Moduli of stable curves of genus zero

Neil Strickland

March 24, 2009

Overview

Overview

- This talk describes results from the PhD thesis of Daniel Singh.

Overview

- This talk describes results from the PhD thesis of Daniel Singh.
- Let S be a finite set. Put $S_+ = S \amalg \{+\}$. We will describe a class of objects called *stable S_+-marked curves of genus zero*.

Overview

- This talk describes results from the PhD thesis of Daniel Singh.
- Let S be a finite set. Put $S_+ = S \amalg \{+\}$. We will describe a class of objects called *stable* S_+*-marked curves of genus zero*.
- (**Convention:** the set S, and many subsets of S that we will mention, are required to have size at least two. We will not say this explicitly.)

Overview

- This talk describes results from the PhD thesis of Daniel Singh.
- Let S be a finite set. Put $S_+ = S \amalg \{+\}$. We will describe a class of objects called *stable S_+-marked curves of genus zero*.
- (**Convention:** the set S, and many subsets of S that we will mention, are required to have size at least two. We will not say this explicitly.)
- We will write $\mathcal{X}_S$ for the space of isomorphism classes of such objects. It is a compact complex manifold of dimension $|S| - 2$. It has been studied extensively, especially for applications in quantum cohomology.

Overview

- This talk describes results from the PhD thesis of Daniel Singh.
- Let S be a finite set. Put $S_+ = S \amalg \{+\}$. We will describe a class of objects called *stable S_+-marked curves of genus zero*.
- (**Convention:** the set S, and many subsets of S that we will mention, are required to have size at least two. We will not say this explicitly.)
- We will write $\mathcal{X}_S$ for the space of isomorphism classes of such objects. It is a compact complex manifold of dimension $|S| - 2$. It has been studied extensively, especially for applications in quantum cohomology.
- There are various constructions of $\mathcal{X}_S$ in the literature, using abstract methods from algebraic geometry (geometric invariant theory, Chow quotients, iterated blowups). We will describe a more elementary model.

Overview

- This talk describes results from the PhD thesis of Daniel Singh.
- Let S be a finite set. Put $S_+ = S \amalg \{+\}$. We will describe a class of objects called *stable S_+-marked curves of genus zero*.
- (**Convention:** the set S, and many subsets of S that we will mention, are required to have size at least two. We will not say this explicitly.)
- We will write $\mathcal{X}_S$ for the space of isomorphism classes of such objects. It is a compact complex manifold of dimension $|S| - 2$. It has been studied extensively, especially for applications in quantum cohomology.
- There are various constructions of $\mathcal{X}_S$ in the literature, using abstract methods from algebraic geometry (geometric invariant theory, Chow quotients, iterated blowups). We will describe a more elementary model.
- One way to think about it: instead of Kapranov's carefully constructed sequence of blowups depending on some arbitrary choices, we perform all possible blowups simultaneously. Miraculously, this does not mess things up.

Overview

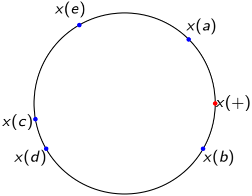
- This talk describes results from the PhD thesis of Daniel Singh.
- Let S be a finite set. Put $S_+ = S \amalg \{+\}$. We will describe a class of objects called *stable S_+-marked curves of genus zero*.
- (**Convention:** the set S, and many subsets of S that we will mention, are required to have size at least two. We will not say this explicitly.)
- We will write $\mathcal{X}_S$ for the space of isomorphism classes of such objects. It is a compact complex manifold of dimension $|S| - 2$. It has been studied extensively, especially for applications in quantum cohomology.
- There are various constructions of $\mathcal{X}_S$ in the literature, using abstract methods from algebraic geometry (geometric invariant theory, Chow quotients, iterated blowups). We will describe a more elementary model.
- One way to think about it: instead of Kapranov's carefully constructed sequence of blowups depending on some arbitrary choices, we perform all possible blowups simultaneously. Miraculously, this does not mess things up.
- The cohomology of $\mathcal{X}_S$ was described by Sean Keel. We will give an alternative description that fits more neatly with Singh's geometric description of the space.

Generic S_+-marked curves of genus zero

Generic S_+-marked curves of genus zero

- A *generic S_+-marked curve of genus zero* is a pair (C, x), where C is an algebraic curve isomorphic to $\mathbb{C}P^1$, and $x\colon S_+ \to C$ is an injective map.

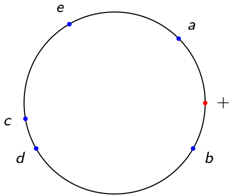
$$S = \{a, b, c, d, e\}$$



Generic S_+-marked curves of genus zero

- A *generic S_+-marked curve of genus zero* is a pair (C, x), where C is an algebraic curve isomorphic to $\mathbb{C}P^1$, and $x\colon S_+ \to C$ is an injective map.

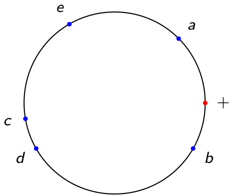
$$S = \{a, b, c, d, e\}$$



Generic S_+-marked curves of genus zero

- A *generic S_+-marked curve of genus zero* is a pair (C, x), where C is an algebraic curve isomorphic to $\mathbb{C}P^1$, and $x\colon S_+ \to C$ is an injective map.

$$S = \{a, b, c, d, e\}$$

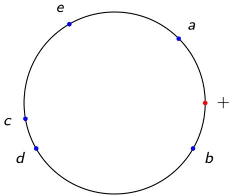


- We write $\mathcal{X}'_S$ for the set of isomorphism classes of such objects.

Generic S_+-marked curves of genus zero

- A *generic S_+-marked curve of genus zero* is a pair (C, x), where C is an algebraic curve isomorphic to $\mathbb{C}P^1$, and $x\colon S_+ \to C$ is an injective map.

$$S = \{a, b, c, d, e\}$$



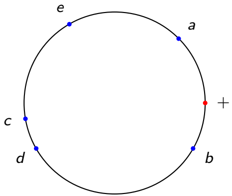
- We write $\mathcal{X}'_S$ for the set of isomorphism classes of such objects.
-

$$V_S = \mathrm{Map}(S, \mathbb{C})/\mathbb{C}$$

Generic S_+-marked curves of genus zero

- A *generic S_+-marked curve of genus zero* is a pair (C, x), where C is an algebraic curve isomorphic to $\mathbb{C}P^1$, and $x\colon S_+ \to C$ is an injective map.

$$S = \{a, b, c, d, e\}$$



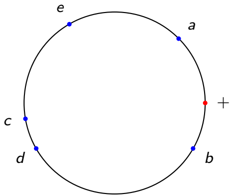
- We write $\mathcal{X}'_S$ for the set of isomorphism classes of such objects.
-

$$V_S = \mathrm{Map}(S, \mathbb{C})/\mathbb{C}$$
$$PV_S = \{L \leq V_S \mid \dim(L) = 1\} \simeq \mathbb{C}P^{|S|-2}$$

Generic S_+-marked curves of genus zero

- A *generic S_+-marked curve of genus zero* is a pair (C, x), where C is an algebraic curve isomorphic to $\mathbb{C}P^1$, and $x\colon S_+ \to C$ is an injective map.

$$S = \{a, b, c, d, e\}$$



- We write $\mathcal{X}'_S$ for the set of isomorphism classes of such objects.

