Semi-formal verification as a routine tool

Neil Strickland

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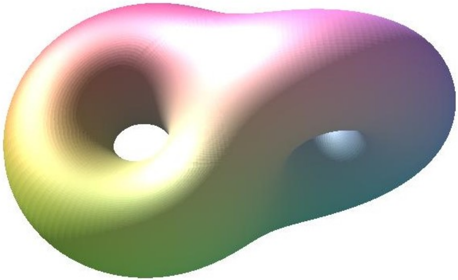
- However, I have used a kind of semi-formal verification based on Maple for a wide range of projects, some of which are large.
- In this talk I will discuss my experience of this, in the hope that it might provoke new ideas about the move to fully formal verification.

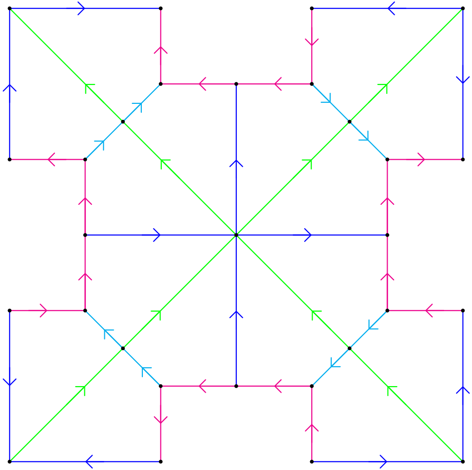
Geometry of embedded surfaces



$$EX^* = \{x \in S^3 \mid (3x_3^2 - 2)x_4 + \sqrt{2}(x_1^2 - x_2^2)x_3 = 0\},$$

This is a large and complex project, with a 160 page monograph, 30000 lines of Maple code, and 2500 assertions checked by Maple. It involves Riemannian geometry of surfaces embedded in S^3

Geometry of embedded surfaces



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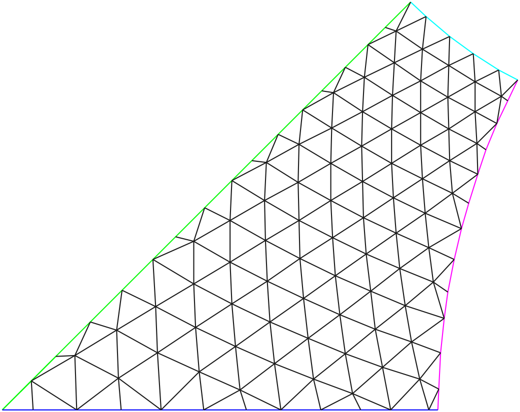

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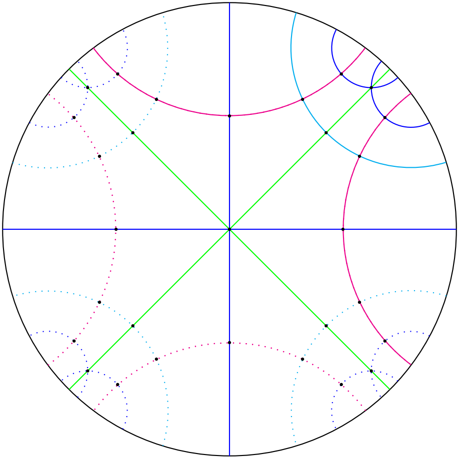

. . . hyperbolic geometry and combinatorial group theory of Fuchsian groups

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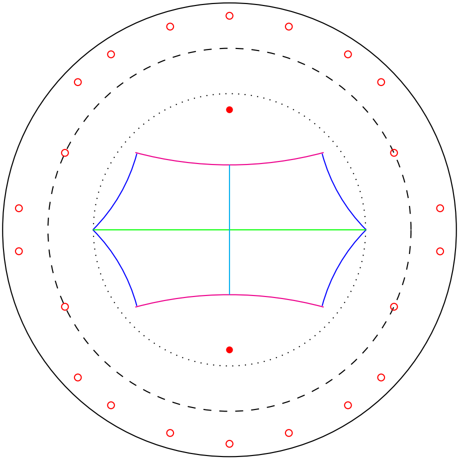

