Double subdivision of relative categories

Neil Strickland

February 9, 2024

Introduction

- Recall: a *quasicategory* is simplicial set with fillers for all inner horns.
- For $n \in \mathbb{N}$ we have a poset $[n] = \{0, \ldots, n\}$. Posets can be regarded as categories, with one morphism from x to y if $x \leq y$, and none otherwise.
- For any category $\mathcal{C}$, we have a simplicial set $N\mathcal{C}$ with $(N\mathcal{C})_n = \mathrm{Cat}([n], \mathcal{C})$.
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- For any simplicial set X, we have a homotopy category $\mathrm{Ho}(X)$ with $\mathrm{obj}(\mathrm{Ho}(X)) = X_0$, morphisms generated by X_1, one relation $d_1(u) = d_0(u) \circ d_2(u)$ for each $u \in X_2$.

- This satisfies $\mathrm{Cat}(\mathrm{Ho}(X), \mathcal{C}) = \mathrm{sSet}(X, N\mathcal{C})$ for all categories $\mathcal{C}$, i.e. $\mathrm{Ho}\colon \mathrm{sSet} \to \mathrm{Cat}$ is left adjoint to $N\colon \mathrm{Cat} \to \mathrm{sSet}$. Also $\mathrm{Ho}(N\mathcal{C}) \simeq \mathcal{C}$.
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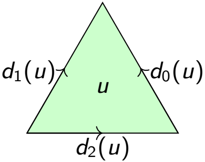

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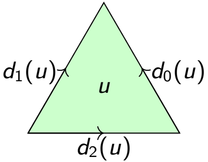

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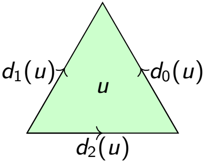

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- Problem: construct examples of quasicategories from natural input data.
- One construction is the *coherent nerve* of a simplicial/topological/differential graded category. But that is only appropriate when *all* objects of $\mathcal{C}$ are homotopically well-behaved.
- Often we start with a *relative category*, i.e. a category $\mathcal{C}$ with a class $\text{we} \subseteq \text{mor}(\mathcal{C})$ of *weak equivalences*
(containing all identities and closed under composition).
- We want to construct a *relative nerve* $N\mathcal{C}$ which should be a quasicategory with $\text{Ho}(N\mathcal{C}) = \mathcal{C}[\text{we}^{-1}]$.
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We seek a more direct argument.

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The relative posets Ξ_n

- Ξ_n is the set of sets of the form $\theta = \{\sigma_0, \sigma_1, \ldots, \sigma_r\}$, where
$$\emptyset \neq \sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_r \subseteq [n].$$
- Order this by $\theta \leq \theta'$ iff $\theta \subseteq \theta'$, and so regard Ξ_n as a category.
- Define nondecreasing $\pi\colon \Xi_n \to [n]$ by $\pi(\theta) = \min(\sigma_0) = \min(\bigcap \theta)$.
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- Suppose that $\mathcal{C}$ is discrete, i.e. we $= \{1_c \mid c \in \mathrm{obj}(\mathcal{C})\}$. Then any relative functor $\Xi_n \to \mathcal{C}$ factors uniquely through $\pi\colon \Xi_n \to [n]$, so $N\mathcal{C}$ is just the ordinary nerve.

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- Ξ_n is the set of sets of the form $\theta = \{\sigma_0, \sigma_1, \dots, \sigma_r\}$, where

$$\emptyset \neq \sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_r \subseteq [n].$$

- Order this by $\theta \leq \theta'$ iff $\theta \subseteq \theta'$, and so regard Ξ_n as a category.
- Define nondecreasing $\pi\colon \Xi_n \to [n]$ by $\pi(\theta) = \min(\sigma_0) = \min(\bigcap \theta)$.
- For $\theta \leq \theta'$, declare that $\theta \to \theta'$ is a weak equivalence iff $\pi(\theta) = \pi(\theta')$. This makes Ξ_n a relative category.
- For $u \in \mathbf{\Delta}(n, m)$ and $\emptyset \neq \sigma \subseteq [n]$ define $u_*(\sigma) = \{u(i) \mid i \in \sigma\}$.
- Then for $\theta \in \Xi_n$ put $u_\#(\theta) = \{u_*(\sigma) \mid \sigma \in \theta\}$. This is a relative functor $\Xi_n \to \Xi_m$ (with $\pi(u_\#(\theta)) = u(\pi(\theta))$).
- This makes Ξ_* into a cosimplicial object in relative categories.
- Thus, for a relative category $\mathcal{C}$ we can define a simplicial set $N\mathcal{C}$ by $(N\mathcal{C})_n = \mathrm{RelCat}(\Xi_n, \mathcal{C})$: this is the relative nerve.
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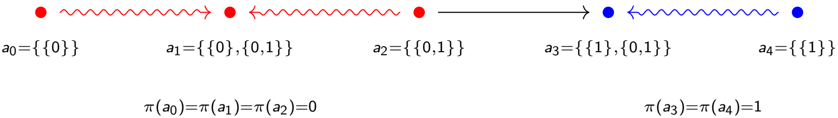

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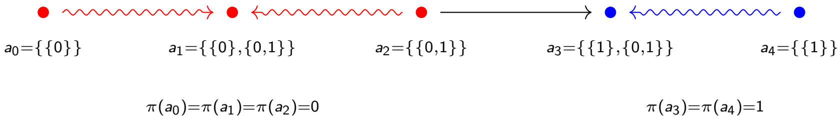

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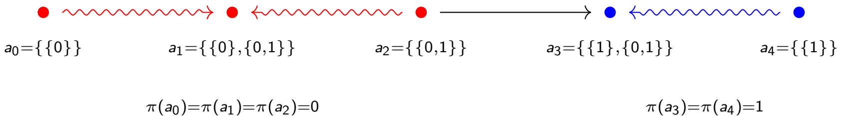

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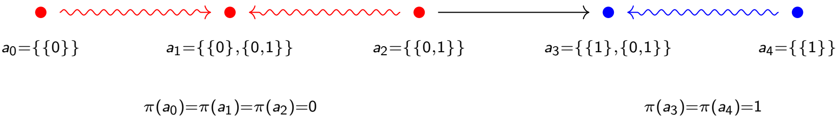

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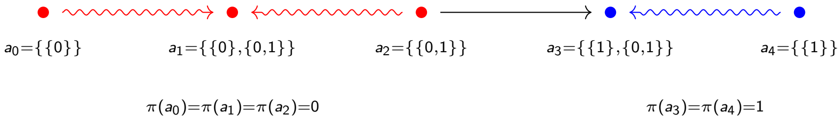

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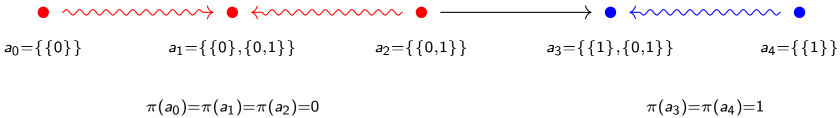