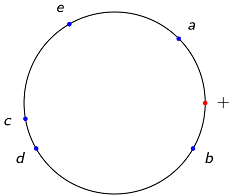
$$V_S = \mathrm{Map}(S, \mathbb{C})/\mathbb{C} \qquad \widetilde{U}_S = \mathrm{Inj}(S, \mathbb{C})/\mathbb{C} \subset V_S$$

$$PV_S = \{L \leq V_S \mid \dim(L) = 1\} \simeq \mathbb{C}P^{|S|-2}$$

Generic S_+-marked curves of genus zero

- A *generic S_+-marked curve of genus zero* is a pair (C, x), where C is an algebraic curve isomorphic to $\mathbb{C}P^1$, and $x\colon S_+ \to C$ is an injective map.

$$S = \{a, b, c, d, e\}$$



- We write $\mathcal{X}'_S$ for the set of isomorphism classes of such objects.
-

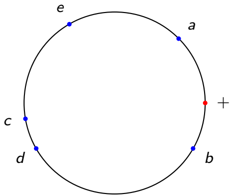
$$V_S = \mathrm{Map}(S, \mathbb{C})/\mathbb{C} \qquad \widetilde{U}_S = \mathrm{Inj}(S, \mathbb{C})/\mathbb{C} \subset V_S$$

$$PV_S = \{L \leq V_S \mid \dim(L) = 1\} \simeq \mathbb{C}P^{|S|-2} \qquad U_S = \text{ image of } \widetilde{U}_S \text{ in } PV_S.$$

Generic S_+-marked curves of genus zero

- A *generic S_+-marked curve of genus zero* is a pair (C, x), where C is an algebraic curve isomorphic to $\mathbb{C}P^1$, and $x\colon S_+ \to C$ is an injective map.

$S = \{a, b, c, d, e\}$



- We write $\mathcal{X}'_S$ for the set of isomorphism classes of such objects.

$$V_S = \mathrm{Map}(S, \mathbb{C})/\mathbb{C} \qquad \widetilde{U}_S = \mathrm{Inj}(S, \mathbb{C})/\mathbb{C} \subset V_S$$

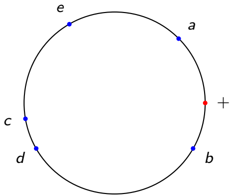
$$PV_S = \{L \leq V_S \mid \dim(L) = 1\} \simeq \mathbb{C}P^{|S|-2} \qquad U_S = \text{ image of } \widetilde{U}_S \text{ in } PV_S.$$

- Let $M(C, x)$ be the (2-dimensional) space of holomorphic maps $f\colon C \setminus \{x(+)\} \to \mathbb{C}$ with at worst a simple pole at $x(+)$.

Generic S_+-marked curves of genus zero

- A *generic S_+-marked curve of genus zero* is a pair (C, x), where C is an algebraic curve isomorphic to $\mathbb{C}P^1$, and $x \colon S_+ \to C$ is an injective map.

$$S = \{a, b, c, d, e\}$$



- We write $\mathcal{X}'_S$ for the set of isomorphism classes of such objects.

$$V_S = \mathrm{Map}(S, \mathbb{C})/\mathbb{C} \qquad \widetilde{U}_S = \mathrm{Inj}(S, \mathbb{C})/\mathbb{C} \subset V_S$$

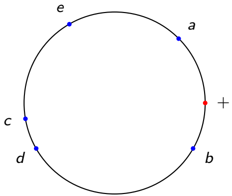
$$PV_S = \{L \leq V_S \mid \dim(L) = 1\} \simeq \mathbb{C}P^{|S|-2} \qquad U_S = \text{ image of } \widetilde{U}_S \text{ in } PV_S.$$

- Let $M(C, x)$ be the (2-dimensional) space of holomorphic maps $f \colon C \setminus \{x(+)\} \to \mathbb{C}$ with at worst a simple pole at $x(+)$. Let $\lambda(C, x)$ be the image of the composite $M(C, x)/\mathbb{C} \xrightarrow{x^*} V_S$.

Generic S_+-marked curves of genus zero

- A *generic S_+-marked curve of genus zero* is a pair (C, x), where C is an algebraic curve isomorphic to $\mathbb{C}P^1$, and $x\colon S_+ \to C$ is an injective map.

$$S = \{a, b, c, d, e\}$$



- We write $\mathcal{X}'_S$ for the set of isomorphism classes of such objects.
-

$$V_S = \mathrm{Map}(S, \mathbb{C})/\mathbb{C} \qquad \widetilde{U}_S = \mathrm{Inj}(S, \mathbb{C})/\mathbb{C} \subset V_S$$

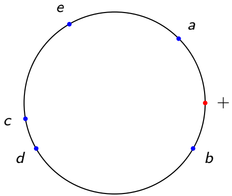
$$PV_S = \{L \leq V_S \mid \dim(L) = 1\} \simeq \mathbb{C}P^{|S|-2} \qquad U_S = \text{ image of } \widetilde{U}_S \text{ in } PV_S.$$

- Let $M(C, x)$ be the (2-dimensional) space of holomorphic maps $f\colon C \setminus \{x(+)\} \to \mathbb{C}$ with at worst a simple pole at $x(+)$. Let $\lambda(C, x)$ be the image of the composite $M(C, x)/\mathbb{C} \xrightarrow{x^*} V_S$. This defines a bijection $\lambda\colon \mathcal{X}'_S \to U_S \subset PV_S$.

Generic S_+-marked curves of genus zero

- A *generic S_+-marked curve of genus zero* is a pair (C, x), where C is an algebraic curve isomorphic to $\mathbb{C}P^1$, and $x\colon S_+ \to C$ is an injective map.

$$S = \{a, b, c, d, e\}$$



- We write $\mathcal{X}'_S$ for the set of isomorphism classes of such objects.
-

$$V_S = \mathrm{Map}(S, \mathbb{C})/\mathbb{C} \qquad \widetilde{U}_S = \mathrm{Inj}(S, \mathbb{C})/\mathbb{C} \subset V_S$$

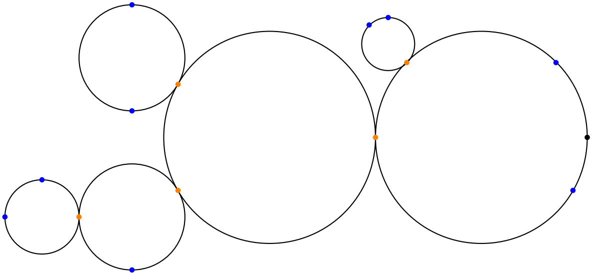
$$PV_S = \{L \leq V_S \mid \dim(L) = 1\} \simeq \mathbb{C}P^{|S|-2} \qquad U_S = \text{ image of } \widetilde{U}_S \text{ in } PV_S.$$

- Let $M(C, x)$ be the (2-dimensional) space of holomorphic maps $f\colon C \setminus \{x(+)\} \to \mathbb{C}$ with at worst a simple pole at $x(+)$. Let $\lambda(C, x)$ be the image of the composite $M(C, x)/\mathbb{C} \xrightarrow{x^*} V_S$. This defines a bijection $\lambda\colon \mathcal{X}'_S \to U_S \subset PV_S$. Note that $\mathcal{X}'_S \simeq U_S$ is not compact.

