle = \langle R\rangle \wedge \langle M\rangle \leq \langle R\rangle$.

(b) Let K be a ring spectrum such that all nonzero homogeneous elements of K_* are invertible. Then for any X we have either $K_*X = 0$ and $\langle K\rangle \wedge \langle X\rangle = 0$, or $K_*X \neq 0$ and $\langle K\rangle \wedge \langle X\rangle = \langle K\rangle$ and $\langle X\rangle \geq \langle K\rangle$.

(c) Let X be a spectrum, and let $v\colon \Sigma^d X \to X$ be a self-map with cofibre X/v and telescope $X[v^{-1}]$. Then $\langle X\rangle = \langle X/v\rangle \vee \langle X[v^{-1}]\rangle$.

(d) Let T and X be spectra such that the homotopy groups of X are finitely generated over $\mathbb{Z}_{(p)}$. Then $T \wedge IX = 0$ iff $T \wedge I(X/p) = 0$ iff $F(T, X/p) = 0$.

(e) Suppose again that the homotopy groups of X are finitely generated over $\mathbb{Z}_{(p)}$, and that they are not all torsion groups. Then $\langle X\rangle = \langle X_p^\wedge\rangle = \langle H\mathbb{Q}\rangle \vee \langle X/p\rangle$.

General facts about Bousfield classes

(a) If R is a ring spectrum then $\langle R\rangle \wedge \langle R\rangle = \langle R\rangle$. Moreover, if M is any R-module spectrum then $\langle M\rangle = \langle R\rangle \wedge \langle M\rangle \leq \langle R\rangle$.

(b) Let K be a ring spectrum such that all nonzero homogeneous elements of K_* are invertible. Then for any X we have either $K_*X = 0$ and $\langle K\rangle \wedge \langle X\rangle = 0$, or $K_*X \neq 0$ and $\langle K\rangle \wedge \langle X\rangle = \langle K\rangle$ and $\langle X\rangle \geq \langle K\rangle$.

(c) Let X be a spectrum, and let $v\colon \Sigma^d X \to X$ be a self-map with cofibre X/v and telescope $X[v^{-1}]$. Then $\langle X\rangle = \langle X/v\rangle \vee \langle X[v^{-1}]\rangle$.

(d) Let T and X be spectra such that the homotopy groups of X are finitely generated over $\mathbb{Z}_{(p)}$. Then $T \wedge IX = 0$ iff $T \wedge I(X/p) = 0$ iff $F(T, X/p) = 0$.

(e) Suppose again that the homotopy groups of X are finitely generated over $\mathbb{Z}_{(p)}$, and that they are not all torsion groups. Then $\langle X\rangle = \langle X_p^{\wedge}\rangle = \langle H\mathbb{Q}\rangle \vee \langle X/p\rangle$.

General facts about Bousfield classes

(a) If R is a ring spectrum then $\langle R\rangle \wedge \langle R\rangle = \langle R\rangle$. Moreover, if M is any R-module spectrum then $\langle M\rangle = \langle R\rangle \wedge \langle M\rangle \leq \langle R\rangle$.

(b) Let K be a ring spectrum such that all nonzero homogeneous elements of K_* are invertible. Then for any X we have either $K_*X = 0$ and $\langle K\rangle \wedge \langle X\rangle = 0$, or $K_*X \neq 0$ and $\langle K\rangle \wedge \langle X\rangle = \langle K\rangle$ and $\langle X\rangle \geq \langle K\rangle$.

(c) Let X be a spectrum, and let $v\colon \Sigma^d X \to X$ be a self-map with cofibre X/v and telescope $X[v^{-1}]$. Then $\langle X\rangle = \langle X/v\rangle \vee \langle X[v^{-1}]\rangle$.

(d) Let T and X be spectra such that the homotopy groups of X are finitely generated over $\mathbb{Z}_{(p)}$. Then $T \wedge IX = 0$ iff $T \wedge I(X/p) = 0$ iff $F(T, X/p) = 0$.

(e) Suppose again that the homotopy groups of X are finitely generated over $\mathbb{Z}_{(p)}$, and that they are not all torsion groups. Then $\langle X\rangle = \langle X_p^\wedge\rangle = \langle H\mathbb{Q}\rangle \vee \langle X/p\rangle$.