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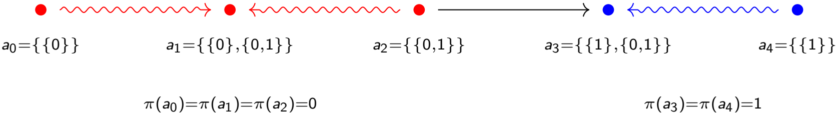

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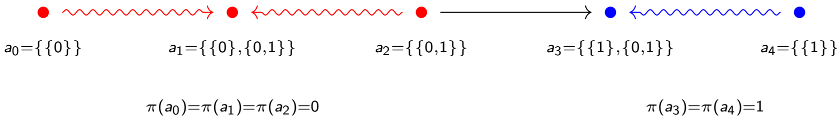

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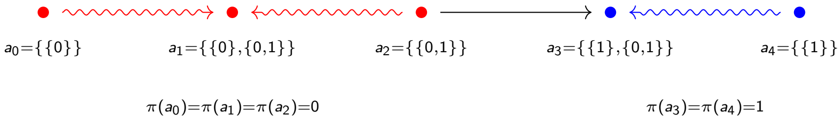

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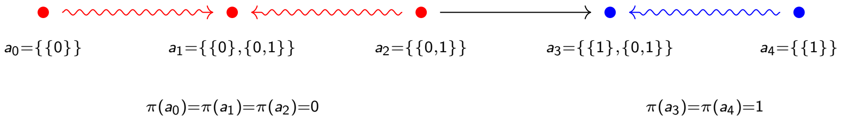

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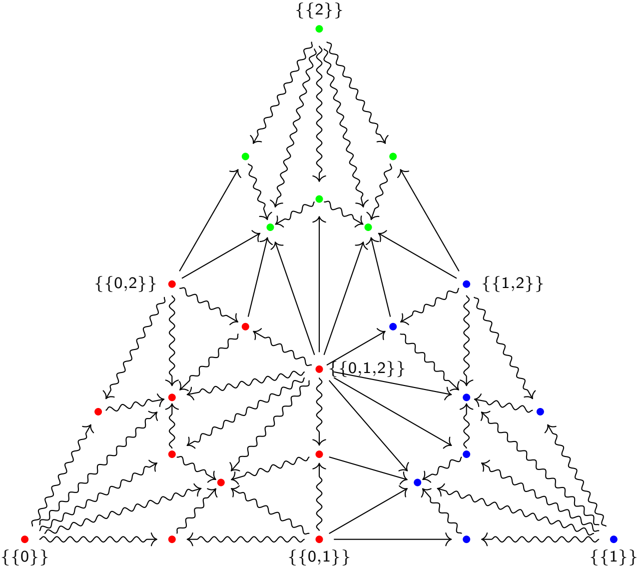
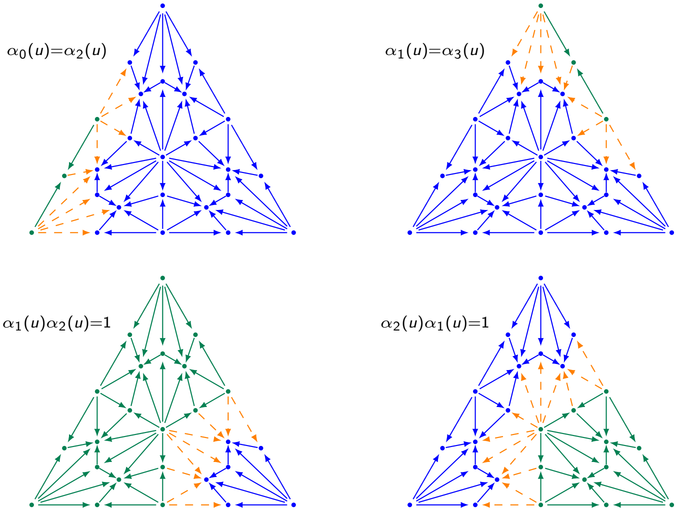


Relations in $\mathrm{Ho}(N\mathcal{C})$

The universal example of a relative category with a weak equivalence is $i[1]$.
Any morphism $\Xi_2 \to i[1]$ gives a relation in $\mathrm{Ho}(N(i[1]))$.



The gluing relation

- Any edge $u \in (N\mathcal{C})_1$ gives morphisms $\bullet \xrightarrow{u_0} \bullet \xleftarrow{u_1} \bullet \xrightarrow{u_2} \bullet \xleftarrow{u_3} \bullet$ in $\mathcal{C}$.
- Claim: in $\mathrm{Ho}(N\mathcal{C})$ we have

$$u = \alpha(u_3)^{-1}\alpha(u_2)\alpha(u_1)^{-1}\alpha(u_0) = \alpha_3(u_3)\alpha_2(u_2)\alpha_1(u_1)\alpha_0(u_0).$$

- To prove a claim like this about the composite of 4 edges, we need a 4-simplex incorporating those edges.
- Define $g\colon \Xi_4 \to \Xi_1$ as follows.
  Consider an element $\theta \in \Xi_4$, and let σ_0 be the smallest set in θ.
  - If $\theta = \{\{0\}\}$, we put $g(\theta) = a_0$
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$\mathrm{Ho}(\Xi_n) \simeq [n]$

- The functor $\pi\colon \Xi_n \to [n]$ induces $\mathrm{Ho}(\Xi_n) \to [n]$. It is easy to guess that this is an equivalence, but not trivial to prove.
- Define $\omega\colon [n] \to \Xi_n$ by $\omega(k) = \{[j,n] \mid 0 \leq j \leq k\}$, so for $n = 3$:

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- This is a poset map with $\pi \circ \omega = 1$. The map π is cosimplicial but ω is not.
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- Then $p_k, q_k \in \mathrm{RelPos}(\Xi_n, \Xi_n)$ with $\pi \circ p_k = \pi \circ q_k = \pi$ and

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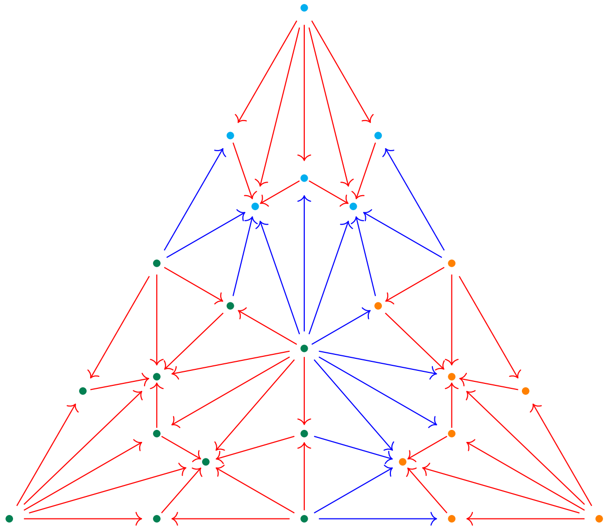