Stable S_+-marked curves of genus zero

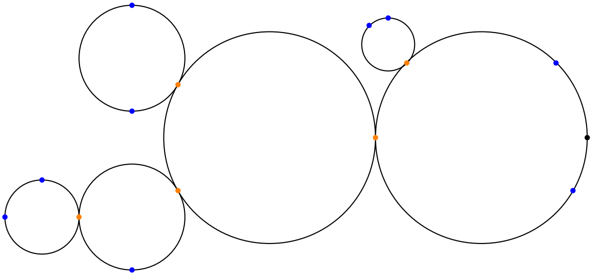
Stable S_+-marked curves of genus zero

- A *stable S_+-marked curve of genus zero* is a pair (C, x), where C is an algebraic curve and $x \colon S_+ \to C$ is an injective map, subject to certain conditions.



Stable S_+-marked curves of genus zero

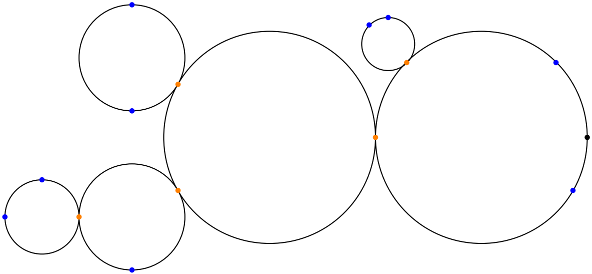
- A *stable S_+-marked curve of genus zero* is a pair (C, x), where C is an algebraic curve and $x\colon S_+ \to C$ is an injective map, subject to certain conditions.



- Each irreducible component must isomorphic to $\mathbb{C}P^1$.

Stable S_+-marked curves of genus zero

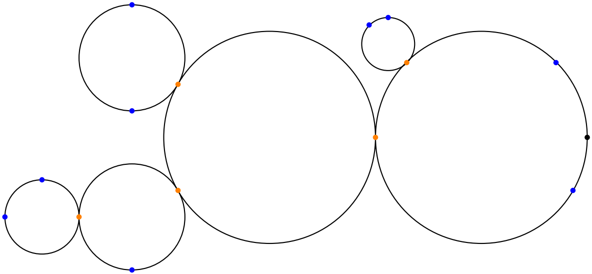
- A *stable S_+-marked curve of genus zero* is a pair (C, x), where C is an algebraic curve and $x\colon S_+ \to C$ is an injective map, subject to certain conditions.



- Each irreducible component must isomorphic to $\mathbb{C}P^1$.
- Any singularities must be ordinary double points ($\sim \mathbb{C}[\![x, y]\!]/(xy)$).

Stable S_+-marked curves of genus zero

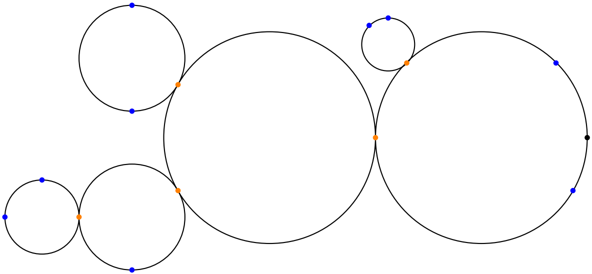
- A *stable S_+-marked curve of genus zero* is a pair (C, x), where C is an algebraic curve and $x\colon S_+ \to C$ is an injective map, subject to certain conditions.



- Each irreducible component must isomorphic to $\mathbb{C}P^1$.
- Any singularities must be ordinary double points ($\sim \mathbb{C}[\![x, y]\!]/(xy)$).
- All marked points must be smooth.

Stable S_+-marked curves of genus zero

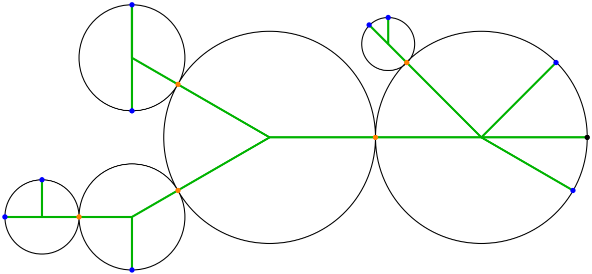
- A *stable S_+-marked curve of genus zero* is a pair (C, x), where C is an algebraic curve and $x\colon S_+ \to C$ is an injective map, subject to certain conditions.



- Each irreducible component must isomorphic to $\mathbb{C}P^1$.
- Any singularities must be ordinary double points ($\sim \mathbb{C}[\![x, y]\!]/(xy)$).
- All marked points must be smooth.
- Each $\mathbb{C}P^1$ must contain ≥ 3 points that are either singular or marked.

Stable S_+-marked curves of genus zero

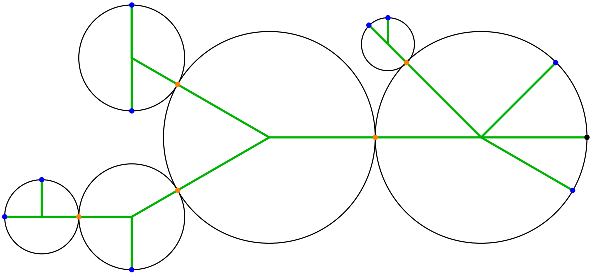
- A *stable S_+-marked curve of genus zero* is a pair (C, x), where C is an algebraic curve and $x\colon S_+ \to C$ is an injective map, subject to certain conditions.



- Each irreducible component must isomorphic to $\mathbb{C}P^1$.
- Any singularities must be ordinary double points ($\sim \mathbb{C}[\![x,y]\!]/(xy)$).
- All marked points must be smooth.
- Each $\mathbb{C}P^1$ must contain ≥ 3 points that are either singular or marked.
- The graph of components must be a tree.

Stable S_+-marked curves of genus zero

- A *stable S_+-marked curve of genus zero* is a pair (C, x), where C is an algebraic curve and $x\colon S_+ \to C$ is an injective map, subject to certain conditions.



- Each irreducible component must isomorphic to $\mathbb{C}P^1$.
- Any singularities must be ordinary double points ($\sim \mathbb{C}[\![x, y]\!]/(xy)$).
- All marked points must be smooth.
- Each $\mathbb{C}P^1$ must contain ≥ 3 points that are either singular or marked.
- The graph of components must be a tree.
- We write $\mathcal{X}_S$ for the set of isomorphism classes of such objects.

The projective model

The projective model

- For $U \subseteq T \subseteq S$ we have a restriction map $\mathrm{Map}(T, \mathbb{C}) \twoheadrightarrow \mathrm{Map}(U, \mathbb{C})$ inducing a map $\rho^T_U \colon V_T \to V_U$.