General facts about Bousfield classes

(a) If R is a ring spectrum then $\langle R\rangle \wedge \langle R\rangle = \langle R\rangle$. Moreover, if M is any R-module spectrum then $\langle M\rangle = \langle R\rangle \wedge \langle M\rangle \leq \langle R\rangle$.

(b) Let K be a ring spectrum such that all nonzero homogeneous elements of K_* are invertible. Then for any X we have either $K_*X = 0$ and $\langle K\rangle \wedge \langle X\rangle = 0$, or $K_*X \neq 0$ and $\langle K\rangle \wedge \langle X\rangle = \langle K\rangle$ and $\langle X\rangle \geq \langle K\rangle$.

(c) Let X be a spectrum, and let $v\colon \Sigma^d X \to X$ be a self-map with cofibre X/v and telescope $X[v^{-1}]$. Then $\langle X\rangle = \langle X/v\rangle \vee \langle X[v^{-1}]\rangle$.

(d) Let T and X be spectra such that the homotopy groups of X are finitely generated over $\mathbb{Z}_{(p)}$. Then $T \wedge IX = 0$ iff $T \wedge I(X/p) = 0$ iff $F(T, X/p) = 0$.

(e) Suppose again that the homotopy groups of X are finitely generated over $\mathbb{Z}_{(p)}$, and that they are not all torsion groups. Then $\langle X\rangle = \langle X_p^\wedge\rangle = \langle H\mathbb{Q}\rangle \vee \langle X/p\rangle$.

(a) If R is a ring spectrum then $\langle R\rangle \wedge \langle R\rangle = \langle R\rangle$. Moreover, if M is any R-module spectrum then $\langle M\rangle = \langle R\rangle \wedge \langle M\rangle \leq \langle R\rangle$.

(b) Let K be a ring spectrum such that all nonzero homogeneous elements of K_* are invertible. Then for any X we have either $K_*X = 0$ and $\langle K\rangle \wedge \langle X\rangle = 0$, or $K_*X \neq 0$ and $\langle K\rangle \wedge \langle X\rangle = \langle K\rangle$ and $\langle X\rangle \geq \langle K\rangle$.

(c) Let X be a spectrum, and let $v\colon \Sigma^d X \to X$ be a self-map with cofibre X/v and telescope $X[v^{-1}]$. Then $\langle X\rangle = \langle X/v\rangle \vee \langle X[v^{-1}]\rangle$.

(d) Let T and X be spectra such that the homotopy groups of X are finitely generated over $\mathbb{Z}_{(p)}$. Then $T \wedge IX = 0$ iff $T \wedge I(X/p) = 0$ iff $F(T, X/p) = 0$.

(e) Suppose again that the homotopy groups of X are finitely generated over $\mathbb{Z}_{(p)}$, and that they are not all torsion groups. Then $\langle X\rangle = \langle X_p^\wedge\rangle = \langle H\mathbb{Q}\rangle \vee \langle X/p\rangle$.

(a) If R is a ring spectrum then $\langle R\rangle \wedge \langle R\rangle = \langle R\rangle$. Moreover, if M is any R-module spectrum then $\langle M\rangle = \langle R\rangle \wedge \langle M\rangle \leq \langle R\rangle$.

(b) Let K be a ring spectrum such that all nonzero homogeneous elements of K_* are invertible. Then for any X we have either $K_*X = 0$ and $\langle K\rangle \wedge \langle X\rangle = 0$, or $K_*X \neq 0$ and $\langle K\rangle \wedge \langle X\rangle = \langle K\rangle$ and $\langle X\rangle \geq \langle K\rangle$.

(c) Let X be a spectrum, and let $v\colon \Sigma^d X \to X$ be a self-map with cofibre X/v and telescope $X[v^{-1}]$. Then $\langle X\rangle = \langle X/v\rangle \vee \langle X[v^{-1}]\rangle$.

(d) Let T and X be spectra such that the homotopy groups of X are finitely generated over $\mathbb{Z}_{(p)}$. Then $T \wedge IX = 0$ iff $T \wedge I(X/p) = 0$ iff $F(T, X/p) = 0$.