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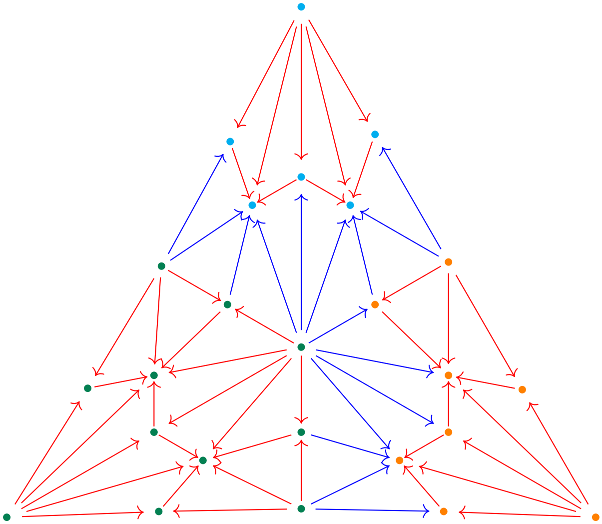

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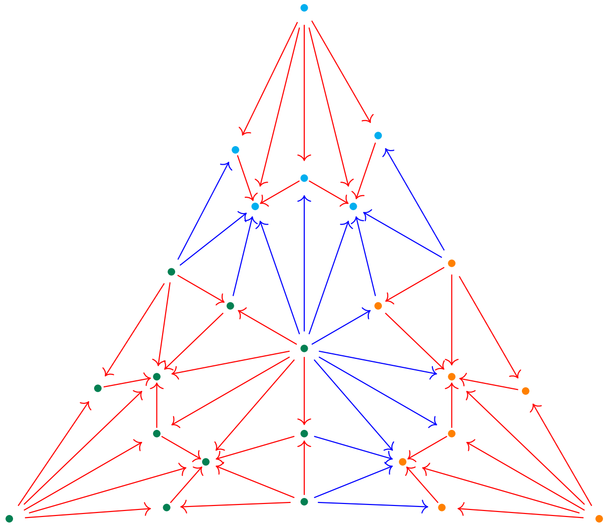

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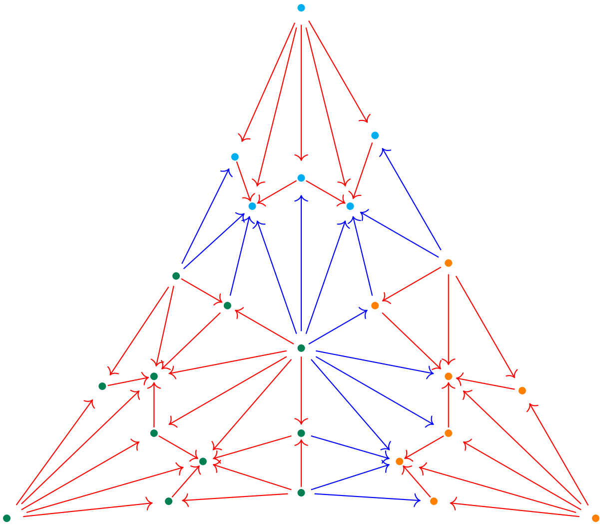

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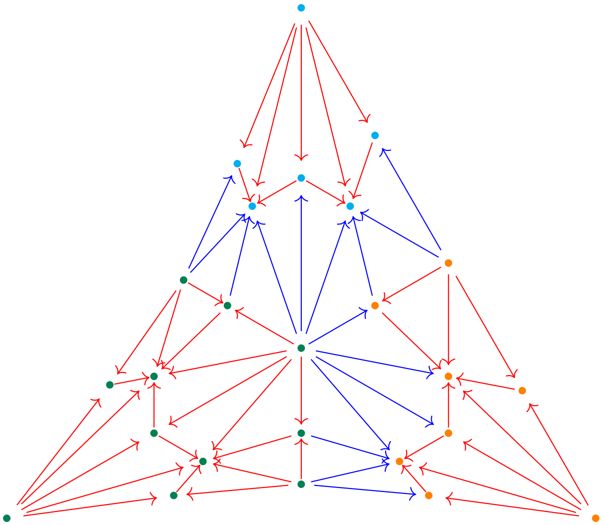

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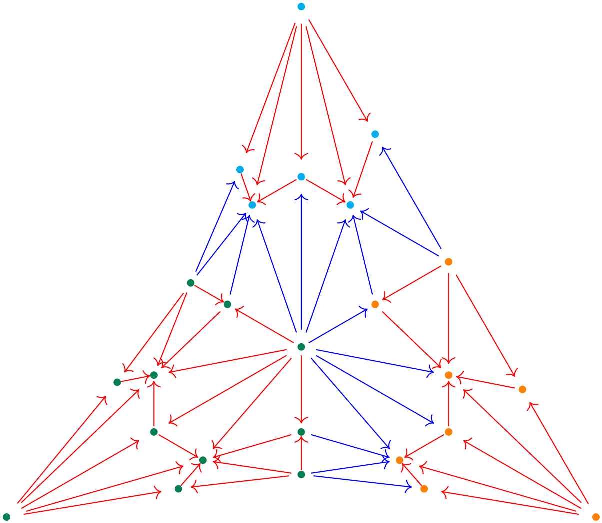

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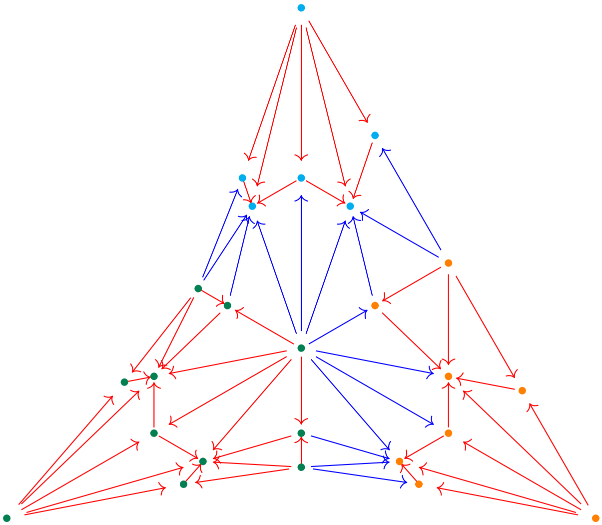

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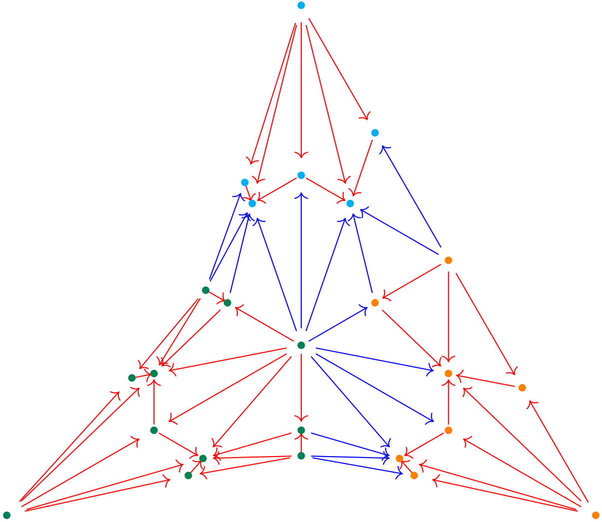