The projective model

- For $U \subseteq T \subseteq S$ we have a restriction map $\mathrm{Map}(T, \mathbb{C}) \twoheadrightarrow \mathrm{Map}(U, \mathbb{C})$ inducing a map $\rho_U^T \colon V_T \to V_U$.
- Consider an element $M = (M_T)_{T \subseteq S}$ in the product $\mathcal{P}_S = \prod_{T \subseteq S} PV_T$. We say that M is *coherent* if for all $U \subseteq T$ we have $M_T \leq (\rho_U^T)^{-1}(M_U)$ or equivalently $\rho_U^T(M_T) \in \{0, M_U\}$.

The projective model

- For $U \subseteq T \subseteq S$ we have a restriction map $\mathrm{Map}(T, \mathbb{C}) \twoheadrightarrow \mathrm{Map}(U, \mathbb{C})$ inducing a map $\rho^T_U \colon V_T \to V_U$.
- Consider an element $M = (M_T)_{T \subseteq S}$ in the product $\mathcal{P}_S = \prod_{T \subseteq S} PV_T$. We say that M is *coherent* if for all $U \subseteq T$ we have $M_T \leq (\rho^T_U)^{-1}(M_U)$ or equivalently $\rho^T_U(M_T) \in \{0, M_U\}$.
- We write $\mathcal{M}_S$ for the subspace of coherent points in $\mathcal{P}_S$. This is a kind of inverse limit of a diagram involving partially defined maps $PV_T \to PV_U$.

The projective model

- For $U \subseteq T \subseteq S$ we have a restriction map $\mathrm{Map}(T, \mathbb{C}) \twoheadrightarrow \mathrm{Map}(U, \mathbb{C})$ inducing a map $\rho_U^T \colon V_T \to V_U$.
- Consider an element $M = (M_T)_{T \subseteq S}$ in the product $\mathcal{P}_S = \prod_{T \subseteq S} PV_T$. We say that M is *coherent* if for all $U \subseteq T$ we have $M_T \leq (\rho_U^T)^{-1}(M_U)$ or equivalently $\rho_U^T(M_T) \in \{0, M_U\}$.
- We write $\mathcal{M}_S$ for the subspace of coherent points in $\mathcal{P}_S$. This is a kind of inverse limit of a diagram involving partially defined maps $PV_T \to PV_U$.
- **Theorem:** the scheme $\mathcal{X}_S$ is naturally isomorphic to $\mathcal{M}_S$.

The projective model

- For $U \subseteq T \subseteq S$ we have a restriction map $\mathrm{Map}(T, \mathbb{C}) \twoheadrightarrow \mathrm{Map}(U, \mathbb{C})$ inducing a map $\rho^T_U \colon V_T \to V_U$.
- Consider an element $M = (M_T)_{T \subseteq S}$ in the product $\mathcal{P}_S = \prod_{T \subseteq S} PV_T$. We say that M is *coherent* if for all $U \subseteq T$ we have $M_T \leq (\rho^T_U)^{-1}(M_U)$ or equivalently $\rho^T_U(M_T) \in \{0, M_U\}$.
- We write $\mathcal{M}_S$ for the subspace of coherent points in $\mathcal{P}_S$. This is a kind of inverse limit of a diagram involving partially defined maps $PV_T \to PV_U$.
- **Theorem:** the scheme $\mathcal{X}_S$ is naturally isomorphic to $\mathcal{M}_S$.
- There is a projection map $\pi \colon \mathcal{M}_{S_+} \to \mathcal{M}_S$, and each fibre $\pi^{-1}\{x\}$ is naturally an S_+-marked stable curve of genus 0. We thus have a map $\mu \colon \mathcal{M}_S \to \mathcal{X}_S$ sending x to the isomorphism type of $\pi^{-1}\{x\}$.

The projective model

- For $U \subseteq T \subseteq S$ we have a restriction map $\mathrm{Map}(T, \mathbb{C}) \twoheadrightarrow \mathrm{Map}(U, \mathbb{C})$ inducing a map $\rho^T_U \colon V_T \to V_U$.
- Consider an element $M = (M_T)_{T \subseteq S}$ in the product $\mathcal{P}_S = \prod_{T \subseteq S} PV_T$. We say that M is *coherent* if for all $U \subseteq T$ we have $M_T \leq (\rho^T_U)^{-1}(M_U)$ or equivalently $\rho^T_U(M_T) \in \{0, M_U\}$.
- We write $\mathcal{M}_S$ for the subspace of coherent points in $\mathcal{P}_S$. This is a kind of inverse limit of a diagram involving partially defined maps $PV_T \to PV_U$.
- **Theorem:** the scheme $\mathcal{X}_S$ is naturally isomorphic to $\mathcal{M}_S$.
- There is a projection map $\pi \colon \mathcal{M}_{S_+} \to \mathcal{M}_S$, and each fibre $\pi^{-1}\{x\}$ is naturally an S_+-marked stable curve of genus 0. We thus have a map $\mu \colon \mathcal{M}_S \to \mathcal{X}_S$ sending x to the isomorphism type of $\pi^{-1}\{x\}$.
- The map $\lambda \colon \mathcal{X}'_S \to U_S \subset PV_S$ extends uniquely (via the same definition) to give a map $\lambda \colon \mathcal{X}_S \to PV_S$.

The projective model

- For $U \subseteq T \subseteq S$ we have a restriction map $\mathrm{Map}(T,\mathbb{C}) \twoheadrightarrow \mathrm{Map}(U,\mathbb{C})$ inducing a map $\rho^T_U\colon V_T \to V_U$.
- Consider an element $M = (M_T)_{T\subseteq S}$ in the product $\mathcal{P}_S = \prod_{T\subseteq S} PV_T$. We say that M is *coherent* if for all $U \subseteq T$ we have $M_T \leq (\rho^T_U)^{-1}(M_U)$ or equivalently $\rho^T_U(M_T) \in \{0, M_U\}$.
- We write $\mathcal{M}_S$ for the subspace of coherent points in $\mathcal{P}_S$. This is a kind of inverse limit of a diagram involving partially defined maps $PV_T \to PV_U$.
- **Theorem:** the scheme $\mathcal{X}_S$ is naturally isomorphic to $\mathcal{M}_S$.
- There is a projection map $\pi\colon \mathcal{M}_{S_+} \to \mathcal{M}_S$, and each fibre $\pi^{-1}\{x\}$ is naturally an S_+-marked stable curve of genus 0. We thus have a map $\mu\colon \mathcal{M}_S \to \mathcal{X}_S$ sending x to the isomorphism type of $\pi^{-1}\{x\}$.
- The map $\lambda\colon \mathcal{X}'_S \to U_S \subset PV_S$ extends uniquely (via the same definition) to give a map $\lambda\colon \mathcal{X}_S \to PV_S$.
- There is a "stable forgetting" map $\mathcal{X}_S \to \mathcal{X}_T$ as follows: given $(C, x) \in \mathcal{X}_S$ take $(C, x|_T)$ and collapse to a point any irreducible component that does not contain at least thee points that are marked or singular.