. . . analytic and geometric theory of differential equations, Schwarzian derivatives and conformal mappings

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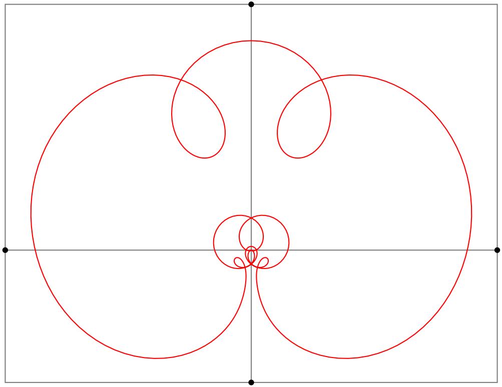

. . . theory of automorphic forms

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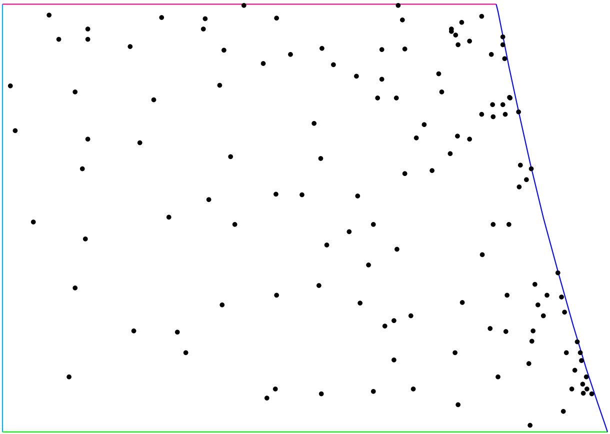

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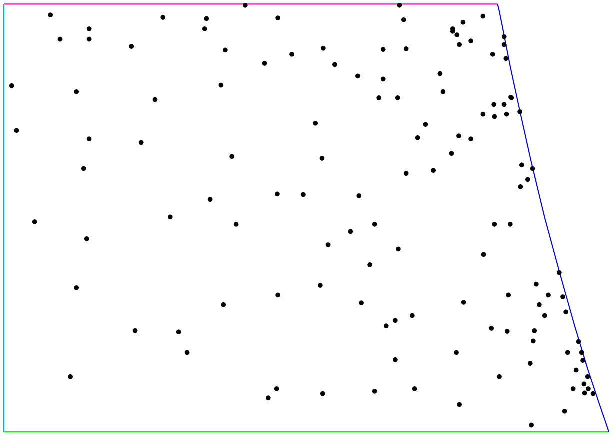

. . . brute force solution of multivariable diophantine equations

. . . and various other ingredients.

Examples of data and formulae

	χ_0	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9
1	1	1	1	1	1	1	1	1	2	2
λ^2	1	1	1	1	1	1	1	1	-2	-2
$\mu\nu$	1	1	-1	-1	1	1	-1	-1	2	-2
$\lambda^2\mu\nu$	1	1	-1	-1	1	1	-1	-1	-2	2
$\lambda^{\pm 1}$	1	-1	1	-1	-1	1	-1	1	0	0
$\mu, \lambda^2\mu$	1	1	-1	-1	-1	-1	1	1	0	0
$\lambda^{\pm 1}\mu$	1	-1	-1	1	1	-1	-1	1	0	0
$\nu, \lambda^2\nu$	1	1	1	1	-1	-1	-1	-1	0	0
$\lambda^{\pm 1}\nu$	1	-1	1	-1	1	-1	1	-1	0	0
$\lambda^{\pm 1}\mu\nu$	1	-1	-1	1	-1	1	1	-1	0	0