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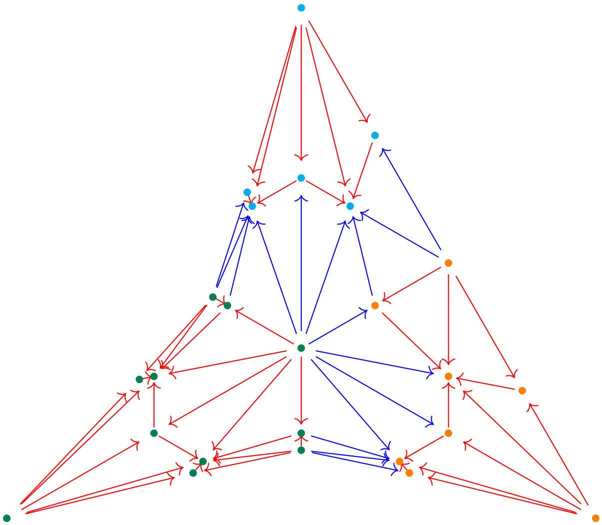

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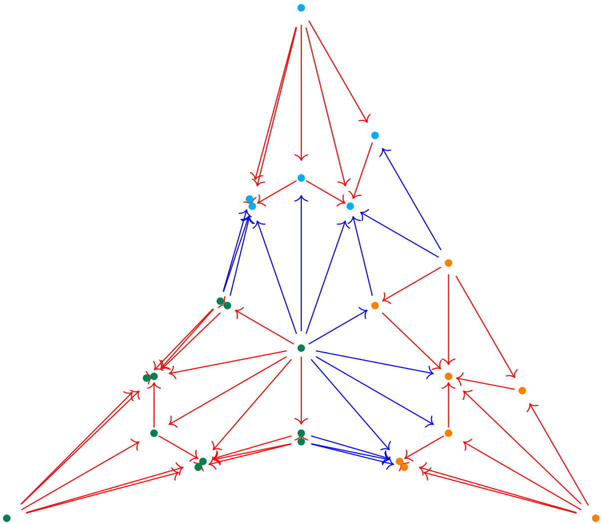

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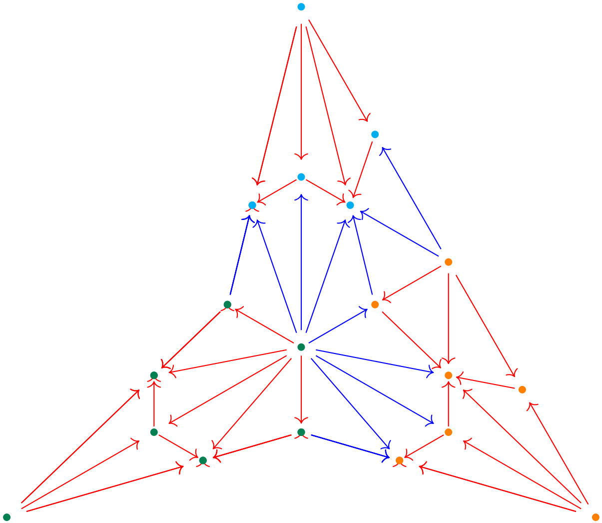

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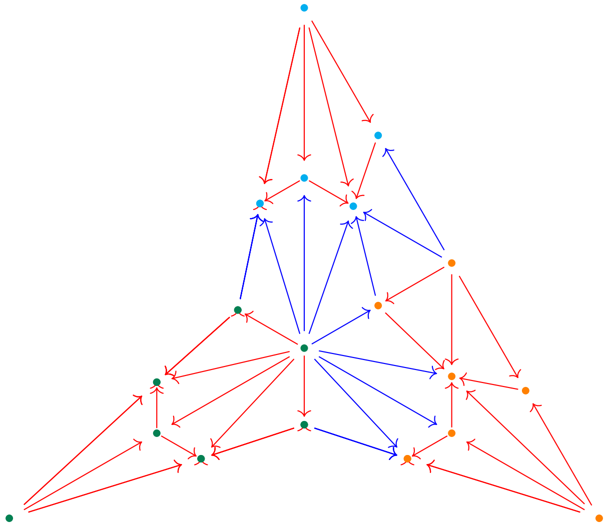

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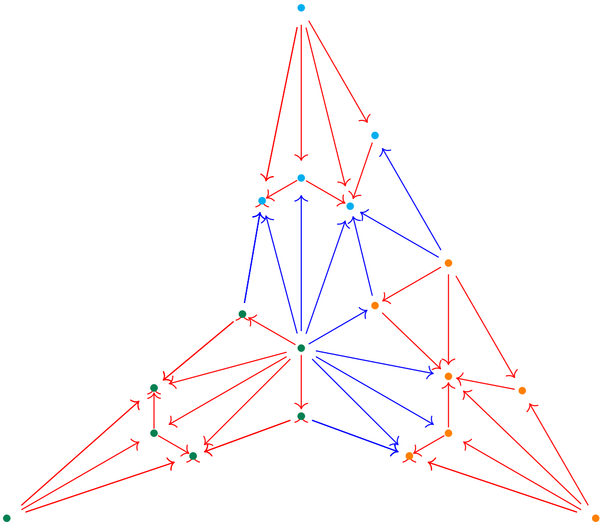

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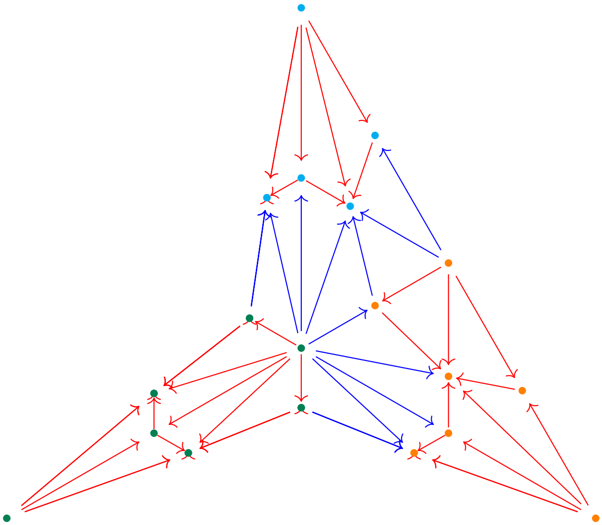

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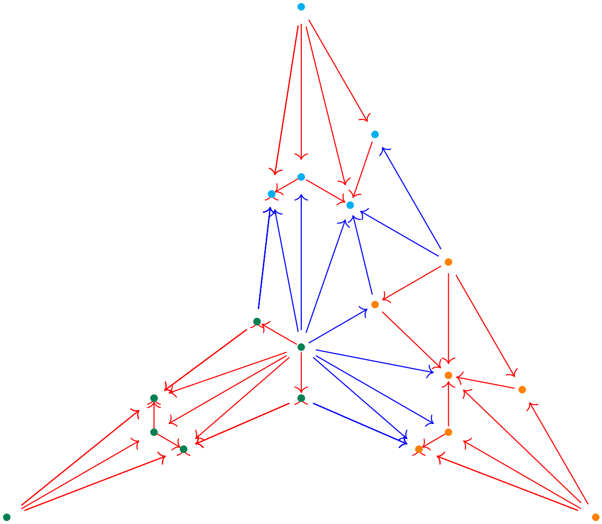

