# Introduction to chromatic homotopy

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# Ordinary cohomology

- For any space $X$ we have a cohomology ring $H^*(X)$
- For many spaces this can be described explicitly: for example, if $X = \{$ two-dimensional subspaces of $\mathbb{C}^4\}$ then $H^*(X) = \mathbb{Z}[c_1, c_2]/(c_1^3 - 2c_1c_2, c_1^2c_2 - c_2^2)$.
- We can also consider the scheme $X_H = \text{spec}(H^*(X))$, so $H^*(X)$ is the ring of functions on $X_H$.
- Now $f\colon X \to Y$ gives $f_H\colon X_H \to Y_H$ (depending only on the homotopy class) and $(X \amalg Y)_H = X_H \amalg Y_H$ and $(X \times Y)_H \sim X_H \times Y_H$.
- How good an invariant is this?
  - If $f_H\colon X_H \to Y_H$ is an isomorphism then $f$ is a homotopy equivalence (subject to mild conditions).
  - The map $[X, Y] \to \text{Schemes}(X_H, Y_H) = \text{Rings}(H^*(Y), H^*(X))$ is typically far from being injective or surjective.
  - If $X_H \simeq Y_H$, that is only weak evidence for $X \simeq Y$.
- How to find better invariants?
  (a) Use Steenrod operations on $H^*(X; \mathbb{F}_p)$
  (b) Use generalised cohomology theories.
- But (a) is really part of (b).

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- We often work with *even periodic theories* where $E^1(1) = 0$ and $E^{-2}(1)$ contains a unit. Here it is natural to focus on $E^0(X)$.
- Given an even periodic theory $E$ we put $X_E = \text{spf}(E^0X)$.
- There is an even periodic theory $KU$ with $KU^*(1) = \mathbb{Z}[u, u^{-1}]$ (where $|u| = -2$) and $KU^0(X)$ is the ring of virtual complex vector bundles on $X$.
- Put $MP(n) = \{(v, V) \mid v \in V \leq \mathbb{C}^{2n}\}_\infty$ and $\Sigma^m X = (\mathbb{R}^m \times X)_\infty$ and $MP^k(X) = \varinjlim_n [\Sigma^{2n-k}X, MP(n)]$.

  This gives an even periodic theory with $MP^*(1) = \mathbb{Z}[a_1, a_2, a_3, \dots]$. This is called *periodic complex cobordism*.
- The Nilpotence (pre)Theorem of Hopkins-Devinatz-Smith: if $MP^*(u) = 0$ then $u^k = 0$ for large $k$. This is the most powerful known theorem of the type algebra $\Rightarrow$ topology.
- Fix a prime $p$ and an integer $n > 0$. There is then an even periodic theory $K(p, n)$ with $K(p, n)^*(1) = \mathbb{F}_p[u, u^{-1}]$. This is called *Morava K-theory*.
- The $K(p, n)$'s together carry roughly the same information as $MP$.

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- The $K(p, n)$'s together carry roughly the same information as $MP$.

# Generalised cohomology

- A *generalised cohomology theory* is a contravariant, homotopy invariant functor $E^*\colon$ Spaces $\to$ Rings$^*$ with properties similar to $H^*$, but $E^*(1)$ need not be $\mathbb{Z}$. It takes work to provide interesting examples.
- We often work with *even periodic theories* where $E^1(1) = 0$ and $E^{-2}(1)$ contains a unit. Here it is natural to focus on $E^0(X)$.
- Given an even periodic theory $E$ we put $X_E = \mathrm{spf}(E^0 X)$.
- There is an even periodic theory $KU$ with $KU^*(1) = \mathbb{Z}[u, u^{-1}]$ (where $|u| = -2$) and $KU^0(X)$ is the ring of virtual complex vector bundles on $X$.
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# Formal groups — what are they good for?