(e) Suppose again that the homotopy groups of X are finitely generated over $\mathbb{Z}_{(p)}$, and that they are not all torsion groups. Then $\langle X\rangle = \langle X_p^\wedge\rangle = \langle H\mathbb{Q}\rangle \vee \langle X/p\rangle$.

Some intermediate results

- The classes $t(q)$, $f(n)$, $t(q;n)$ and $k'(n)$ are represented by ring spectra and so are idempotent. The class $k(U)$ is also idempotent.
- For any x, the class $k(n) \wedge x$ is 0 or $k(n)$, and standard arguments tell us which possibility holds for all our x.
- From this it is easy to understand $k(U) \wedge x$ and also $k(U) \vee x$, except for the fact that $k(U) \vee j(m, S) = k(U \cup S)$ when U is big.
- We have ring maps $T(0) \to T(1) \to \cdots \to T(\infty) = BP \to K(n)$, giving $t(0) \geq t(1) \geq \cdots \geq t(\infty) \geq k(n)$.
- However, $j(0) \leq j(1) \leq \cdots \leq j(\omega) \leq j(\infty)$ by applying Brown-Comenetz duality to a $T(q+1)$-based Adams resolution of $T(q)$.
- A similar argument with generalised Adams resolutions gives $t(q) \wedge j(m) = 0$ for $q > m$. However, if $q \leq m$ then $J(m)$ is a $T(q)$-module and so $t(q) \wedge j(m) = j(m)$.
- The spectrum $J(q)$ is bounded above with torsion homotopy groups and so satisfies $\langle J(q) \rangle \leq \langle H/p \rangle$, or $j(q) \leq k(\infty)$.
- There are various equations $u \vee v = x$ and $u \wedge v = y$ that hold by definition in $\mathcal{A}$; we need to show that they also hold in $\overline{\mathcal{L}}$. In the cases where x and y are not of the form $t(q, T)$, we now have enough information to see that the relevant equations hold already in $\mathcal{L}$.

Some intermediate results

- The classes $t(q)$, $f(n)$, $t(q;n)$ and $k'(n)$ are represented by ring spectra and so are idempotent. The class $k(U)$ is also idempotent.
- For any x, the class $k(n) \wedge x$ is 0 or $k(n)$, and standard arguments tell us which possibility holds for all our x.
- From this it is easy to understand $k(U) \wedge x$ and also $k(U) \vee x$, except for the fact that $k(U) \vee j(m,S) = k(U \cup S)$ when U is big.
- We have ring maps $T(0) \to T(1) \to \cdots \to T(\infty) = BP \to K(n)$, giving $t(0) \geq t(1) \geq \cdots \geq t(\infty) \geq k(n)$.
- However, $j(0) \leq j(1) \leq \cdots \leq j(\omega) \leq j(\infty)$ by applying Brown-Comenetz duality to a $T(q+1)$-based Adams resolution of $T(q)$.
- A similar argument with generalised Adams resolutions gives $t(q) \wedge j(m) = 0$ for $q > m$. However, if $q \leq m$ then $J(m)$ is a $T(q)$-module and so $t(q) \wedge j(m) = j(m)$.
- The spectrum $J(q)$ is bounded above with torsion homotopy groups and so satisfies $\langle J(q) \rangle \leq \langle H/p \rangle$, or $j(q) \leq k(\infty)$.
- There are various equations $u \vee v = x$ and $u \wedge v = y$ that hold by definition in $\mathcal{A}$; we need to show that they also hold in $\overline{\mathcal{L}}$. In the cases where x and y are not of the form $t(q,T)$, we now have enough information to see that the relevant equations hold already in $\mathcal{L}$.