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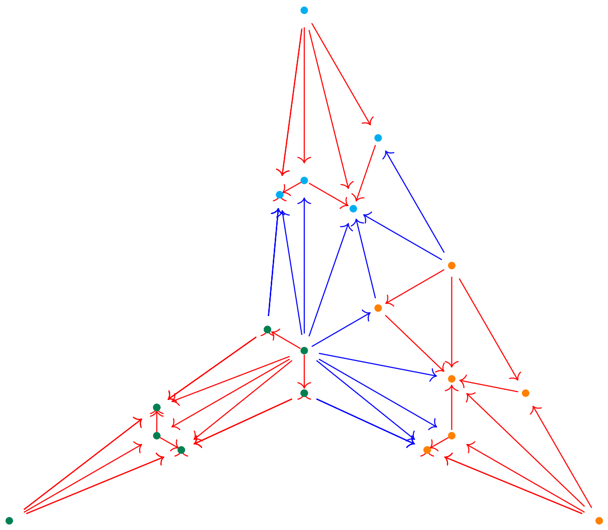

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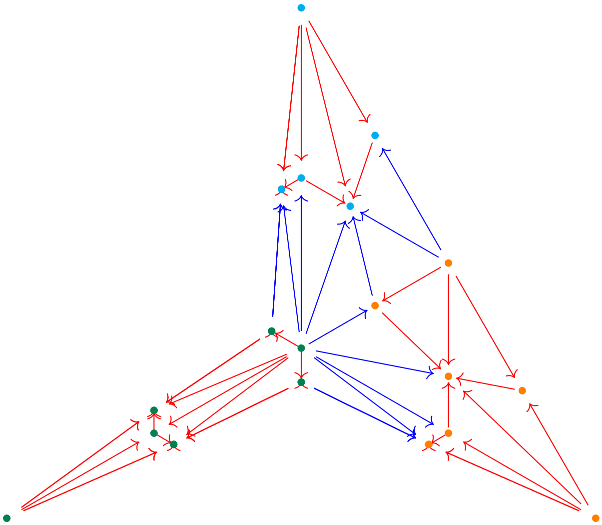

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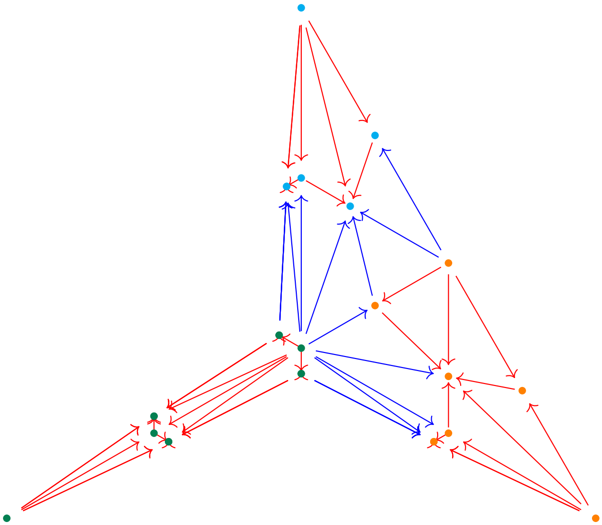

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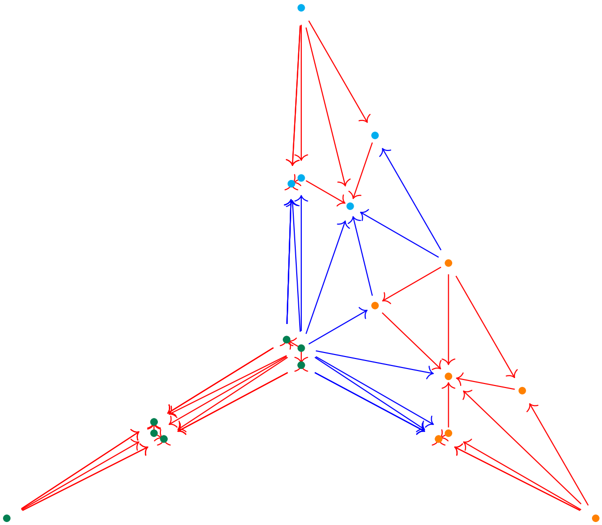

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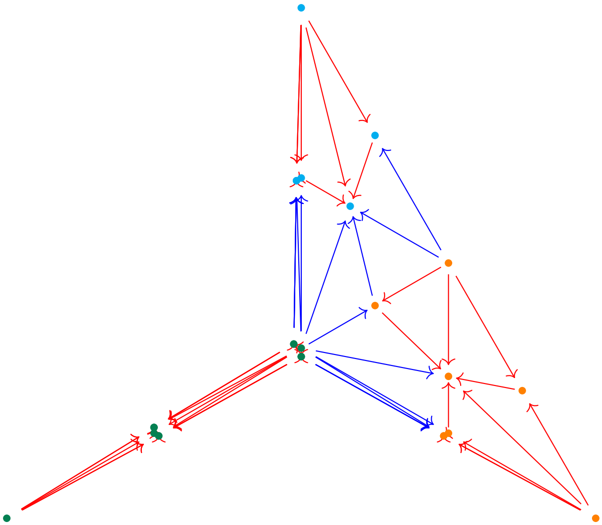

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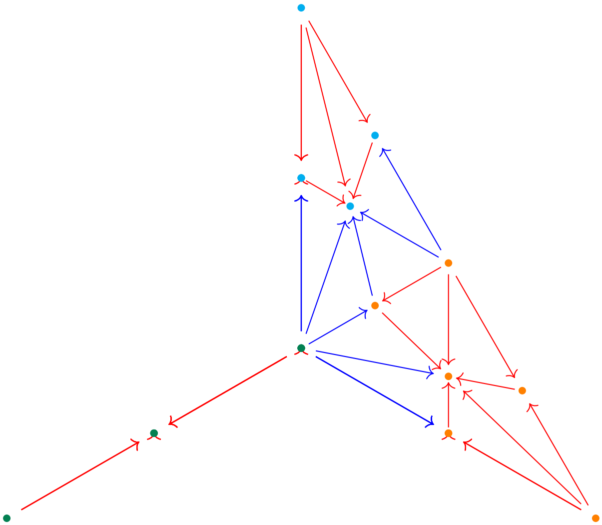

