Overview

- For each topological space X, we have a graded ring $Ell^*(X)$, the elliptic cohomology of X.
- There are various different versions, & a unifying axiomatic framework. Our versions have $Ell^{*+2}(X) \simeq Ell^*(X)$.
- The current definition is by homotopy theory, but one hopes for an analytic/geometric/physical construction.
- $Ell^*(\text{point})$ is closely related to the ring of modular forms over $\mathbb{Z}$.
- For manifolds M satisfying some low-dimensional conditions, we have a cobordism invariant $\sigma(M) \in Ell^{-\dim(M)}(\text{point})$, called the elliptic genus.
- Witten has interpreted $\sigma(M)$ in terms of a theory of strings moving in M, and the index of the Dirac operator on the free loop space $LM = Map(S^1, M)$.
- For many spaces X, the scheme $spec(Ell^*(X))$ has a natural interpretation in terms of (universal families of) elliptic curves.

The chromatic picture

- Stable homotopy theory is naturally filtered by chromatic layers.
- De Rahm cohomology sees only layer 0.
- K-theory sees layers 0 & 1.
- Elliptic cohomology sees layers 0, 1 & 2.
- Higher layers are seen by the Morava K-theories $K(p,n)$. These are constructed separately for each prime p, and are not expected to be very geometric.
- The theories H, K & Ell have associated one-dimensional algebraic groups: the additive group, the multiplicative group, and (the family of all) elliptic curves.
- The theories $K(p,n)$ only have an associated formal group (similar to the part of an elliptic curve infinitesimally close to zero).

Comparison with K-theory

- $K^0(X)$ is the set of formal differences $P-Q$, where P & Q are isomorphism classes of complex vector bundles over X.
- This is a ring under $\oplus$ and $\otimes$, and it fits into a graded ring $K^*(X)$ with $K^{*+2}(X) \simeq K^*(X)$.
- ARITHMETIC:
  - K theory can be used to count vector fields on spheres, or detect elements in stable homotopy groups. The calculations involve arithmetic properties of Bernoulli numbers.
- ANALYSIS AND PHYSICS:
  An elliptic PDE on a manifold M has an invariant in $K^*(M)$, and the Atiyah-Singer index theorem relates this to the dimension of the solution space. By considering the Dirac equation for electrons in M we obtain a cobordism invariant $\tau(M) \in \mathbb{Z}$ called the Todd genus (for complex M).
- REPRESENTATION THEORY:
  The equivariant K-theory of G-bundles over G-spaces includes the representation theory of G. This involves all of the multiplicative group; nonequivariantly we see only the infinitesimal neighborhood of 1.
- HOMOTOPY THEORY:
  $K^0(X) = [X, \mathbb{Z} \times BU]$

  $K^*(X) = K^*(\text{point}) \otimes_{MP^*(\text{point})} MP^*(X)$ $\qquad$ MP = complex cobordism (2-periodic version)

Spectra

- There is a category $\mathcal{S}$ of objects called spectra. Roughly speaking, it is obtained from the category of spaces by introducing spheres of negative dimension.
- We write $h\mathcal{S}$ for the associated homotopy category. This is (miraculously) an additive category.
- There is a smash product operation on $\mathcal{S}$ written $(X,Y) \mapsto X \wedge Y$. This is analogous to a tensor product.
- For each of the theories $E^* = H^*, K^*, Ell^*, MP^*$ there is a representing spectrum $E \in \mathcal{S}$ such that $E^n X = h\mathcal{S}(X_+, S^n \wedge E)$.
- In each case, the ring structure on $E^* X$ comes from a map $\mu : E \wedge E \to E$, which is commutative & associative up to homotopy. Thus, E is a ring spectrum.
- Let E be a 2-periodic ring spectrum with $\pi_1 E = 0$. We put $S = S_E = \mathrm{spec}(E^0(\text{point}))$

$$G = G_E = \varinjlim_n \mathrm{spec}(E^0(\mathbb{C}P^n)).$$

Then G is a formal group over S.

Elliptic curves

- We need the arithmetic definition of elliptic curves, in a form that is valid in all characteristics including 2&3.
- Let k be a ring, with associated scheme $S = \mathrm{spec}(k)$. An elliptic curve over S is a certain kind of group scheme over S, subject to some technical axioms.
- If C is an elliptic curve, then locally on S one can choose parameters x, y satisfying

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

for some $a_1, a_2, a_3, a_4, a_6 \in k$. The functions x and y have poles (of orders 2 and 3) at the zero-section of C, but are regular elsewhere. This is called a Weierstrass parametrisation of C.
- If 6 is invertible in k, we can choose a parametrisation of the form $y^2 = x^3 + a_4 x + a_6$.
- The rational function $t = x/y$ is a local parameter on C near the zero section.
- We let $\hat{C}$ be the part of C infinitesimally close to zero, so $\hat{C} = \varinjlim_n \mathrm{spec}(k[t]/t^n)$.