Some intermediate results

- The classes $t(q)$, $f(n)$, $t(q;n)$ and $k'(n)$ are represented by ring spectra and so are idempotent. The class $k(U)$ is also idempotent.
- For any x, the class $k(n) \wedge x$ is 0 or $k(n)$, and standard arguments tell us which possibility holds for all our x.
- From this it is easy to understand $k(U) \wedge x$ and also $k(U) \vee x$, except for the fact that $k(U) \vee j(m, S) = k(U \cup S)$ when U is big.
- We have ring maps $T(0) \to T(1) \to \cdots \to T(\infty) = BP \to K(n)$, giving $t(0) \geq t(1) \geq \cdots \geq t(\infty) \geq k(n)$.
- However, $j(0) \leq j(1) \leq \cdots \leq j(\omega) \leq j(\infty)$ by applying Brown-Comenetz duality to a $T(q+1)$-based Adams resolution of $T(q)$.
- A similar argument with generalised Adams resolutions gives $t(q) \wedge j(m) = 0$ for $q > m$. However, if $q \leq m$ then $J(m)$ is a $T(q)$-module and so $t(q) \wedge j(m) = j(m)$.
- The spectrum $J(q)$ is bounded above with torsion homotopy groups and so satisfies $\langle J(q) \rangle \leq \langle H/p \rangle$, or $j(q) \leq k(\infty)$.
- There are various equations $u \vee v = x$ and $u \wedge v = y$ that hold by definition in $\mathcal{A}$; we need to show that they also hold in $\overline{\mathcal{L}}$. In the cases where x and y are not of the form $t(q, T)$, we now have enough information to see that the relevant equations hold already in $\mathcal{L}$.

Some intermediate results

- The classes $t(q)$, $f(n)$, $t(q;n)$ and $k'(n)$ are represented by ring spectra and so are idempotent. The class $k(U)$ is also idempotent.
- For any x, the class $k(n) \wedge x$ is 0 or $k(n)$, and standard arguments tell us which possibility holds for all our x.
- From this it is easy to understand $k(U) \wedge x$ and also $k(U) \vee x$, except for the fact that $k(U) \vee j(m,S) = k(U \cup S)$ when U is big.
- We have ring maps $T(0) \to T(1) \to \cdots \to T(\infty) = BP \to K(n)$, giving $t(0) \geq t(1) \geq \cdots \geq t(\infty) \geq k(n)$.
- However, $j(0) \leq j(1) \leq \cdots \leq j(\omega) \leq j(\infty)$ by applying Brown-Comenetz duality to a $T(q+1)$-based Adams resolution of $T(q)$.
- A similar argument with generalised Adams resolutions gives $t(q) \wedge j(m) = 0$ for $q > m$. However, if $q \leq m$ then $J(m)$ is a $T(q)$-module and so $t(q) \wedge j(m) = j(m)$.
- The spectrum $J(q)$ is bounded above with torsion homotopy groups and so satisfies $\langle J(q) \rangle \leq \langle H/p \rangle$, or $j(q) \leq k(\infty)$.
- There are various equations $u \vee v = x$ and $u \wedge v = y$ that hold by definition in $\mathcal{A}$; we need to show that they also hold in $\overline{\mathcal{L}}$. In the cases where x and y are not of the form $t(q,T)$, we now have enough information to see that the relevant equations hold already in $\mathcal{L}$.

Some intermediate results

- The classes $t(q)$, $f(n)$, $t(q;n)$ and $k'(n)$ are represented by ring spectra and so are idempotent. The class $k(U)$ is also idempotent.
- For any x, the class $k(n) \wedge x$ is 0 or $k(n)$, and standard arguments tell us which possibility holds for all our x.
- From this it is easy to understand $k(U) \wedge x$ and also $k(U) \vee x$, except for the fact that $k(U) \vee j(m,S) = k(U \cup S)$ when U is big.
- We have ring maps $T(0) \to T(1) \to \cdots \to T(\infty) = BP \to K(n)$, giving $t(0) \geq t(1) \geq \cdots \geq t(\infty) \geq k(n)$.
- However, $j(0) \leq j(1) \leq \cdots \leq j(\omega) \leq j(\infty)$ by applying Brown-Comenetz duality to a $T(q+1)$-based Adams resolution of $T(q)$.
- A similar argument with generalised Adams resolutions gives $t(q) \wedge j(m) = 0$ for $q > m$. However, if $q \leq m$ then $J(m)$ is a $T(q)$-module and so $t(q) \wedge j(m) = j(m)$.
- The spectrum $J(q)$ is bounded above with torsion homotopy groups and so satisfies $\langle J(q) \rangle \leq \langle H/p \rangle$, or $j(q) \leq k(\infty)$.
- There are various equations $u \vee v = x$ and $u \wedge v = y$ that hold by definition in $\mathcal{A}$; we need to show that they also hold in $\overline{\mathcal{L}}$. In the cases where x and y are not of the form $t(q,T)$, we now have enough information to see that the relevant equations hold already in $\mathcal{L}$.