Examples of data and formulae

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0			0	$\frac{\pi}{2}$	π	$-\frac{\pi}{2}$	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$-\frac{3\pi}{4}$	$-\frac{\pi}{4}$				
1	0	π					$\frac{\pi}{2}$		$-\frac{\pi}{2}$					
2	0	π						$\frac{\pi}{2}$		$-\frac{\pi}{2}$				
3				$\frac{\pi}{2}$		$-\frac{\pi}{2}$						0		π
4			$-\frac{\pi}{2}$		$\frac{\pi}{2}$						0		π	
5	0											π		
6	0										π			
7		0												π
8		0											π	

$$
\begin{array}{lll}
\lambda(c_0(t)) = c_0(t+\pi/2) & \mu(c_0(t)) = c_0(-t) & \nu(c_0(t)) = c_0(-t) \\
\lambda(c_1(t)) = c_2(t) & \mu(c_1(t)) = c_2(t+\pi) & \nu(c_1(t)) = c_2(-t) \\
\lambda(c_2(t)) = c_1(-t) & \mu(c_2(t)) = c_1(t+\pi) & \nu(c_2(t)) = c_1(-t) \\
\lambda(c_3(t)) = c_4(t) & \mu(c_3(t)) = c_3(t+\pi) & \nu(c_3(t)) = c_3(-t) \\
\lambda(c_4(t)) = c_3(-t) & \mu(c_4(t)) = c_4(-t-\pi) & \nu(c_4(t)) = c_4(t) \\
\lambda(c_5(t)) = c_6(t) & \mu(c_5(t)) = c_7(t) & \nu(c_5(t)) = c_5(t) \\
\lambda(c_6(t)) = c_5(-t) & \mu(c_6(t)) = c_8(-t) & \nu(c_6(t)) = c_6(-t) \\
\lambda(c_7(t)) = c_8(t) & \mu(c_7(t)) = c_5(t) & \nu(c_7(t)) = c_7(t) \\
\lambda(c_8(t)) = c_7(-t) & \mu(c_8(t)) = c_6(-t) & \nu(c_8(t)) = c_8(-t)
\end{array}
$$

Examples of data and formulae

$$
\begin{aligned}
c_0(t) &= j'(-\sqrt{(a^{-1}-a)^2+4\sin^2(2t)},\ e^{it},\ -e^{-it}) \\
c_1(t) &= j'\left(\frac{1+i}{8\sqrt{2}}\sin(t)\sqrt{16\cos(t)^2+(a+a^{-1})^2\sin(t)^4},\ \frac{1+\cos(t)}{2},\ \frac{1-\cos(t)}{2}i\right) \\
c_2(t) &= \lambda(c_1(t)) \\
c_3(t) &= j'\left(-i\frac{a^{-1}-a}{8}\sin(t)\sqrt{(1+a)^4-(1-a)^4\cos(t)^2}\sqrt{(1+a)^2-(1-a)^2\cos(t)^2},\right. \\
&\qquad \left.\frac{(1+a)+(1-a)\cos(t)}{2},\ \frac{(1+a)-(1-a)\cos(t)}{2}\right) \\
c_4(t) &= \lambda(c_3(t)) \\
c_5(t) &= j\left(\frac{\sin(t)}{8}\sqrt{2a(3-\cos(t))(4-a^4(1-\cos(t))^2)},\ a\frac{1-\cos(t)}{2}\right) \\
c_6(t) &= \lambda(c_5(t)) \\
c_7(t) &= \mu(c_5(t)) \\
c_8(t) &= \lambda\mu(c_5(t)).
\end{aligned}
$$

Here $a \in (0,1)$ so all square roots are of positive quantities. Maple is able to take this into account when simplifying but there is no documentation of the algorithm used.

Examples of data and formulae

There are morphisms of elliptic curves

$$E^+(a) \xrightarrow{\pi^+} E^-(a) \xrightarrow{\pi^-} E^+(a)$$

given generically by

$$\pi^+(j(y,x)) = j\left(\frac{\sqrt{2}y((1-x)^2+b_-^2x^2)}{((1-x)^2-b_-^2x^2)^2}, \frac{2x(x-1)}{((1-x)^2-b_-^2x^2)}\right)$$
$$\pi^-(j(y,x)) = j\left(\frac{\sqrt{2}y((1-x)^2-b_+^2x^2)}{((1-x)^2+b_+^2x^2)^2}, \frac{2x(x-1)}{((1-x)^2+b_+^2x^2)}\right).$$

We need to verify that the denominators do not cause trouble. For this we use homogeneous coordinates, check some polynomial identities that were found by Gröbner methods, and apply some logic.