Axiomatics of elliptic cohomology

- An elliptic spectrum consists of a ring spectrum E (with associated formal group $G = \varinjlim_n \operatorname{spec}(E^0 \mathbb{C}P^n)$ over $S = \operatorname{spec}(E^0(\text{point}))$) together with an elliptic curve C over S and a specified isomorphism $G \simeq \hat{C}$ of formal groups.
- For large classes of elliptic curves C/S, there is a canonical (or even unique) way to construct E as above; so the category of elliptic spectra is close to the category of elliptic curves.
- Any elliptic spectrum should be thought of as a version of elliptic cohomology.
- There is no universal elliptic curve, but there are universal examples of elliptic curves with mild extra properties (eg j and $j-1728$ invertible) and/or extra structure (eg a level structure or invariant differential form). These are defined over rings such as $k = \mathbb{Z}[j][(j(1728-j))^{-1}]$ or $k = \mathbb{Z}[\frac{1}{6}, a_4, a_6][\Delta^{-1}]$. These quasiuniversal cases give the most popular versions of elliptic cohomology.
- Many elliptic spectra are Landweber exact, ie there is a natural isomorphism $MP^*(X) \otimes_{MP^*(pt)} E^*(pt) \simeq E^*(X)$. Indeed, they can be constructed from this description by the Landweber exact functor theorem.

Strict commutativity

- Methods discussed so far construct elliptic spectra E only up to homotopy, and the product $\mu : E \wedge E \to E$ is only commutative and associative up to homotopy.
- The spectra H, K and MP (and many other spectra with analytic or geometric meaning) can be constructed more rigidly, with strictly commutative & associative products. Can this be done for elliptic cohomology?
- Strict commutativity has a remarkable algebraic consequence: Given a point $a \in S$ and a finite flat subgroup scheme $H \leq G_a$, there is another point $b \in S$ and a canonical isogeny $q : G_a \twoheadrightarrow G_b$ with kernel H. The construction $(a, H) \mapsto (b, q)$ has good naturality & transitivity properties.
- When 2 is inverted, Hopkins & Goerss have constructed many strictly commutative elliptic spectra. Inversion of 2 is expected to be unnecessary. The method is indirect: one can compute the homotopy groups of the space of strictly commutative spectra of the required type, and one finds that the space is contractible (and so nonempty).
- The method works when S is étale over the moduli stack. This means roughly that we have the universal example of an elliptic curve with extra structure, and there are only a finite number of well-separated choices for that structure.

Elliptic genera

- For any elliptic spectrum E, there is a canonical map $\sigma : MString \to E$, which is natural for maps between elliptic spectra. This is called the σ-orientation.
- Here $MString = MO\langle 8\rangle$ is a spectrum related to cobordism of manifolds with a trivialisation of the tangent bundle restricted to the 7-skeleton. This fits in a sequence with $MO\langle 1\rangle = MO$, $MO\langle 2\rangle = MSO$, $MO\langle 4\rangle = MSpin$.
- For particular E and with slightly modified domain, the σ-orientation goes by various other names: the elliptic genus, the Ochanine genus, or the Witten genus.
- For any formal group G we have a line bundle L corresponding to the invertible ideal sheaf of functions vanishing at zero. We let $\Theta^3(L)$ be the line bundle over $G\times G\times G$ whose fibre at (a,b,c) is

$$\Theta^3(L)_{abc} = L_0 \otimes L_a^* \otimes L_b^* \otimes L_c^* \otimes L_{a+b} \otimes L_{b+c} \otimes L_{c+a} \otimes L_{a+b+c}^*$$

Ring maps $MString \to E$ biject with trivialisations of $\Theta^3(L)$ with certain symmetry properties. Precisely one such trivialisation extends from $G\times G\times G \simeq \hat{C}\times\hat{C}\times\hat{C}$ to all of $C\times C\times C$; this one corresponds to $\sigma : MString \to E$.

The proof uses Weil pairings and cubical structures.