Some intermediate results

- The classes $t(q)$, $f(n)$, $t(q;n)$ and $k'(n)$ are represented by ring spectra and so are idempotent. The class $k(U)$ is also idempotent.
- For any x, the class $k(n) \wedge x$ is 0 or $k(n)$, and standard arguments tell us which possibility holds for all our x.
- From this it is easy to understand $k(U) \wedge x$ and also $k(U) \vee x$, except for the fact that $k(U) \vee j(m, S) = k(U \cup S)$ when U is big.
- We have ring maps $T(0) \to T(1) \to \cdots \to T(\infty) = BP \to K(n)$, giving $t(0) \geq t(1) \geq \cdots \geq t(\infty) \geq k(n)$.
- However, $j(0) \leq j(1) \leq \cdots \leq j(\omega) \leq j(\infty)$ by applying Brown-Comenetz duality to a $T(q+1)$-based Adams resolution of $T(q)$.
- A similar argument with generalised Adams resolutions gives $t(q) \wedge j(m) = 0$ for $q > m$. However, if $q \leq m$ then $J(m)$ is a $T(q)$-module and so $t(q) \wedge j(m) = j(m)$.
- The spectrum $J(q)$ is bounded above with torsion homotopy groups and so satisfies $\langle J(q) \rangle \leq \langle H/p \rangle$, or $j(q) \leq k(\infty)$.
- There are various equations $u \vee v = x$ and $u \wedge v = y$ that hold by definition in $\mathcal{A}$; we need to show that they also hold in $\overline{\mathcal{L}}$. In the cases where x and y are not of the form $t(q, T)$, we now have enough information to see that the relevant equations hold already in $\mathcal{L}$.

Some intermediate results

- The classes $t(q)$, $f(n)$, $t(q;n)$ and $k'(n)$ are represented by ring spectra and so are idempotent. The class $k(U)$ is also idempotent.
- For any x, the class $k(n) \wedge x$ is 0 or $k(n)$, and standard arguments tell us which possibility holds for all our x.
- From this it is easy to understand $k(U) \wedge x$ and also $k(U) \vee x$, except for the fact that $k(U) \vee j(m, S) = k(U \cup S)$ when U is big.
- We have ring maps $T(0) \to T(1) \to \cdots \to T(\infty) = BP \to K(n)$, giving $t(0) \geq t(1) \geq \cdots \geq t(\infty) \geq k(n)$.
- However, $j(0) \leq j(1) \leq \cdots \leq j(\omega) \leq j(\infty)$ by applying Brown-Comenetz duality to a $T(q+1)$-based Adams resolution of $T(q)$.
- A similar argument with generalised Adams resolutions gives $t(q) \wedge j(m) = 0$ for $q > m$. However, if $q \leq m$ then $J(m)$ is a $T(q)$-module and so $t(q) \wedge j(m) = j(m)$.
- The spectrum $J(q)$ is bounded above with torsion homotopy groups and so satisfies $\langle J(q) \rangle \leq \langle H/p \rangle$, or $j(q) \leq k(\infty)$.
- There are various equations $u \vee v = x$ and $u \wedge v = y$ that hold by definition in $\mathcal{A}$; we need to show that they also hold in $\overline{\mathcal{L}}$. In the cases where x and y are not of the form $t(q, T)$, we now have enough information to see that the relevant equations hold already in $\mathcal{L}$.

