Chromatic methods in equivariant stable homotopy

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23 February 2009

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$\mathcal{B}_G$
G-spectra

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$$\underset{\text{based } G\text{-spaces}}{\mathcal{T}_G} \xrightarrow[\Sigma^\infty]{} \underset{G\text{-spectra}}{\mathcal{B}_G}$$

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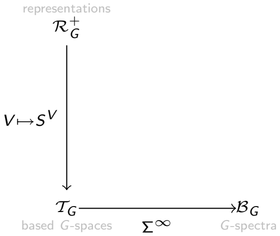

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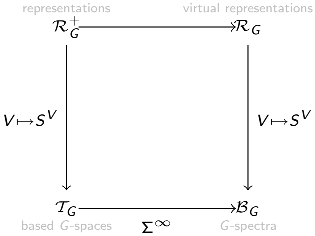

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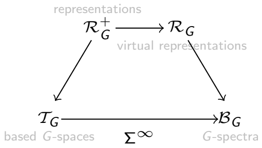

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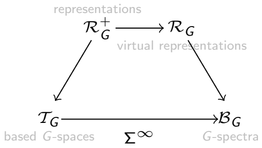

$\mathcal{B}_G(\Sigma^\infty X, \Sigma^\infty Y) = [\Sigma^\infty X, \Sigma^\infty Y]^G = [S^{n\mathbb{R}[G]} \wedge X, S^{n\mathbb{R}[G]} \wedge Y]$
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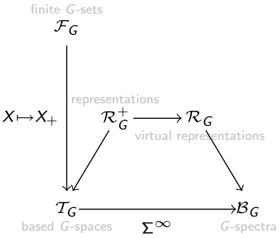

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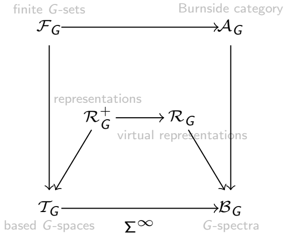

Objects of $\mathcal{A}_G$ are finite G-sets

$$\begin{aligned}\mathcal{A}_G(X, Y) &= \text{ Grothendieck group of finite } G\text{-sets over } X \times Y \\ &= \mathbb{Z}\{ \text{ iso classes of orbits over } X \times X\}\end{aligned}$$

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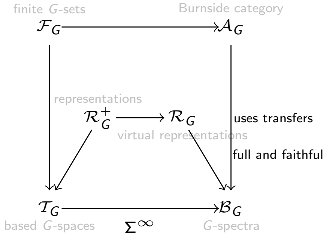

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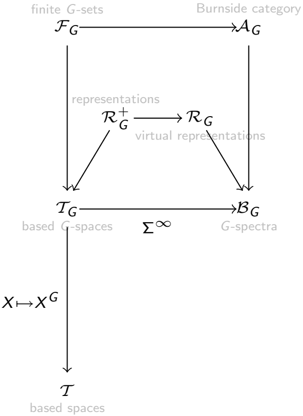

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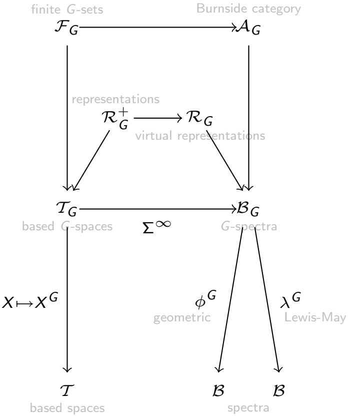

$\phi^G \epsilon^* X = X$ $\qquad \phi^G(X \wedge Y) = \phi^G(X) \wedge \phi^G(Y)$ $\qquad \phi^G \Sigma^\infty X = \Sigma^\infty X^G$

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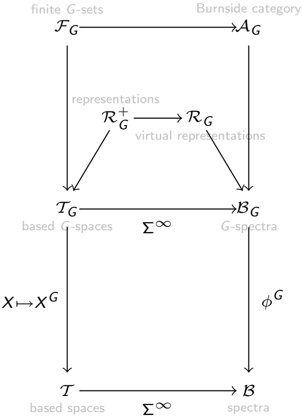