Examples of data and formulae

Fuchsian group with generators $\beta_0, \ldots, \beta_7, \lambda, \mu, \nu$ and relations

$$\beta_{k+4} = \beta_k^{-1} \qquad \beta_0\beta_1\beta_2\beta_3\beta_4\beta_5\beta_6\beta_7 = 1 \qquad \lambda^4 = \mu^2 = \nu^2 = (\lambda\nu)^2 = 1$$

$$(\lambda\mu)^2 = \beta_7\beta_6 \qquad (\nu\mu)^2 = \beta_6\beta_0\beta_7\beta_6 \qquad \lambda\beta_k\lambda^{-1} = \lambda_*(\beta_k) \qquad \mu\beta_k\mu = \mu_*(\beta_k) \qquad \nu\beta_k\nu = \nu_*(\beta_k).$$

	β_0	β_1	β_2	β_3	β_4	β_5	β_6	β_7
λ_*	β_2	β_3	β_4	β_5	β_6	β_7	β_0	β_1
μ_*	$\beta_2\beta_0\beta_1$	$\beta_5\beta_4\beta_3$	$\beta_0\beta_7\beta_6$	$\beta_2\beta_3\beta_1$	$\beta_5\beta_4\beta_6$	$\beta_7\beta_0\beta_1$	$\beta_2\beta_3\beta_4$	$\beta_5\beta_7\beta_6$
ν_*	β_0	$\beta_2\beta_1\beta_2$	β_6	$\beta_0\beta_7\beta_0$	β_4	$\beta_6\beta_5\beta_6$	β_2	$\beta_4\beta_3\beta_4$

Fix $b \in (0,1)$ with $b_\pm = \sqrt{1 \pm b^2}$ and consider action on unit disc by

$$\lambda(z) = iz \qquad \beta_0(z) = \frac{b_+ z + 1}{z + b_+}$$

$$\mu(z) = \frac{b_+ z - b^2 - i}{(b^2 - i)z - b_+} \qquad \beta_1(z) = \frac{b_+^3 z - (2+i)b^2 - i}{((i-2)b^2 + i)z + b_+^3}$$

$$\nu(z) = \overline{z} \qquad \beta_{2n}(z) = i^n \beta_0(z/i^n)$$

$$\beta_{2n+1}(z) = i^n \beta_1(z/i^n).$$

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- Expressions with trig functions, radicals, positivity conditions: trust Maple, more cautiously. Also check random examples.
- More complex proofs: isolate independent (in)equalities as far as possible, check them as above or with random examples, with high precision numerics where appropriate.

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- Maple allows, but does not enforce, various kinds of type checking. This limits the level of rigour that can be achieved, but in practice it seems a good compromise.
- Automated testing definitely catches many errors. A large fraction are just transcription problems, or arise from adjustments in one part of the project not properly carried over to other parts. Some others are more subtle and interesting.

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- There is code for various rings, often arising as (generalised) cohomology rings of specific topological spaces. This typically involves Gröbner bases.
- There is code for many morphisms and (co)operad structures.

Chained preorders

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- Example: given a finite set A and $N > 0$ let $\mathrm{ICP}_N(A)$ be the set of chains $(Q_1, \dots, Q_N)$ of preorders on A such that Q_1 is total, Q_N is separated, any two elements are Q_i-comparable iff Q_{i-1}-equivalent. This set is itself finite, and partially ordered.

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- There is code to enumerate the elements of $\mathrm{ICP}_N(A)$ (for small (A, N)). Translation would implicitly involve a formal proof of correctness; probably substantial extra work.

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- There is code to generate random elements of $\mathrm{ICP}_N(A)$ (tractable for larger (A, N)). The algorithm overweights special cases. This is extremely valuable for exploration and hypothesis testing. I do not know how to fit anything like that in standard proof assistants.

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- We are interested in the homotopy theory of (the geometric realisation of) $\mathrm{ICP}_N(A)$ and related posets. There is code for various general constructions in this theory (shellings, discrete Morse theory, the geometric realisation itself).
- The discipline of coding all posets, morphisms and operad structures helps ensure that everything is properly specified. We can test statements on a complete list or random selection of elements to ensure that edge cases are handled correctly.