Some intermediate results

- The classes $t(q)$, $f(n)$, $t(q;n)$ and $k'(n)$ are represented by ring spectra and so are idempotent. The class $k(U)$ is also idempotent.
- For any x, the class $k(n) \wedge x$ is 0 or $k(n)$, and standard arguments tell us which possibility holds for all our x.
- From this it is easy to understand $k(U) \wedge x$ and also $k(U) \vee x$, except for the fact that $k(U) \vee j(m, S) = k(U \cup S)$ when U is big.
- We have ring maps $T(0) \to T(1) \to \cdots \to T(\infty) = BP \to K(n)$, giving $t(0) \geq t(1) \geq \cdots \geq t(\infty) \geq k(n)$.
- However, $j(0) \leq j(1) \leq \cdots \leq j(\omega) \leq j(\infty)$ by applying Brown-Comenetz duality to a $T(q+1)$-based Adams resolution of $T(q)$.
- A similar argument with generalised Adams resolutions gives $t(q) \wedge j(m) = 0$ for $q > m$. However, if $q \leq m$ then $J(m)$ is a $T(q)$-module and so $t(q) \wedge j(m) = j(m)$.
- The spectrum $J(q)$ is bounded above with torsion homotopy groups and so satisfies $\langle J(q) \rangle \leq \langle H/p \rangle$, or $j(q) \leq k(\infty)$.
- There are various equations $u \vee v = x$ and $u \wedge v = y$ that hold by definition in $\mathcal{A}$; we need to show that they also hold in $\overline{\mathcal{L}}$. In the cases where x and y are not of the form $t(q, T)$, we now have enough information to see that the relevant equations hold already in $\mathcal{L}$.

Some intermediate results

- The classes $t(q)$, $f(n)$, $t(q;n)$ and $k'(n)$ are represented by ring spectra and so are idempotent. The class $k(U)$ is also idempotent.
- For any x, the class $k(n) \wedge x$ is 0 or $k(n)$, and standard arguments tell us which possibility holds for all our x.
- From this it is easy to understand $k(U) \wedge x$ and also $k(U) \vee x$, except for the fact that $k(U) \vee j(m,S) = k(U \cup S)$ when U is big.
- We have ring maps $T(0) \to T(1) \to \cdots \to T(\infty) = BP \to K(n)$, giving $t(0) \geq t(1) \geq \cdots \geq t(\infty) \geq k(n)$.
- However, $j(0) \leq j(1) \leq \cdots \leq j(\omega) \leq j(\infty)$ by applying Brown-Comenetz duality to a $T(q+1)$-based Adams resolution of $T(q)$.
- A similar argument with generalised Adams resolutions gives $t(q) \wedge j(m) = 0$ for $q > m$. However, if $q \leq m$ then $J(m)$ is a $T(q)$-module and so $t(q) \wedge j(m) = j(m)$.
- The spectrum $J(q)$ is bounded above with torsion homotopy groups and so satisfies $\langle J(q) \rangle \leq \langle H/p \rangle$, or $j(q) \leq k(\infty)$.
- There are various equations $u \vee v = x$ and $u \wedge v = y$ that hold by definition in $\mathcal{A}$; we need to show that they also hold in $\overline{\mathcal{L}}$. In the cases where x and y are not of the form $t(q,T)$, we now have enough information to see that the relevant equations hold already in $\mathcal{L}$.

Adjusted equations in $\mathcal{L}$

- Recall that $t(q, T; n) = t(q) \wedge f(n) \vee k(T)$ for sufficiently large n.
- The following rules are valid in $\mathcal{L}$ (provided that n is large enough for the terms on the left to be defined):

$$
\begin{aligned}
t(q, T; n) \wedge t(q', T'; n) &= t(\max(q, q'), T \cap T'; n)) \\
t(q, T; n) \vee t(q', T'; n) &= t(\min(q, q'), T \cup T'; n) \\
t(q, T; n) \vee j(m', S') &= t(q, T \cup S'; n) \\
t(q, T; n) \vee k(U') &= t(q, T \cup U'; n).
\end{aligned}
$$

- In the quotient $\overline{\mathcal{L}}$, the class $t(q, T; n)$ is independent of n. (Increasing n by 1 swaps a $k'(n)$ for a $k(n)$.)

Adjusted equations in $\mathcal{L}$

- Recall that $t(q, T; n) = t(q) \wedge f(n) \vee k(T)$ for sufficiently large n.
- The following rules are valid in $\mathcal{L}$ (provided that n is large enough for the terms on the left to be defined):

$$\begin{aligned} t(q, T; n) \wedge t(q', T'; n) &= t(\max(q, q'), T \cap T'; n)) \\ t(q, T; n) \vee t(q', T'; n) &= t(\min(q, q'), T \cup T'; n) \\ t(q, T; n) \vee j(m', S') &= t(q, T \cup S'; n) \\ t(q, T; n) \vee k(U') &= t(q, T \cup U'; n). \end{aligned}$$

- In the quotient $\overline{\mathcal{L}}$, the class $t(q, T; n)$ is independent of n. (Increasing n by 1 swaps a $k'(n)$ for a $k(n)$.)