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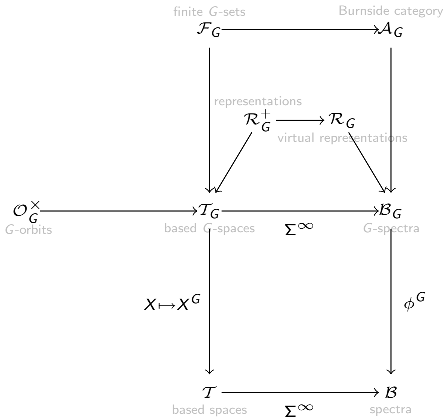

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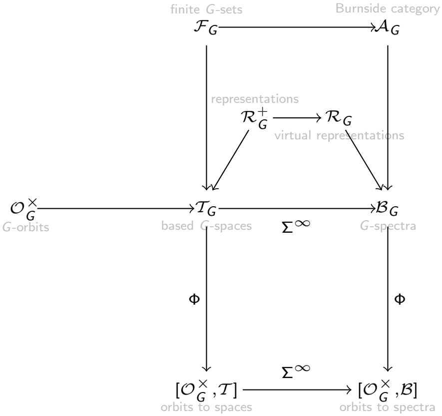

$$\Phi(X)(G/H) = \mathrm{Map}_G(G/H, X) = X^H$$

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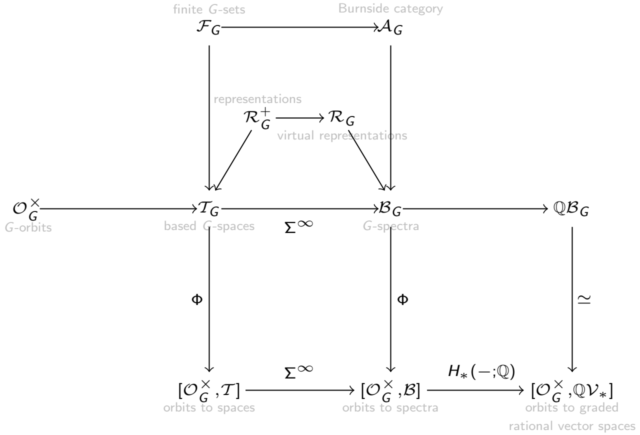

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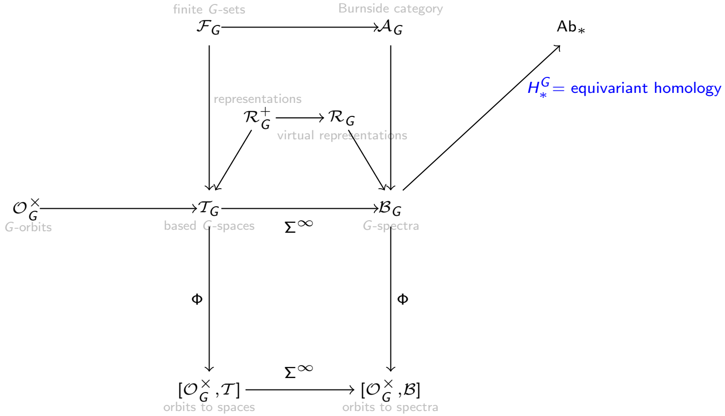

$$H_*^G(\Sigma^\infty X) = \widetilde{H}_*(X/G)$$

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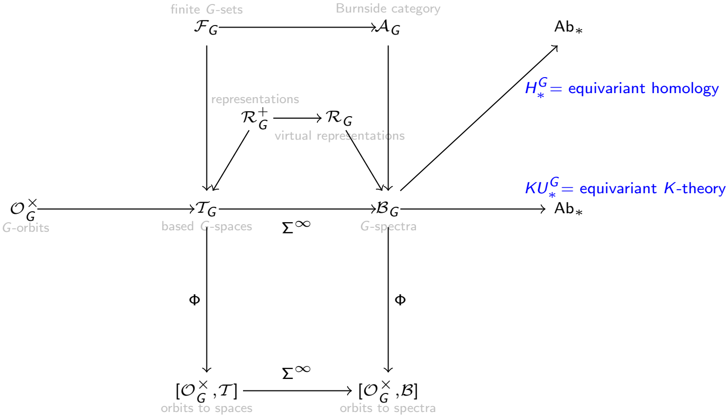