- Every even periodic theory $E$ gives a formal group $P_E$.
- The functor $E \mapsto P_E$ is not too far from being an equivalence.
- The most elementary examples of formal groups are the additive and multiplicative formal groups; these correspond to $HP$ and $KU$. (Here $HP^i(X) = \prod_j H^{i+2j}(X)$.)
- Steenrod operations in $HP^0(X; \mathbb{F}_p)$ and Adams operations in $KU^0(X)$ are closely related to endomorphisms of the associated formal groups.
- The ring $MP^0(1)$ is naturally isomorphic to the Lazard ring, which plays a central role in formal group theory.
- The Morava $K$-theories $K(p, n)$ all have different formal groups.
- Together with $HP^0(X; \mathbb{F}_p)$ and $HP^0(X; \mathbb{Q})$ this gives all formal groups over fields up to Galois twisting.
- For many spaces $X$ the scheme $X_E$ can be described naturally in terms of $P_E$. For example, if $X = BU(n) = \{n - \text{dimensional subspaces of } \mathbb{C}^\infty\}$ then $X_E = (P_E^n)/\Sigma_n$.

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# Examples of formal groups

- For any ring $R$ we define commutative groups as follows:
  - $G_a(R) = \{a \in R \mid a \text{ is nilpotent }\}$ (under addition)
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- The functors $G_i$ are *formal groups*; the power series $F_i$ are *formal group laws*.
- Axioms: $F(s, 0) = s$, $F(s, t) = F(t, s)$ and $F(F(s, t), u) = F(s, F(t, u))$.
- More general version: we have a ground ring $k$, and $G(R)$ is only functorial for $k$-algebras, and $F(s, t) \in k[\![s, t]\!]$.
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- For $E = K(p, n)$ we have $\pi^*(x) = x^{p^n}$ and the height is $n$.
- Over an algebraically closed field of characteristic $p$, any two formal groups of the same height are isomorphic.

# Formal groups from even periodic theories

- $P = (\mathbb{C}[t] \setminus \{0\})/\mathbb{C}^\times = \{1 - \text{dim subspaces of } \mathbb{C}[t]\} = \mathbb{C}P^\infty$.
- This is a commutative topological monoid (with inverses up to homotopy).
- So $P_E$ is a formal group scheme over $1_E = \text{spec}(E^0(1))$.
- We can calculate $E^*(\mathbb{C}P^n)$ by induction on $n$ using Mayer-Vietoris. It follows that there exists $x$ with $E^0(P) = E^0(1)[\![x]\!]$ (but there is no canonical choice of $x$).
- This gives $E^0(P \times P) = E^0(1)[\![x_1, x_2]\!]$. The multiplication map $\mu \colon P \times P \to P$ has $\mu^*(x) = F(x_1, x_2)$ for some formal group law $F$.
- Now fix a prime $p$ and let $\pi \colon P \to P$ be the $p$'th power map and put $B = (\mathbb{C}[t] \setminus \{0\})/C_p$.
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## The Lazard ring

- Consider a formal power series $F(s,t) = \sum_{i,j} b_{ij} s^i t^j \in k[\![s,t]\!]$. When is this an FGL?
- For $F(s,0) = s$ we need $b_{i0} = \delta_{i,1}$. For $F(s,t) = F(t,s)$ we need $b_{ij} = b_{ji}$.
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- There are no more relations: any power series of the above form satisfies the FGL conditions up to errors of order 5.
- Lazard's theorem: we can continue to define $a_4, a_5, \dots$ so that $F(s,t)$ can be expressed in terms of the $a_i$, and no further relations are required to make the associativity axiom hold.
- Reformulation: over the Lazard ring $L = \mathbb{Z}[a_1, a_2, \dots]$ there is a universal formal group law $F_u$ such that the resulting map $\mathrm{Rings}(L,k) \to \mathrm{FGL}(k)$ is bijective for all $k$.