The Tate Curve

- Given $\tau \in \mathbb{C}\setminus\mathbb{R}$ we have an analytic elliptic curve $C = \mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau) \simeq \mathbb{C}^\times/q^{\mathbb{Z}}$ (where $q = e^{2\pi i\tau}$)
We can expand out the structure formulae in terms of q and interpret everything in terms of formal power series to define an elliptic curve C_{Tate} over the ring $\mathbb{Z}[\![q]\!]$.
- Put $K^*_{\mathrm{Tate}}(X) = K^*(X)[\![q]\!]$. There is a natural way to interpret $(K_{\mathrm{Tate}}, C_{\mathrm{Tate}})$ as an elliptic spectrum.
- Given a vector bundle V over X, we put $\lambda_t(V) = \sum_n t^n \lambda^n(V) \in K^0(V)[t]$, where $\lambda^n(V)$ is the n'th exterior power. We then put
$$\gamma(V) = \prod_{m>0} \lambda_{-q^m}(V)\,\lambda_{q^m}(V^*) \in K^0_{\mathrm{Tate}}(X).$$
This is an exponential characteristic class, ie $\gamma(V\oplus W) = \gamma(V)\gamma(W)$, and is related to the construction of Fock spaces in quantum field theory.
- We can twist the Todd genus $MP \to K \to K_{\mathrm{Tate}}$ by γ to get a new ring map $\phi : MP \to K_{\mathrm{Tate}}$, which is essentially the σ-orientation.
- The naturality of σ implies that $\phi(M) \in K^{\dim(M)}_{\mathrm{Tate}}(pt) \simeq \mathbb{Z}[\![q]\!]$ is the q-expansion of a modular form over $\mathbb{Z}$. This is mysterious from the definition of ϕ, but is also explained by Witten's heuristic calculation of $\phi(M)$ as a partition function for the nonlinear σ-model.

Modular forms

- For any elliptic curve C over S, we write ω_C for the space of holomorphic differentials on C, which is locally a free module of rank one over $\mathcal{O}_S$.
- A modular form of weight n is a rule that assigns to each C an element $f(C) \in \omega_C^{\otimes n}$, that is natural in a suitable sense.
- To make contact with the classical analytic picture, we have $f(\mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau) = g(\tau)(dz)^{\otimes n}$ for some function $g(\tau)$ satisfying $g\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^n g(\tau)$.
- The ring MF_* of modular forms can be described as $\varprojlim_{(C,S)} \bigoplus_n \omega_C^{\otimes n}$. Its structure is $MF_* = \mathbb{Z}[c_4, c_6, \Delta, \Delta^{-1}]/(1728\Delta - c_4^3 + c_6^2)$

  A slight modification of the definition gives the ring

  $$mf_* = \mathbb{Z}[c_4, c_6, \Delta]/(1728\Delta - c_4^3 + c_6^2)$$

Topological modular forms

- We define TMF to be the homotopy inverse limit of all elliptic spectra. For this to make sense, we need to work in the category of strictly commutative ring spectra. (Details are not quite in place, but there are ad hoc workarounds.)
- The spectrum TMF is not itself elliptic (essentially because there is no universal elliptic curve). It is periodic of period 576. A slightly modified definition gives a spectrum tmf with $\pi_n tmf = 0$ for $n < 0$.
- There are natural maps $\pi_* tmf \to mf_*$ & $\pi_* TMF \to MF_*$, which become isomorphisms after inverting 6.
- There is a map $MString \to tmf$ and maps $tmf \to E$ for all elliptic spectra E, such that $MString \to tmf \to E$ is the σ-orientation.
- The hoped-for geometric construction of elliptic cohomology should probably produce tmf rather than an elliptic spectrum.
- The triple (tmf, TMF, Ell) is analogous to (kO, KO, KU)
- The homotopy ring $\pi_* tmf$ is known, by elaborate calculations of Hopkins & Mahowald. It encodes & reflects a great deal of what is known about stable homotopy groups of spheres at $p=2$.

K-theory & K-theory

- Quillen defined an algebraic K-theory spectrum $K(R)$ for any ring R. This can be generalised to define $K(R)$ for any strictly commutative ring spectrum R.
- In particular, we can consider $K(kU)$, where kU is the connective complex K-theory spectrum.
- Rognes has calculated that $K(kU)$ detects some phenomena of chromatic level 2, and argued that $K(kU)$ should be related to tmf.

  More precisely, we ask if the following diagram can be filled in:

$$\begin{tikzcd}
\Sigma^\infty K(\mathbb{Z},3)_+ \arrow[r] \arrow[d] & K(kU) \arrow[r] \arrow[d, dotted] & K(\mathbb{Z}) \arrow[d] \\
MString \arrow[r] \arrow[ur, dotted] & tmf \arrow[r] & kO
\end{tikzcd}$$

  This would make part of Witten's approach mathematically rigorous.
- There are other attempts to make rigorous connections with the physics of conformal field theory and vertex algebras (Segal ; Stolz-Teichner ; Tamanoi).

Equivariant theories

- The spectra H, K and MP all have good G-equivariant versions for all compact Lie groups G, as do other spectra with geometric significance.
- Any elliptic spectrum E can be made A-equivariant for any finite abelian group A, subject to some mild technicalities. Here $\mathrm{spec}(E^*_A(pt)) = \mathrm{Hom}(A^*, C)$.
- If G is finite and $|G|$ is invertible in $\pi_0 E$, then E can be made G-equivariant.
- If $\pi_0 E$ is a $\mathbb{Q}$-algebra, then E can be made S^1-equivariant. This is the first point where we see nontrivial higher cohomology of sheaves over elliptic curves.
- Grojnowski has constructed analytic G-equivariant sheaf-valued theories, for all compact Lie groups G. Here all cohomology rings are $\mathbb{C}$-algebras.