The projective model

- For $U \subseteq T \subseteq S$ we have a restriction map $\mathrm{Map}(T, \mathbb{C}) \twoheadrightarrow \mathrm{Map}(U, \mathbb{C})$ inducing a map $\rho^T_U \colon V_T \to V_U$.
- Consider an element $M = (M_T)_{T \subseteq S}$ in the product $\mathcal{P}_S = \prod_{T \subseteq S} PV_T$. We say that M is *coherent* if for all $U \subseteq T$ we have $M_T \leq (\rho^T_U)^{-1}(M_U)$ or equivalently $\rho^T_U(M_T) \in \{0, M_U\}$.
- We write $\mathcal{M}_S$ for the subspace of coherent points in $\mathcal{P}_S$. This is a kind of inverse limit of a diagram involving partially defined maps $PV_T \to PV_U$.
- **Theorem:** the scheme $\mathcal{X}_S$ is naturally isomorphic to $\mathcal{M}_S$.
- There is a projection map $\pi \colon \mathcal{M}_{S_+} \to \mathcal{M}_S$, and each fibre $\pi^{-1}\{x\}$ is naturally an S_+-marked stable curve of genus 0. We thus have a map $\mu \colon \mathcal{M}_S \to \mathcal{X}_S$ sending x to the isomorphism type of $\pi^{-1}\{x\}$.
- The map $\lambda \colon \mathcal{X}'_S \to U_S \subset PV_S$ extends uniquely (via the same definition) to give a map $\lambda \colon \mathcal{X}_S \to PV_S$.
- There is a "stable forgetting" map $\mathcal{X}_S \to \mathcal{X}_T$ as follows: given $(C, x) \in \mathcal{X}_S$ take $(C, x|_T)$ and collapse to a point any irreducible component that does not contain at least thee points that are marked or singular.
- By combining these stable forgetting maps with the maps $\lambda_T \colon \mathcal{X}_T \to PV_T$ we obtain a canonical map $\nu \colon \mathcal{X}_S \to \mathcal{M}_S$.

The projective model

- For $U \subseteq T \subseteq S$ we have a restriction map $\mathrm{Map}(T,\mathbb{C}) \twoheadrightarrow \mathrm{Map}(U,\mathbb{C})$ inducing a map $\rho_U^T\colon V_T \to V_U$.
- Consider an element $M = (M_T)_{T\subseteq S}$ in the product $\mathcal{P}_S = \prod_{T\subseteq S} PV_T$. We say that M is *coherent* if for all $U \subseteq T$ we have $M_T \leq (\rho_U^T)^{-1}(M_U)$ or equivalently $\rho_U^T(M_T) \in \{0, M_U\}$.
- We write $\mathcal{M}_S$ for the subspace of coherent points in $\mathcal{P}_S$. This is a kind of inverse limit of a diagram involving partially defined maps $PV_T \to PV_U$.
- **Theorem:** the scheme $\mathcal{X}_S$ is naturally isomorphic to $\mathcal{M}_S$.
- There is a projection map $\pi\colon \mathcal{M}_{S_+} \to \mathcal{M}_S$, and each fibre $\pi^{-1}\{x\}$ is naturally an S_+-marked stable curve of genus 0. We thus have a map $\mu\colon \mathcal{M}_S \to \mathcal{X}_S$ sending x to the isomorphism type of $\pi^{-1}\{x\}$.
- The map $\lambda\colon \mathcal{X}'_S \to U_S \subset PV_S$ extends uniquely (via the same definition) to give a map $\lambda\colon \mathcal{X}_S \to PV_S$.
- There is a "stable forgetting" map $\mathcal{X}_S \to \mathcal{X}_T$ as follows: given $(C,x) \in \mathcal{X}_S$ take $(C, x|_T)$ and collapse to a point any irreducible component that does not contain at least thee points that are marked or singular.
- By combining these stable forgetting maps with the maps $\lambda_T\colon \mathcal{X}_T \to PV_T$ we obtain a canonical map $\nu\colon \mathcal{X}_S \to \mathcal{M}_S$. It works out that ν is an isomorphism of varieties, with inverse μ.

Vector bundles and cohomology

Vector bundles and cohomology

- Let L_T be the line bundle over $\mathcal{M}_S$ whose fibre at M is M_T.

Vector bundles and cohomology

- Let L_T be the line bundle over $\mathcal{M}_S$ whose fibre at M is M_T.
- Let $x_T \in H^2(\mathcal{M}_S)$ be the Euler class of L_T.

Vector bundles and cohomology

- Let L_T be the line bundle over $\mathcal{M}_S$ whose fibre at M is M_T.
- Let $x_T \in H^2(\mathcal{M}_S)$ be the Euler class of L_T.
- If T and U overlap then the map $V_{T\cup U} \to V_T \oplus V_U$ is injective

Vector bundles and cohomology

- Let L_T be the line bundle over $\mathcal{M}_S$ whose fibre at M is M_T.
- Let $x_T \in H^2(\mathcal{M}_S)$ be the Euler class of L_T.
- If T and U overlap then the map $V_{T\cup U} \to V_T \oplus V_U$ is injective, so $L_{T\cup U} \leq L_T \oplus L_U$

Vector bundles and cohomology

- Let L_T be the line bundle over $\mathcal{M}_S$ whose fibre at M is M_T.
- Let $x_T \in H^2(\mathcal{M}_S)$ be the Euler class of L_T.
- If T and U overlap then the map $V_{T\cup U} \to V_T \oplus V_U$ is injective, so $L_{T\cup U} \leq L_T \oplus L_U$, so $(x_{T\cup U} - x_T)(x_{T\cup U} - x_U) = 0$.

Vector bundles and cohomology

- Let L_T be the line bundle over $\mathcal{M}_S$ whose fibre at M is M_T.
- Let $x_T \in H^2(\mathcal{M}_S)$ be the Euler class of L_T.
- If T and U overlap then the map $V_{T\cup U} \to V_T \oplus V_U$ is injective, so $L_{T\cup U} \leq L_T \oplus L_U$, so $(x_{T\cup U} - x_T)(x_{T\cup U} - x_U) = 0$.
- Now suppose that $U_1, \dots, U_r$ are disjoint subsets of T. Put $m = (|T| - 1) - \sum_i (|U_i| - 1)$. There is a short exact sequence

$$0 \to \mathbb{C}^m \to V_T \to \bigoplus_i V_{U_i} \to 0.$$

Vector bundles and cohomology

- Let L_T be the line bundle over $\mathcal{M}_S$ whose fibre at M is M_T.
- Let $x_T \in H^2(\mathcal{M}_S)$ be the Euler class of L_T.
- If T and U overlap then the map $V_{T\cup U} \to V_T \oplus V_U$ is injective, so $L_{T\cup U} \leq L_T \oplus L_U$, so $(x_{T\cup U} - x_T)(x_{T\cup U} - x_U) = 0$.
- Now suppose that $U_1, \dots, U_r$ are disjoint subsets of T. Put $m = (|T| - 1) - \sum_i (|U_i| - 1)$. There is a short exact sequence