$KU^0_G(\Sigma^\infty_+ X) =$ Grothendieck group of equivariant vector bundles over X
$KU^0_G(\Sigma^\infty_+ G/H) =$ representation ring of H

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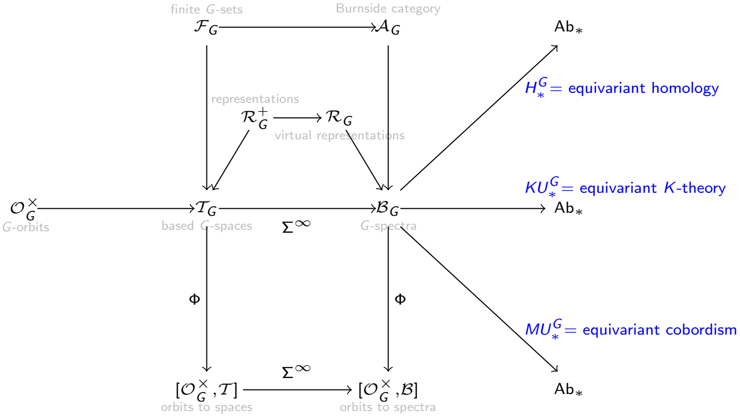

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- When G is abelian we have a partial understanding of a stack of G-equivariant formal groups, which we can attempt to relate to $\mathcal{B}_G$.
- When G is not abelian we have no definition of G-equivariant formal groups, and evidence that there cannot be one. Nevertheless, there is a good chromatic theory.

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Define $\phi\colon A(G) \to B(G)$ by $\phi[X](H) = |X^H|$.

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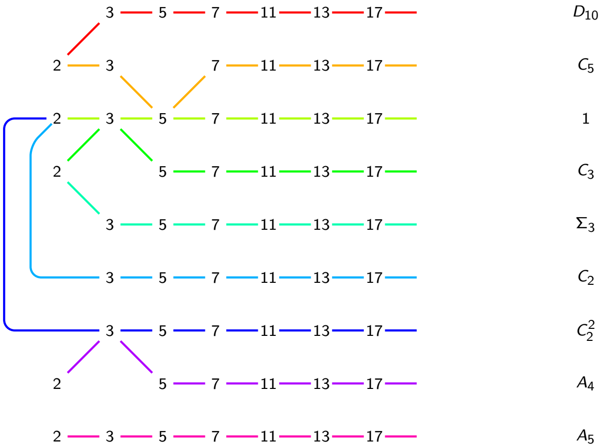
It turns out (by a theorem of Dress) that $\mathrm{spec}(A(G))$ is the quotient of this where (H, p) is identified with (K, p) whenever $\mathcal{O}^p(H)$ is conjugate to $\mathcal{O}^p(K)$. Here $\mathcal{O}^p(H)$ is the smallest normal subgroup of H of p-power index.

The case of A_5



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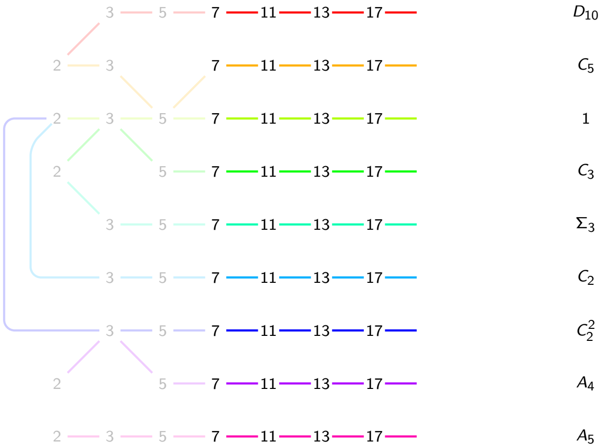

Above every prime not dividing $|G|$ there is a maximal ideal in $A(G)$ for each conjugacy class of subgroups.