Adjusted equations in $\mathcal{L}$

- Recall that $t(q, T; n) = t(q) \wedge f(n) \vee k(T)$ for sufficiently large n.
- The following rules are valid in $\mathcal{L}$ (provided that n is large enough for the terms on the left to be defined):

$$\begin{aligned}
t(q, T; n) \wedge t(q', T'; n) &= t(\max(q, q'), T \cap T'; n)) \\
t(q, T; n) \vee t(q', T'; n) &= t(\min(q, q'), T \cup T'; n) \\
t(q, T; n) \vee j(m', S') &= t(q, T \cup S'; n) \\
t(q, T; n) \vee k(U') &= t(q, T \cup U'; n).
\end{aligned}$$

- In the quotient $\overline{\mathcal{L}}$, the class $t(q, T; n)$ is independent of n. (Increasing n by 1 swaps a $k'(n)$ for a $k(n)$.)

Adjusted equations in $\mathcal{L}$

- Recall that $t(q, T; n) = t(q) \wedge f(n) \vee k(T)$ for sufficiently large n.
- The following rules are valid in $\mathcal{L}$ (provided that n is large enough for the terms on the left to be defined):

$$\begin{aligned} t(q, T; n) \wedge t(q', T'; n) &= t(\max(q, q'), T \cap T'; n)) \\ t(q, T; n) \vee t(q', T'; n) &= t(\min(q, q'), T \cup T'; n) \\ t(q, T; n) \vee j(m', S') &= t(q, T \cup S'; n) \\ t(q, T; n) \vee k(U') &= t(q, T \cup U'; n). \end{aligned}$$

- In the quotient $\overline{\mathcal{L}}$, the class $t(q, T; n)$ is independent of n. (Increasing n by 1 swaps a $k'(n)$ for a $k(n)$.)

Popular Bousfield classes

$$
\begin{aligned}
0 &= k(\emptyset) \\
S = S_p^\wedge = T(0) &= t(0, \mathbb{N}_\infty) \\
S/p = S/p^\infty &= t(0, [1, \infty]) \\
F(n) &= t(0, [n, \infty]) \\
H\mathbb{Q} = S\mathbb{Q} = I(H\mathbb{Q}) &= k(\{0\}) \\
H/p = H/p^\infty = I(H) = I(H/p) = I(BP\langle n\rangle) &= k(\{\infty\}) \\
H &= k(\{0, \infty\}) \\
v_n^{-1}F(n) = K'(n) &\simeq k(\{n\}) \\
T(q) &= t(q, \mathbb{N}) \\
BP = BP_p^\wedge = T(\infty) &= t(\infty, \mathbb{N}) \\
P(n) = BP/I_n &= t(\infty, [n, \infty]) \\
B(n) = v_n^{-1}P(n) = K(n) = M_nS &= k(\{n\}) \\
IB(n) = IK(n) &= k(\{n\})
\end{aligned}
$$

Popular Bousfield classes

$$
\begin{aligned}
E(n) = v_n^{-1}BP\langle n\rangle = v_n^{-1}BP = L_nS &= k([0,n]) \\
\widehat{E(n)} = L_{K(n)}S &= k([0,n]) \\
C_nS &\simeq t(0,[n+1,\infty]) \\
BP\langle n\rangle &= k([0,n]\cup\{\infty\}) \\
BP\langle n\rangle/I_n &= k(\{n,\infty\}) \\
KU = KO &= k(\{0,1\}) \\
kU = kO &= k(\{0,1,\infty\}) \\
Ell = TMF &= k(\{0,1,2\}) \\
I(S) = I(T(0)) = I(F(n)) &= j(0,\emptyset) \\
I(S_p^\wedge) = I(S/p^\infty) &= j(0,\{0\}) \\
I(T(m)) = I(T(m)\wedge F(n)) &= j(m,\emptyset)
\end{aligned}
$$