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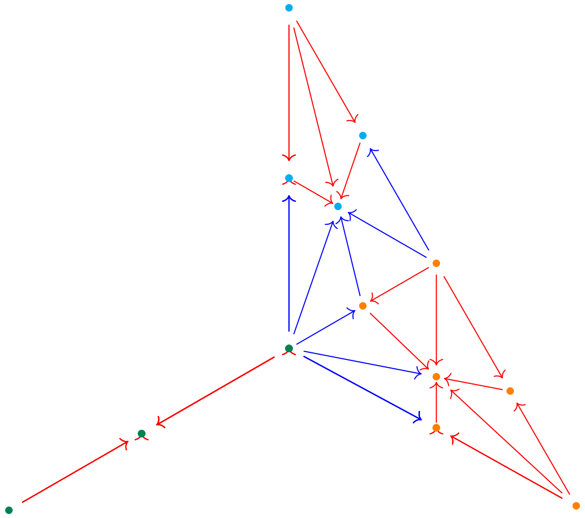

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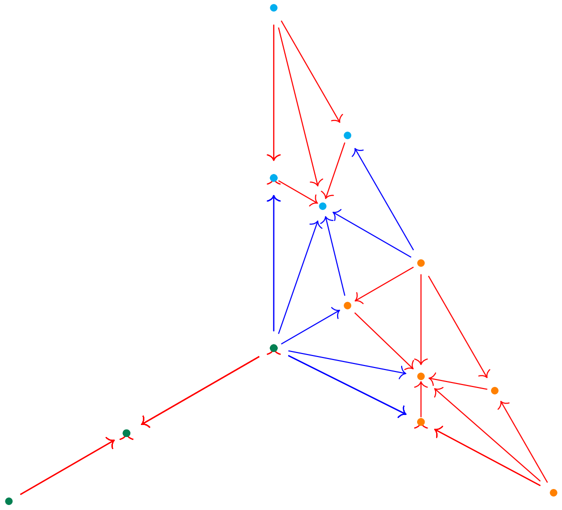

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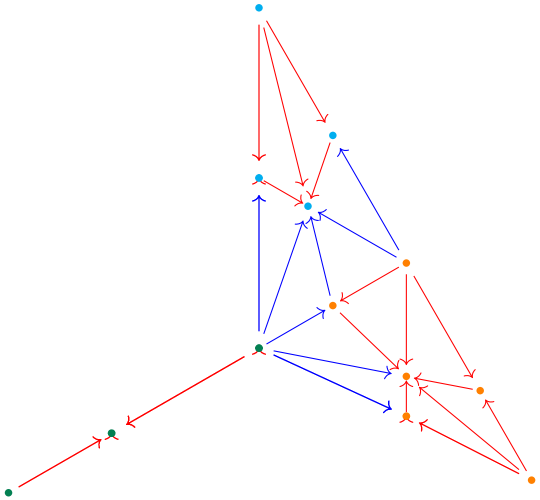

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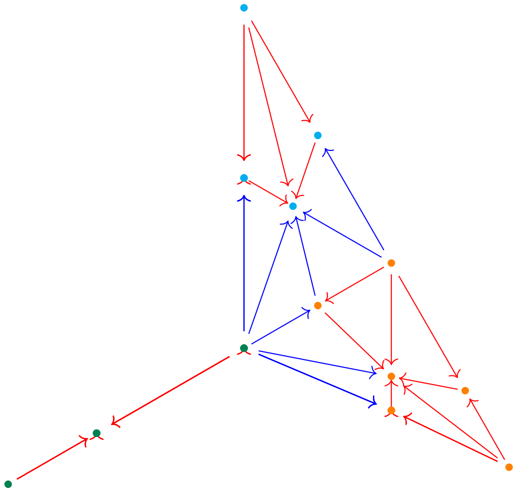

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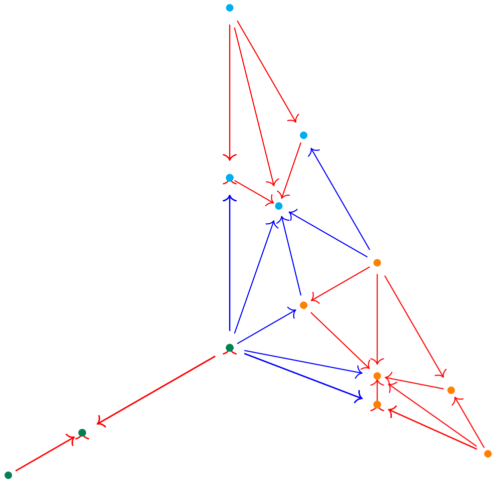

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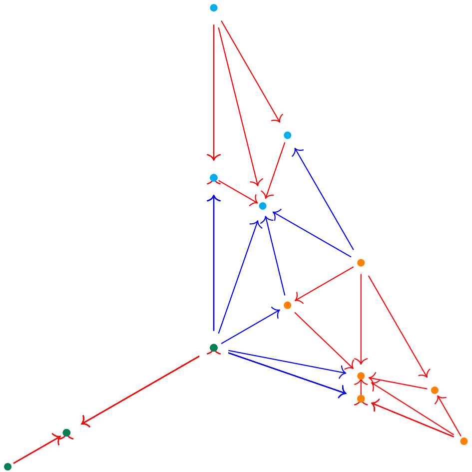

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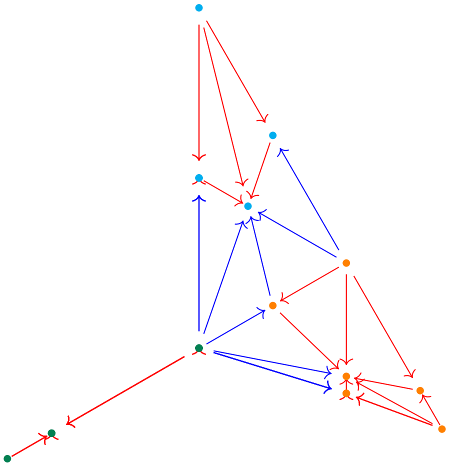

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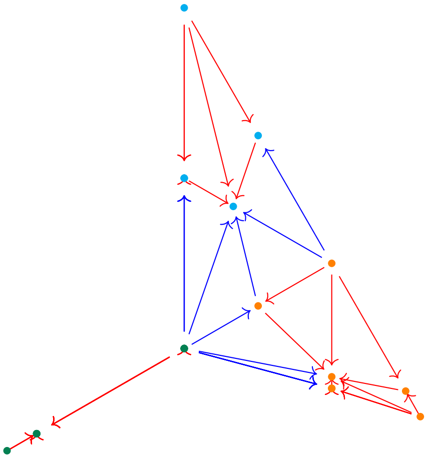

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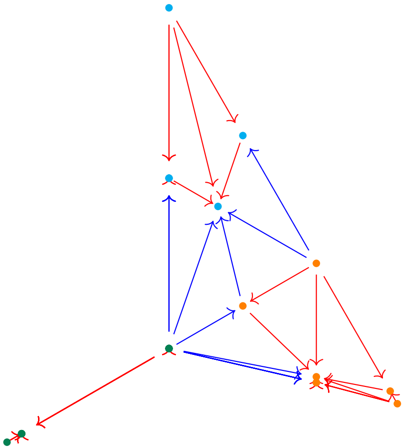