$$0 \to \mathbb{C}^m \to V_T \to \bigoplus_i V_{U_i} \to 0.$$

It follows that $L_T \leq \mathbb{C}^m \oplus \bigoplus_i L_{U_i}$

Vector bundles and cohomology

- Let L_T be the line bundle over $\mathcal{M}_S$ whose fibre at M is M_T.
- Let $x_T \in H^2(\mathcal{M}_S)$ be the Euler class of L_T.
- If T and U overlap then the map $V_{T\cup U} \to V_T \oplus V_U$ is injective, so $L_{T\cup U} \leq L_T \oplus L_U$, so $(x_{T\cup U} - x_T)(x_{T\cup U} - x_U) = 0$.
- Now suppose that $U_1, \dots, U_r$ are disjoint subsets of T. Put $m = (|T| - 1) - \sum_i (|U_i| - 1)$. There is a short exact sequence

$$0 \to \mathbb{C}^m \to V_T \to \bigoplus_i V_{U_i} \to 0.$$

It follows that $L_T \leq \mathbb{C}^m \oplus \bigoplus_i L_{U_i}$, and thus

$$x_T^m \prod_i (x_T - x_{U_i}) = 0.$$

- Let L_T be the line bundle over $\mathcal{M}_S$ whose fibre at M is M_T.
- Let $x_T \in H^2(\mathcal{M}_S)$ be the Euler class of L_T.
- If T and U overlap then the map $V_{T\cup U} \to V_T \oplus V_U$ is injective, so $L_{T\cup U} \leq L_T \oplus L_U$, so $(x_{T\cup U} - x_T)(x_{T\cup U} - x_U) = 0$.
- Now suppose that $U_1, \dots, U_r$ are disjoint subsets of T. Put $m = (|T| - 1) - \sum_i (|U_i| - 1)$. There is a short exact sequence

$$0 \to \mathbb{C}^m \to V_T \to \bigoplus_i V_{U_i} \to 0.$$

It follows that $L_T \leq \mathbb{C}^m \oplus \bigoplus_i L_{U_i}$, and thus

$$x_T^m \prod_i (x_T - x_{U_i}) = 0.$$

- **Theorem:** $H^*(\mathcal{M}_S)$ is generated by the classes x_T subject only to the relations above.

Vector bundles and cohomology

- Let L_T be the line bundle over $\mathcal{M}_S$ whose fibre at M is M_T.
- Let $x_T \in H^2(\mathcal{M}_S)$ be the Euler class of L_T.
- If T and U overlap then the map $V_{T\cup U} \to V_T \oplus V_U$ is injective, so $L_{T\cup U} \leq L_T \oplus L_U$, so $(x_{T\cup U} - x_T)(x_{T\cup U} - x_U) = 0$.
- Now suppose that $U_1, \dots, U_r$ are disjoint subsets of T. Put $m = (|T| - 1) - \sum_i (|U_i| - 1)$. There is a short exact sequence

$$0 \to \mathbb{C}^m \to V_T \to \bigoplus_i V_{U_i} \to 0.$$

  It follows that $L_T \leq \mathbb{C}^m \oplus \bigoplus_i L_{U_i}$, and thus

$$x_T^m \prod_i (x_T - x_{U_i}) = 0.$$

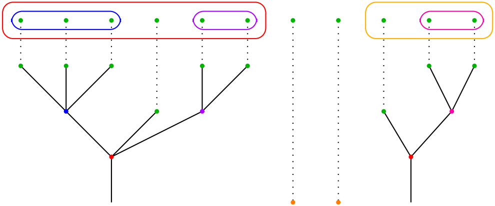
- **Theorem:** $H^*(\mathcal{M}_S)$ is generated by the classes x_T subject only to the relations above.
- For the proof and also for further details of the structure, we need some combinatorics.

Arbology

Arbology

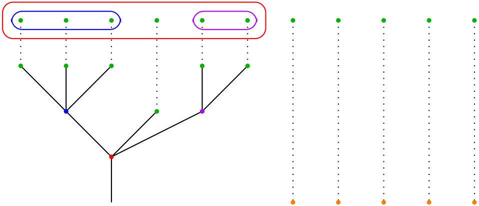
- A *forest* is a collection $\mathcal{F}$ of subsets of S such that for all $U, V \in \mathcal{F}$ we have $U \cap V = \emptyset$ or $U \subseteq V$ or $V \subseteq U$.

- A *forest* is a collection $\mathcal{F}$ of subsets of S such that for all $U, V \in \mathcal{F}$ we have $U \cap V = \emptyset$ or $U \subseteq V$ or $V \subseteq U$.



Arbology

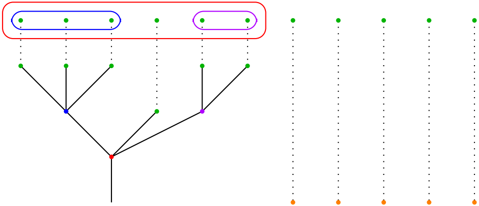
- A *forest* is a collection $\mathcal{F}$ of subsets of S such that for all $U, V \in \mathcal{F}$ we have $U \cap V = \emptyset$ or $U \subseteq V$ or $V \subseteq U$.



- A *tree* is a forest with only one maximal element.

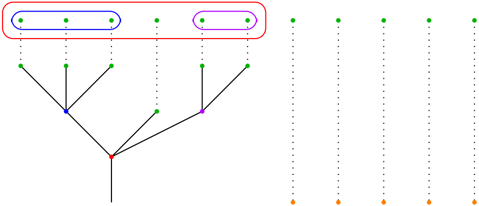
Arbology

- A *forest* is a collection $\mathcal{F}$ of subsets of S such that for all $U, V \in \mathcal{F}$ we have $U \cap V = \emptyset$ or $U \subseteq V$ or $V \subseteq U$.



- A *tree* is a forest with only one maximal element.
- A tree is *autumnal* if there are leaves on the ground, otherwise *vernal*.

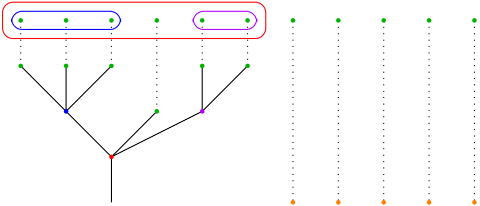
- A *forest* is a collection $\mathcal{F}$ of subsets of S such that for all $U, V \in \mathcal{F}$ we have $U \cap V = \emptyset$ or $U \subseteq V$ or $V \subseteq U$.



- A *tree* is a forest with only one maximal element.
- A tree is *autumnal* if there are leaves on the ground, otherwise *vernal*.
- Suppose that $M \in \mathcal{M}_S$. Say that $T \subseteq S$ is *M-critical* if for all strictly larger sets $U \supset T$ we have $\rho^U_T(M_U) = 0$. Let $\mathrm{type}(M)$ be the collection of all M-critical sets. Then $\mathrm{type}(M)$ is a vernal tree.

Arbology

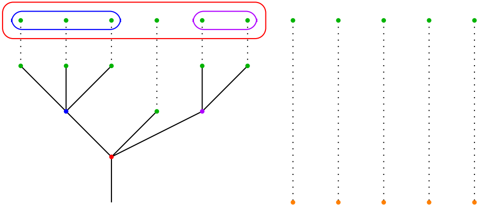
- A *forest* is a collection $\mathcal{F}$ of subsets of S such that for all $U, V \in \mathcal{F}$ we have $U \cap V = \emptyset$ or $U \subseteq V$ or $V \subseteq U$.