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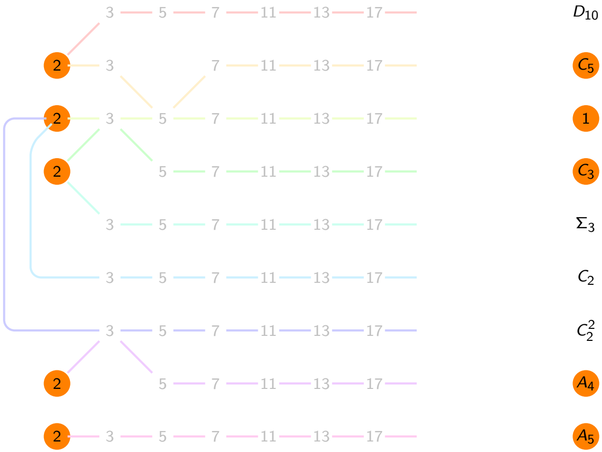

Above the prime 2 there is a maximal ideal in $A(G)$ for each conjugacy class of 2-perfect subgroups.

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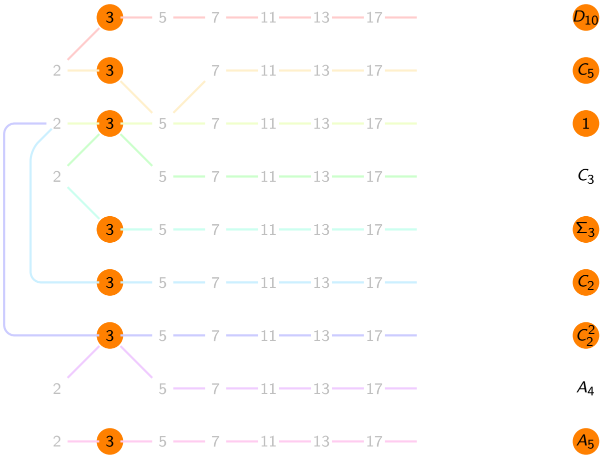

Above the prime 3 there is a maximal ideal in $A(G)$ for each conjugacy class of 3-perfect subgroups.

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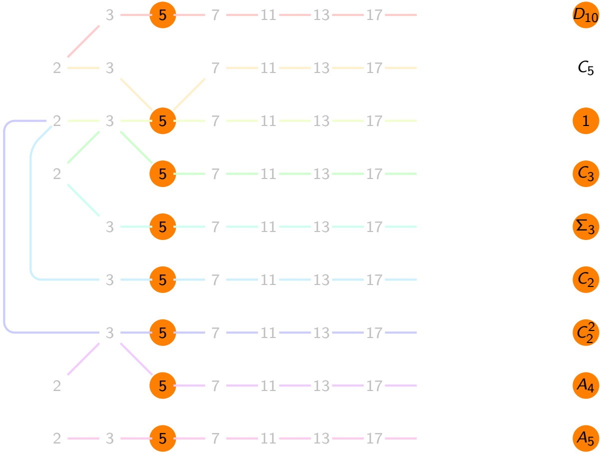

Above the prime 5 there is a maximal ideal in $A(G)$ for each conjugacy class of 5-perfect subgroups.

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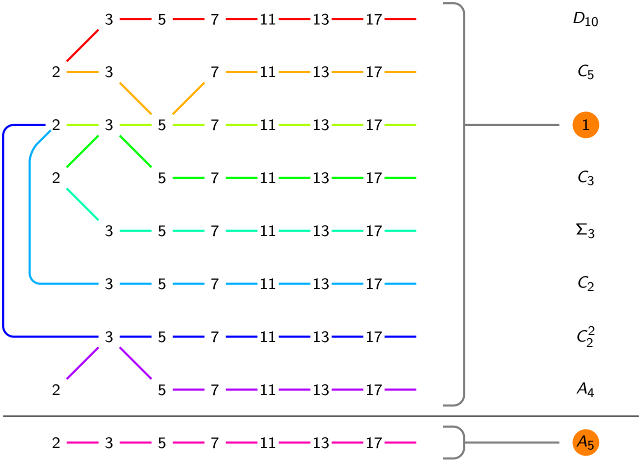

There is one connected component for each conjugacy class of perfect subgroups.