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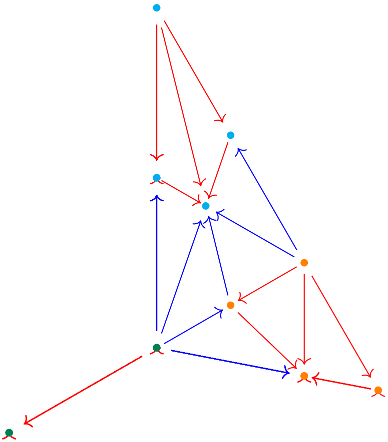

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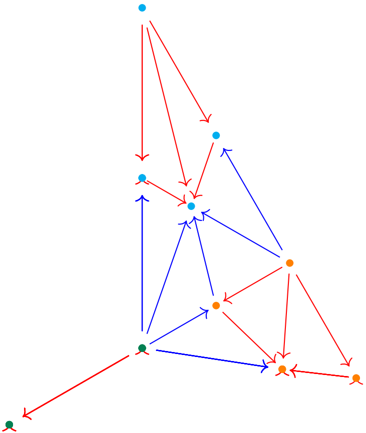

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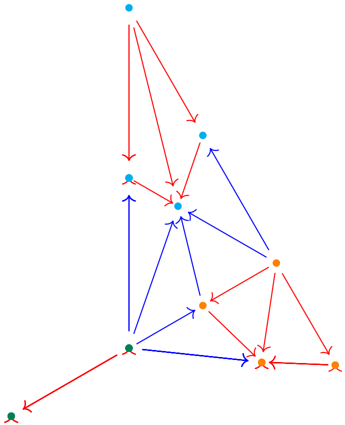

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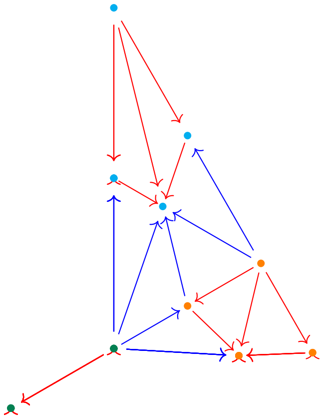

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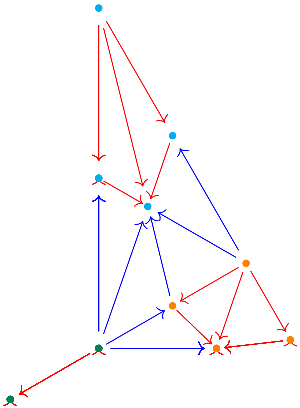

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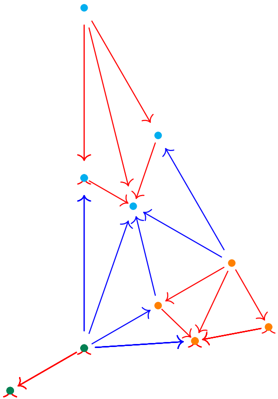

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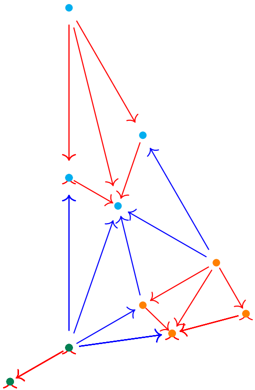

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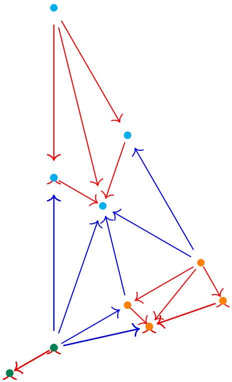

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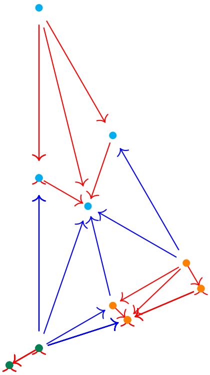

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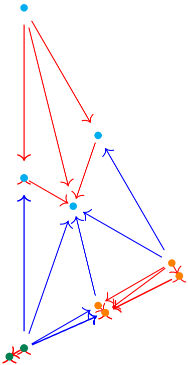

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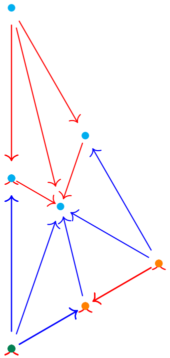

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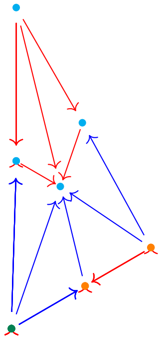

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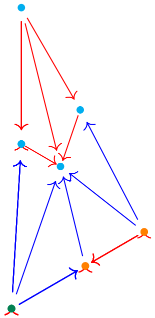

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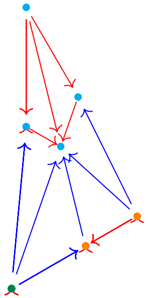

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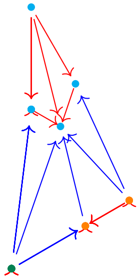

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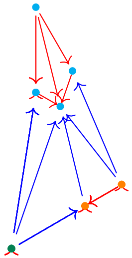

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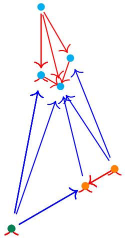

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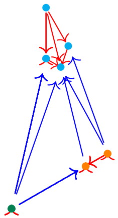

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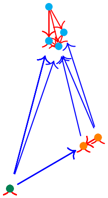

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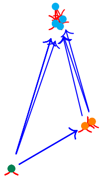

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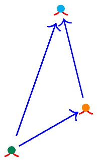

The left adjoint $K\colon \mathrm{sSet} \to \mathrm{RelCat}$

- **Theorem:** There is a functor $K\colon \mathrm{sSet} \to \mathrm{RelCat}$, left adjoint to $N\colon \mathrm{RelCat} \to \mathrm{sSet}$, with $K(X)[\mathrm{we}^{-1}] \simeq \mathrm{Ho}(X)$.
  Moreover, $K(X)$ is actually a poset.
- Morally, $K(X)$ is defined as a certain colimit of Ξ_n's; but colimits of categories are generally hard to handle.
- In this case the final answer is not too bad, although it takes substantial work to prove that.
- Put $\Xi_n^\top = \{\theta \in \Xi_n \mid [n] \in \theta\}$ (the *interior* of Ξ_n).
  Put $ND(X)_n = \{\text{nondegenerate } n-\text{simplices}\}$.
  Then $K(X) = \coprod_n ND(X)_n \times \Xi_n^\top$
  (with appropriate structure as a relative poset).

- Maximally degenerate example: X_n is the set of partitions of $[n]$ into intervals. There is a unique nondegenerate simplex in every degree.

