Higher representations of symmetric groups

Neil Strickland

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The main theorem

Theorem of S.; building on work of
Hunton, Hopkins, Kuhn, Ravenel, Kashiwabara, Wilson

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Connected, $\pi_1(BG) = G$, other $\pi_n(BG) = 0$
Central to equivariant homotopy theory
One model: $B\Sigma_k = \{A \subset \mathbb{R}^\infty \mid |A| = k\}$

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$E^0B(\Sigma_k \times \Sigma_{p^d-k}) \to E^0B\Sigma_{p^d}$ for $0 < k < p^d$.

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$\mathrm{Sub}_d(\mathbb{G})$ is the moduli scheme of finite subgroup schemes of $\mathbb{G}$ of order p^d.

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- R_d is naturally self-dual as an E^0-module, and so is a Gorenstein ring.

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- Calculations for particular groups by Kriz, Lee, Tezuka, Yagita, Schuster, Bakuradze, Priddy.

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- Consider the space $\coprod_k B\Sigma_k$, which has a very rich algebraic structure; the theory of Hopf rings is useful here.
- Compare with QS^0 and QS^2 by the Snaith splitting and the Thom isomorphism. Compare QS^0 with $\Omega^\infty BP$ using work of Kashiwabara and Wilson.