- A *tree* is a forest with only one maximal element.
- A tree is *autumnal* if there are leaves on the ground, otherwise *vernal*.
- Suppose that $M \in \mathcal{M}_S$. Say that $T \subseteq S$ is *M-critical* if for all strictly larger sets $U \supset T$ we have $\rho^U_T(M_U) = 0$. Let type(M) be the collection of all M-critical sets. Then type(M) is a vernal tree. These trees correspond to the component trees of stable curves as drawn previously.

Arbology

- A *forest* is a collection $\mathcal{F}$ of subsets of S such that for all $U, V \in \mathcal{F}$ we have $U \cap V = \emptyset$ or $U \subseteq V$ or $V \subseteq U$.



- A *tree* is a forest with only one maximal element.
- A tree is *autumnal* if there are leaves on the ground, otherwise *vernal*.
- Suppose that $M \in \mathcal{M}_S$. Say that $T \subseteq S$ is *M-critical* if for all strictly larger sets $U \supset T$ we have $\rho_T^U(M_U) = 0$. Let $\text{type}(M)$ be the collection of all M-critical sets. Then $\text{type}(M)$ is a vernal tree. These trees correspond to the component trees of stable curves as drawn previously.
- The stratification by tree type is an important tool for studying the geometry of $\mathcal{M}_S$. The pure strata are products of copies of the spaces $\mathcal{X}'_T \simeq U_T \subset PV_T$.

Shapes and bases

Shapes and bases

- Given a monomial $y = \prod_T x_T^{n_T}$, the *shape* of y is $\{T \mid n_T > 0\}$.

Shapes and bases

- Given a monomial $y = \prod_T x_T^{n_T}$, the *shape* of y is $\{T \mid n_T > 0\}$.
- Given a forest $\mathcal{F}$ and a set $T \in \mathcal{F}$, let $U_1, \dots, U_r$ be the maximal sets in $\{U \in \mathcal{F} \mid U \subset T\}$. Then put

$$m(\mathcal{F}, T) = (|T| - 1) - \sum_i (|U_i| - 1),$$

so we have a relation $x_T^{m(\mathcal{F},T)} \prod_i (x_T - x_{U_i}) = 0$ in $H^*(\mathcal{M}_S)$.

Shapes and bases

- Given a monomial $y = \prod_T x_T^{n_T}$, the *shape* of y is $\{T \mid n_T > 0\}$.
- Given a forest $\mathcal{F}$ and a set $T \in \mathcal{F}$, let $U_1, \ldots, U_r$ be the maximal sets in $\{U \in \mathcal{F} \mid U \subset T\}$. Then put

$$m(\mathcal{F}, T) = (|T| - 1) - \sum_i (|U_i| - 1),$$

  so we have a relation $x_T^{m(\mathcal{F},T)} \prod_i (x_T - x_{U_i}) = 0$ in $H^*(\mathcal{M}_S)$.
- We say that y is *admissible* if shape(y) is a forest and $n_T < m(\text{shape}(y), T)$ for all $T \in \text{shape}(y)$.

Shapes and bases

- Given a monomial $y = \prod_T x_T^{n_T}$, the *shape* of y is $\{T \mid n_T > 0\}$.
- Given a forest $\mathcal{F}$ and a set $T \in \mathcal{F}$, let $U_1, \dots, U_r$ be the maximal sets in $\{U \in \mathcal{F} \mid U \subset T\}$. Then put

$$m(\mathcal{F}, T) = (|T| - 1) - \sum_i (|U_i| - 1),$$

  so we have a relation $x_T^{m(\mathcal{F},T)} \prod_i (x_T - x_{U_i}) = 0$ in $H^*(\mathcal{M}_S)$.
- We say that y is *admissible* if $\mathrm{shape}(y)$ is a forest and $n_T < m(\mathrm{shape}(y), T)$ for all $T \in \mathrm{shape}(y)$.
- **Theorem:** the admissible monomials give a basis for $H^*(\mathcal{M}_S)$ over $\mathbb{Z}$.

Shapes and bases

- Given a monomial $y = \prod_T x_T^{n_T}$, the *shape* of y is $\{T \mid n_T > 0\}$.
- Given a forest $\mathcal{F}$ and a set $T \in \mathcal{F}$, let $U_1, \ldots, U_r$ be the maximal sets in $\{U \in \mathcal{F} \mid U \subset T\}$. Then put

$$m(\mathcal{F}, T) = (|T| - 1) - \sum_i (|U_i| - 1),$$

  so we have a relation $x_T^{m(\mathcal{F},T)} \prod_i (x_T - x_{U_i}) = 0$ in $H^*(\mathcal{M}_S)$.
- We say that y is *admissible* if shape(y) is a forest and $n_T < m(\text{shape}(y), T)$ for all $T \in \text{shape}(y)$.
- **Theorem:** the admissible monomials give a basis for $H^*(\mathcal{M}_S)$ over $\mathbb{Z}$. In particular, $\{x_S^{|S|-2}\}$ is a basis for the top group $H^{2|S|-4}(\mathcal{M}_S)$.

Shapes and bases

- Given a monomial $y = \prod_T x_T^{n_T}$, the *shape* of y is $\{T \mid n_T > 0\}$.
- Given a forest $\mathcal{F}$ and a set $T \in \mathcal{F}$, let $U_1, \ldots, U_r$ be the maximal sets in $\{U \in \mathcal{F} \mid U \subset T\}$. Then put

$$m(\mathcal{F}, T) = (|T| - 1) - \sum_i (|U_i| - 1),$$

  so we have a relation $x_T^{m(\mathcal{F},T)} \prod_i (x_T - x_{U_i}) = 0$ in $H^*(\mathcal{M}_S)$.
- We say that y is *admissible* if shape(y) is a forest and $n_T < m(\text{shape}(y), T)$ for all $T \in \text{shape}(y)$.
- **Theorem:** the admissible monomials give a basis for $H^*(\mathcal{M}_S)$ over $\mathbb{Z}$. In particular, $\{x_S^{|S|-2}\}$ is a basis for the top group $H^{2|S|-4}(\mathcal{M}_S)$.
- We say that y is *strongly inadmissible* if there exists $U \subseteq S$ such that $\sum_{T \subseteq U} n_T > |U| - 2$.

Shapes and bases

- Given a monomial $y = \prod_T x_T^{n_T}$, the *shape* of y is $\{T \mid n_T > 0\}$.
- Given a forest $\mathcal{F}$ and a set $T \in \mathcal{F}$, let $U_1, \dots, U_r$ be the maximal sets in $\{U \in \mathcal{F} \mid U \subset T\}$. Then put
$$m(\mathcal{F}, T) = (|T| - 1) - \sum_i (|U_i| - 1),$$
so we have a relation $x_T^{m(\mathcal{F},T)} \prod_i (x_T - x_{U_i}) = 0$ in $H^*(\mathcal{M}_S)$.
- We say that y is *admissible* if $\text{shape}(y)$ is a forest and $n_T < m(\text{shape}(y), T)$ for all $T \in \text{shape}(y)$.
- **Theorem:** the admissible monomials give a basis for $H^*(\mathcal{M}_S)$ over $\mathbb{Z}$. In particular, $\{x_S^{|S|-2}\}$ is a basis for the top group $H^{2|S|-4}(\mathcal{M}_S)$.
- We say that y is *strongly inadmissible* if there exists $U \subseteq S$ such that $\sum_{T \subseteq U} n_T > |U| - 2$.
- **Theorem:** if y is strongly inadmissible then it is zero in $H^*(\mathcal{M}_S)$. If y is not strongly inadmissible then $x_S^i y = x_S^{|S|-2}$ for $i = |S| - 2 - \deg(y)/2$.