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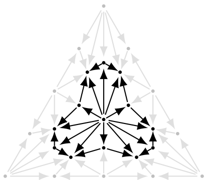

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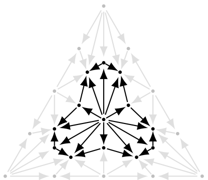

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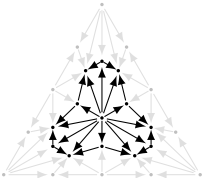

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Sketch of construction of $K(X)$

- Given a simplicial set X, we construct a relative category $\widetilde{K}(X)$ with a class of "strong equivalences" contained in the weak equivalences.
- $K(X)$ is the quotient of $\widetilde{K}(X)$ in which strong equivalences become identities.
- An object $a \in \widetilde{K}(X)$ is a pair (x_a, θ_a) with $x_a \in X_{n_a}$ and $\theta_a \in \Xi_{n_a}$.
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The pullback lemma

Suppose we have morphisms $[n] \xrightarrow{u} [k] \xleftarrow{v} [m]$ in $\mathbf{\Delta}$, where u is injective and v is surjective. Then there is a commutative square in $\mathbf{\Delta}$ as shown on the left below, which is a pullback in $\mathbf{\Delta}$ or in the category of sets; and the resulting diagram as shown on the right is also a pullback.

$$\begin{array}{ccc} [l] & \overset{\widetilde{u}}{\rightarrowtail} & [m] \\ {\scriptstyle \widetilde{v}}\downarrow\!\!\downarrow & & \downarrow\!\!\downarrow{\scriptstyle v} \\ [n] & \underset{u}{\rightarrowtail} & [k] \end{array} \qquad\qquad \begin{array}{ccc} \Xi_l & \overset{\widetilde{u}_\#}{\rightarrowtail} & \Xi_m \\ {\scriptstyle \widetilde{v}_\#}\downarrow\!\!\downarrow & & \downarrow\!\!\downarrow{\scriptstyle v_\#} \\ \Xi_n & \underset{u_\#}{\rightarrowtail} & \Xi_k \end{array}$$

Extension properties

- $\Lambda_k\Xi_n =$ union of faces of Ξ_n except k'th face $= \{\theta \in \Xi_n \mid [n], \{k\}^c \not\in \theta\}$.
  $\Lambda_k^+\Xi_n = \{\theta \in \Xi_n \mid \{k\}^c \not\in \theta\}$.

$\Lambda_1\Xi_2$

$\Lambda_1^+\Xi_2$

- $N\mathcal{C}$ is a quasicategory iff every $u\colon \Lambda_k\Xi_n \to \mathcal{C}$ (with $0 < k < n$) can be extended over Ξ_n.
- $\Lambda_k\Xi_n$ is not a retract of Ξ_n, so $N\mathcal{C}$ is not always a quasicategory.
- However, $\Lambda_k^+\Xi_n$ is a retract of Ξ_n, so $N\mathcal{C}$ is a quasicategory iff every $u\colon \Lambda_k\Xi_n \to \mathcal{C}$ can be extended over $\Lambda_k^+\Xi_n$.
- $\Lambda_k^+\Xi_n$ is $[1] \times \Lambda_k\Xi_n$ union a cone under $\{1\} \times \Lambda_k\Xi_n$.
- If $\mathcal{C}$ has a model structure, we can make the required extension by fibrantly replacing the diagram $u\colon \Lambda_k\Xi_n \to \mathcal{C}$ and taking its inverse limit.
- As we have diagrams of a specific shape, we can assume less than a model structure and be more explicit.

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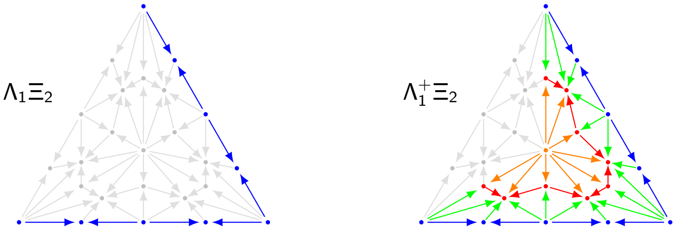

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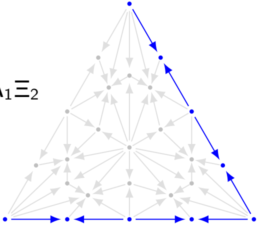

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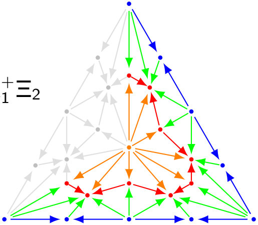

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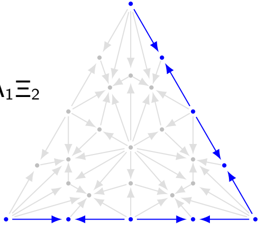

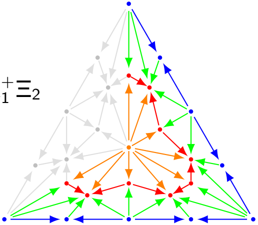

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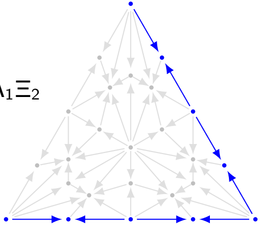

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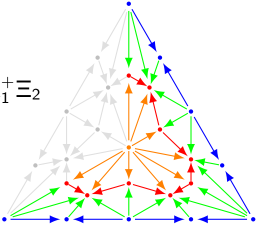

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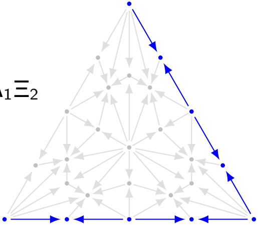

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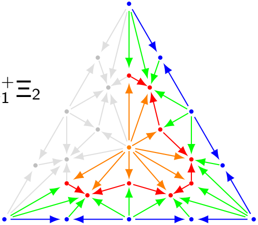

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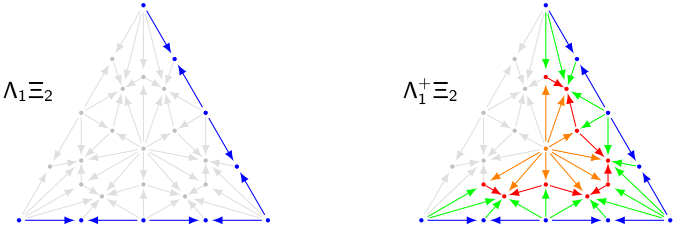

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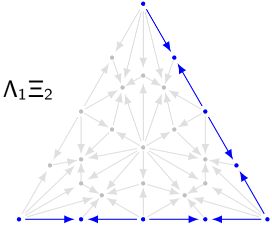

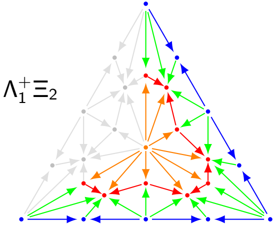

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- Compare this construction with the hammock localisation.
- Compare this construction with the Kan path groupoid.
- Investigate derived functors from this point of view.
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