The case of A_5

There is one minimal prime ideal for each conjugacy class of subgroups. All prime ideals are maximal or minimal, so the Krull dimension is one.

The nonequivariant chromatic picture

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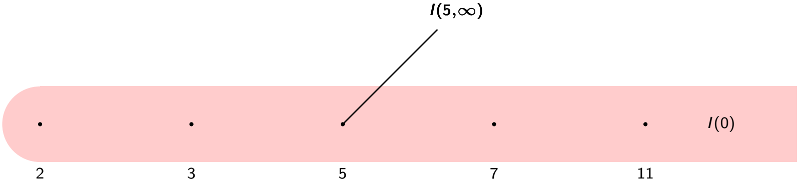

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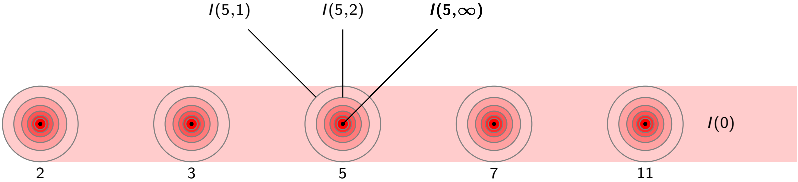

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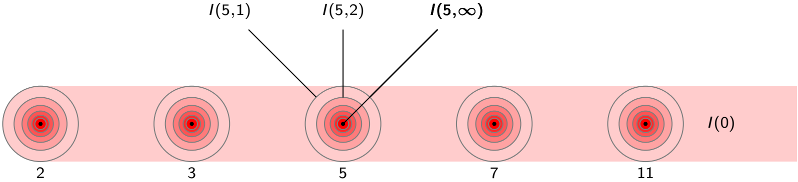

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Prime ideals of spectra

- A *prime ideal of finite spectra* is a subcategory $\mathcal{P}$ such that
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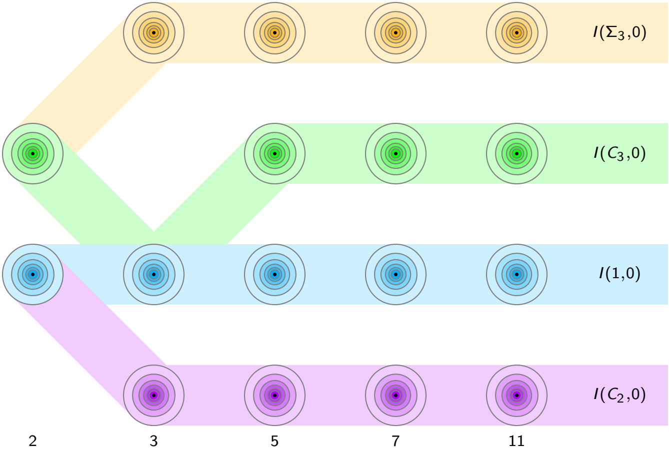
An initial equivariant conjecture



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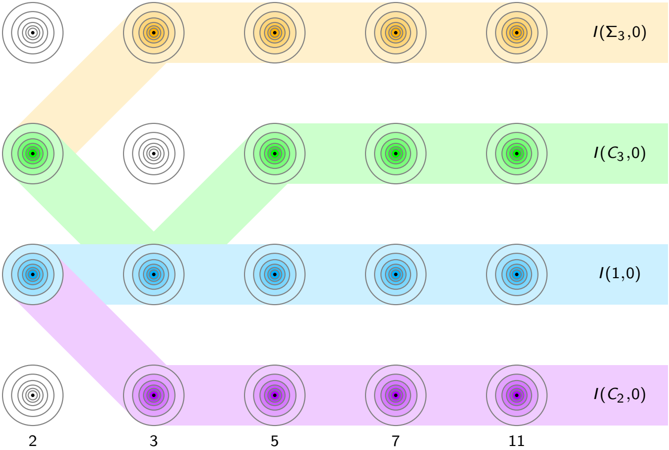

Simply replace each maximal ideal in $A(G)$ by an infinite tower of ideals. We can define $I(H, p, n)$ and $K(H, p, n)$ and $\mathcal{I}(H, p, n)$ for *all* H, p and n. The above picture would be correct if these depended only on $\mathcal{O}^p(H)$.