Thickets

Thickets

- A *thicket* is a collection $\mathcal{L}$ of subsets of S with the following properties: we have $S \in \mathcal{L}$, and if $U, V \in \mathcal{L}$ and $U \cap V \neq \emptyset$ then $U \cup V \in \mathcal{L}$.

Thickets

- A *thicket* is a collection $\mathcal{L}$ of subsets of S with the following properties: we have $S \in \mathcal{L}$, and if $U, V \in \mathcal{L}$ and $U \cap V \neq \emptyset$ then $U \cup V \in \mathcal{L}$.
- Every vernal tree is a thicket, and $\{T \subseteq S \mid |T| > 1\}$ is also a thicket.

Thickets

- A *thicket* is a collection $\mathcal{L}$ of subsets of S with the following properties: we have $S \in \mathcal{L}$, and if $U, V \in \mathcal{L}$ and $U \cap V \neq \emptyset$ then $U \cup V \in \mathcal{L}$.
- Every vernal tree is a thicket, and $\{T \subseteq S \mid |T| > 1\}$ is also a thicket.
- Given a thicket $\mathcal{L}$ we put $\mathcal{P}[\mathcal{L}] = \prod_{T \in \mathcal{L}} PV_T$. We say that a point M in this space is *coherent* if whenever $U, T \in \mathcal{L}$ and $U \subseteq T$ we have $\rho^T_U(M_T) \leq M_U$. We let $\mathcal{M}[\mathcal{L}]$ denote the subspace of coherent points.

Thickets

- A *thicket* is a collection $\mathcal{L}$ of subsets of S with the following properties: we have $S \in \mathcal{L}$, and if $U, V \in \mathcal{L}$ and $U \cap V \neq \emptyset$ then $U \cup V \in \mathcal{L}$.
- Every vernal tree is a thicket, and $\{T \subseteq S \mid |T| > 1\}$ is also a thicket.
- Given a thicket $\mathcal{L}$ we put $\mathcal{P}[\mathcal{L}] = \prod_{T \in \mathcal{L}} PV_T$. We say that a point M in this space is *coherent* if whenever $U, T \in \mathcal{L}$ and $U \subseteq T$ we have $\rho^T_U(M_T) \leq M_U$. We let $\mathcal{M}[\mathcal{L}]$ denote the subspace of coherent points.
- All theorems stated for $\mathcal{M}_S$ can be adapted to be valid for $\mathcal{M}[\mathcal{L}]$. They are proved inductively in this setting by successively discarding minimal elements from $\mathcal{L}$.

Thickets

- A *thicket* is a collection $\mathcal{L}$ of subsets of S with the following properties: we have $S \in \mathcal{L}$, and if $U, V \in \mathcal{L}$ and $U \cap V \neq \emptyset$ then $U \cup V \in \mathcal{L}$.
- Every vernal tree is a thicket, and $\{T \subseteq S \mid |T| > 1\}$ is also a thicket.
- Given a thicket $\mathcal{L}$ we put $\mathcal{P}[\mathcal{L}] = \prod_{T \in \mathcal{L}} PV_T$. We say that a point M in this space is *coherent* if whenever $U, T \in \mathcal{L}$ and $U \subseteq T$ we have $\rho_U^T(M_T) \leq M_U$. We let $\mathcal{M}[\mathcal{L}]$ denote the subspace of coherent points.
- All theorems stated for $\mathcal{M}_S$ can be adapted to be valid for $\mathcal{M}[\mathcal{L}]$. They are proved inductively in this setting by successively discarding minimal elements from $\mathcal{L}$.
- The induction step involves a blowup square

$$\begin{array}{ccc} \mathcal{M}[\overline{\mathcal{L}}] \times PV_T & \rightarrowtail & \mathcal{M}[\mathcal{L}_+] \\ \downarrow & & \downarrow \\ \mathcal{M}[\overline{\mathcal{L}}] & \rightarrowtail & \mathcal{M}[\mathcal{L}] \end{array}$$

where T is minimal in $\mathcal{L}_+$ and $\mathcal{L} = \mathcal{L}_+ \setminus \{T\}$ and $\overline{\mathcal{L}}$ is an induced thicket on S/T.

Trees as thickets

Trees as thickets

- Let $\mathcal{T}$ be a vernal tree, and let $T_1, \ldots, T_r$ be the maximal proper subsets in $\mathcal{T}$. Put $\mathcal{T}_i = \{U \in \mathcal{T} \mid U \subseteq T_i\}$, which is a vernal tree on T_i.

Trees as thickets

- Let $\mathcal{T}$ be a vernal tree, and let $T_1, \ldots, T_r$ be the maximal proper subsets in $\mathcal{T}$. Put $\mathcal{T}_i = \{U \in \mathcal{T} \mid U \subseteq T_i\}$, which is a vernal tree on T_i.
- The space $\mathcal{M}[\mathcal{T}]$ is then the projective bundle associated to a certain vector bundle over $\prod_i \mathcal{M}[\mathcal{T}_i]$, and both the geometry and the cohomology can be analysed easily from this description.

Trees as thickets

- Let $\mathcal{T}$ be a vernal tree, and let $T_1, \dots, T_r$ be the maximal proper subsets in $\mathcal{T}$. Put $\mathcal{T}_i = \{U \in \mathcal{T} \mid U \subseteq T_i\}$, which is a vernal tree on T_i.
- The space $\mathcal{M}[\mathcal{T}]$ is then the projective bundle associated to a certain vector bundle over $\prod_i \mathcal{M}[\mathcal{T}_i]$, and both the geometry and the cohomology can be analysed easily from this description.
- Let $\mathcal{L}$ be a thicket; then we can find many different vernal trees $\mathcal{T} \subseteq \mathcal{L}$.

Trees as thickets

- Let $\mathcal{T}$ be a vernal tree, and let $T_1, \dots, T_r$ be the maximal proper subsets in $\mathcal{T}$. Put $\mathcal{T}_i = \{U \in \mathcal{T} \mid U \subseteq T_i\}$, which is a vernal tree on T_i.
- The space $\mathcal{M}[\mathcal{T}]$ is then the projective bundle associated to a certain vector bundle over $\prod_i \mathcal{M}[\mathcal{T}_i]$, and both the geometry and the cohomology can be analysed easily from this description.
- Let $\mathcal{L}$ be a thicket; then we can find many different vernal trees $\mathcal{T} \subseteq \mathcal{L}$.
- For each such tree, there is a projection map $\mathcal{M}[\mathcal{L}] \to \mathcal{M}[\mathcal{T}]$, which is an isomorphism over a large open subscheme of $\mathcal{M}[\mathcal{T}]$. Some facts are established by this route rather than by induction on $|\mathcal{L}|$.