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- Recall $MP^0(X) = \varinjlim_n [\Sigma^{2n}X, MP(n)]$ (for $X$ a finite complex). Both $P$ and $MP(n)$ are defined using complex linear algebra so it is not hard to give an explicit $x$ with $MP^0(P) = MP^0(1)[\![x]\!]$. (We do not need to know $MP^0(1)$ for this.)
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- Outline of proof:
  - Assemble the spaces $MP(n)$ into a single "spectrum" called $MP$. (This is the start of stable homotopy theory.)
  - There are good methods for calcuating the homology of spaces defined using complex linear algebra, and one can use them to prove that

$$H_*(MP) = \mathbb{Z}[b_0, b_1, b_2, \dots][b_0^{-1}].$$

  - A simple topological construction gives a map $MP^0(1) \to H_*(MP)$. We can push forward the FGL over $MP^0(1)$ to get an FGL over $H_*(MP)$.
  - In fact this is $F(s,t) = f^{-1}(f(s) + f(t))$, where $f(t) = \sum_i b_i t^{i+1}$. So $f$ gives an isomorphism from $F$ to the additive law $F_a(s,t) = s + t$.
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- A *finite spectrum* is an expression $\Sigma^n X$, where $X$ is a based finite simplicial complex, and $n \in \mathbb{Z}$. (This can be interpreted as a space if $n \geq 0$, but not necessarily if $n < 0$.) We write $\mathcal{F}$ for the class of finite spectra.
- We define $\mathcal{F}(\Sigma^n X, \Sigma^m Y) = \varinjlim_k [\Sigma^{n+k} X, \Sigma^{m+k} Y]$. This has a natural structure as a (finitely generated) Abelian group. There is a composition rule making $\mathcal{F}$ an additive category.
- This is an approximation to the homotopy category of finite complexes, and has a rich and interesting structure.
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- Fact: if $K(p,n)_*(X) = 0$, then $K(p,m)_*(X) = 0$ for all $m < n$ (including $K(p,0)_*(X) = H_*(X;\mathbb{Q})$).
- Also, if $K(p,n)_*(X) = 0$ for all $p$ and $n$ then $X = 0$.
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- Periodicity theorem: if $X \in \mathcal{F}(p,n)$ with $n > 0$ then there is a map $v\colon \Sigma^d X \to X$ (for some $d > 0$) giving an isomorphism on $K(p,n)_*(X)$ (and having a number of other properties, making it "almost unique").
- Thick subcategory theorem: if $\mathcal{C}$ is a subcategory of $\mathcal{F}$ satisfying some natural conditions, then it must be one of the categories $\mathcal{F}(p,n)$.
- Chromatic convergence theorem: $\pi_*^S(X) = \mathcal{F}(S^*, X)$ can be built up in layers. The difference between layers $n$ and $n-1$ is in some sense controlled by $K(p,n)$, and consists of families that are periodic of period $2(p^n - 1)p^k$ for large $k$.

## The chromatic filtration

- Fact: if $K(p,n)_*(X)=0$, then $K(p,m)_*(X)=0$ for all $m<n$ (including $K(p,0)_*(X)=H_*(X;\mathbb{Q})$).
- Also, if $K(p,n)_*(X)=0$ for all $p$ and $n$ then $X=0$.
- Say $X$ has *type* $n$ at $p$ if $K(p,n)_*(X)\neq 0$ and $K(p,m)_*(X)=0$ for $m<n$. Let $\mathcal{F}(p,n)$ be the category of $X$ of type at least $n$ at $p$.
- Nilpotence theorem: if $u\colon \Sigma^d X\to X$ and $K(p,n)_*(u)=0$ for all $(p,n)$ then $u^k=0\colon \Sigma^{dk}X\to X$ for $k\gg 0$.
- Periodicity theorem: if $X\in\mathcal{F}(p,n)$ with $n>0$ then there is a map $v\colon \Sigma^d X\to X$ (for some $d>0$) giving an isomorphism on $K(p,n)_*(X)$ (and having a number of other properties, making it "almost unique").
- Thick subcategory theorem: if $\mathcal{C}$ is a subcategory of $\mathcal{F}$ satisfying some natural conditions, then it must be one of the categories $\mathcal{F}(p,n)$.
- Chromatic convergence theorem: $\pi_*^S(X)=\mathcal{F}(S^*,X)$ can be built up in layers. The difference between layers $n$ and $n-1$ is in some sense controlled by $K(p,n)$, and consists of families that are periodic of period $2(p^n-1)p^k$ for large $k$.