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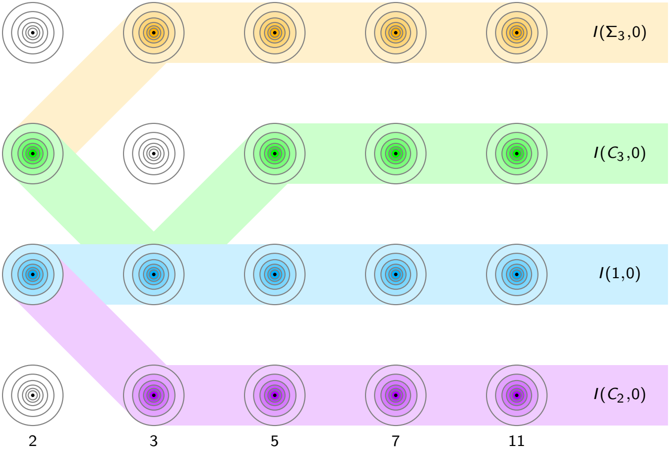

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There is a strong relationship between $I(H,p,n)$ and $I(\mathcal{O}^p(H),p,n)$, but it is not equality.

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- If $\mathcal{C}$ is a thick ideal of equivariant spectra then $\mathcal{C}$ is determined by the set $V(\mathcal{C}) = \{(H, p, n) \mid \mathcal{C} \subseteq \mathcal{I}(H, p, n)\}$.

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- In the abelian case, these relate nicely to the classification of equivariant formal groups over algebraically closed fields.
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- Each Morava K-theory defines a prime ideal of finite G-spectra, $\mathcal{I}(H, p, n) = \{X \mid K(H, p, n)_* X = 0\}$.
- If $\mathcal{C}$ is a thick ideal of equivariant spectra then $\mathcal{C}$ is determined by the set $V(\mathcal{C}) = \{(H, p, n) \mid \mathcal{C} \subseteq \mathcal{I}(H, p, n)\}$.
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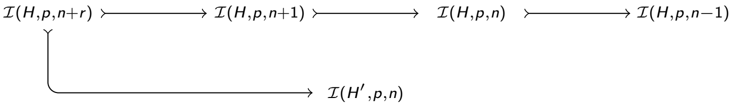

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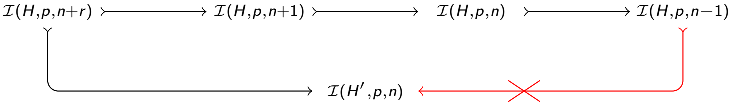

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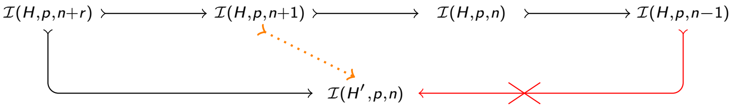

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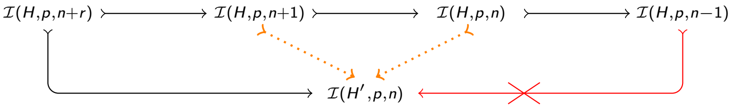

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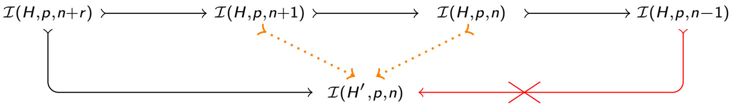

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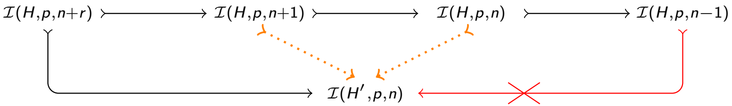

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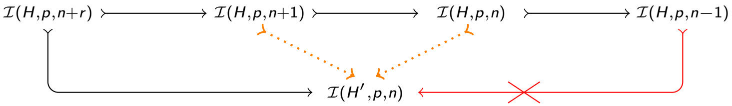

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- In that case we have a complete description of MU^G_*, and an almost-complete comparison with the theory of equivariant formal groups.
- Recently we have considered a new method in equivariant formal group theory that may close the gap.

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