Spaces of linear isometries

Neil Strickland
(joint with Harry Ullman)

June 13, 2013

The problem

Let X be a space, and let U and Z be complex vector bundles over X. Put

$$\mathrm{Inj}(U, Z) = \{(x, \phi) \mid \phi\colon U_x \to Z_x \text{ is linear and injective }\}.$$

The problem

Let X be a space, and let U and Z be complex vector bundles over X. Put

$$\mathrm{Inj}(U, Z) = \{(x, \phi) \mid \phi\colon U_x \to Z_x \text{ is linear and injective } \}.$$

Problem:

Understand the homotopy type and generalised cohomology of $\mathrm{Inj}(U, Z)$.

The problem

Let X be a space, and let U and Z be complex vector bundles over X. Put

$$\mathrm{Inj}(U, Z) = \{(x, \phi) \mid \phi\colon U_x \to Z_x \text{ is linear and injective } \}.$$

Problem:

Understand the homotopy type and generalised cohomology of $\mathrm{Inj}(U, Z)$.
(This is on the edge of what can be understood by formal group theory.)

The problem

Let X be a space, and let U and Z be complex vector bundles over X. Put

$$\mathrm{Inj}(U,Z) = \{(x,\phi) \mid \phi\colon U_x \to Z_x \text{ is linear and injective }\}.$$

Problem:
Understand the homotopy type and generalised cohomology of $\mathrm{Inj}(U,Z)$.
(This is on the edge of what can be understood by formal group theory.)

Example: When $X = \mathbb{C}P^\infty$ and $U = \mathbb{C}^n$ and $Z = \mathbb{C}^n \otimes$ (tautological bundle) we have $\mathrm{Inj}(U,Z) = PU(n)$ (using $S^1 \to U(n) \to PU(n) \to \mathbb{C}P^\infty$).

The problem

Let X be a space, and let U and Z be complex vector bundles over X. Put

$$\mathrm{Inj}(U,Z) = \{(x,\phi) \mid \phi\colon U_x \to Z_x \text{ is linear and injective }\}.$$

Problem:

Understand the homotopy type and generalised cohomology of $\mathrm{Inj}(U,Z)$.
(This is on the edge of what can be understood by formal group theory.)

Example: When $X = \mathbb{C}P^\infty$ and $U = \mathbb{C}^n$ and $Z = \mathbb{C}^n \otimes$ (tautological bundle) we have $\mathrm{Inj}(U,Z) = PU(n)$ (using $S^1 \to U(n) \to PU(n) \to \mathbb{C}P^\infty$).

Example: When X is a point, $\mathrm{Inj}(U,Z)$ is homotopy equivalent to a Stiefel manifold. Many things are well-understood in this case.

The problem

Let X be a space, and let U and Z be complex vector bundles over X. Put

$$\mathrm{Inj}(U, Z) = \{(x, \phi) \mid \phi\colon U_x \to Z_x \text{ is linear and injective } \}.$$

Problem:

Understand the homotopy type and generalised cohomology of $\mathrm{Inj}(U, Z)$. (This is on the edge of what can be understood by formal group theory.)

Example: When $X = \mathbb{C}P^\infty$ and $U = \mathbb{C}^n$ and $Z = \mathbb{C}^n \otimes$ (tautological bundle) we have $\mathrm{Inj}(U, Z) = PU(n)$ (using $S^1 \to U(n) \to PU(n) \to \mathbb{C}P^\infty$).

Example: When X is a point, $\mathrm{Inj}(U, Z)$ is homotopy equivalent to a Stiefel manifold. Many things are well-understood in this case.

If we have a fully functorial understanding of this special case then we can apply it fibrewise to cover the general case.

Let X be a space, and let U and Z be complex vector bundles over X. Put

$$\mathrm{Inj}(U,Z) = \{(x,\phi) \mid \phi\colon U_x \to Z_x \text{ is linear and injective } \}.$$

Problem:
Understand the homotopy type and generalised cohomology of $\mathrm{Inj}(U,Z)$. (This is on the edge of what can be understood by formal group theory.)

Example: When $X = \mathbb{C}P^\infty$ and $U = \mathbb{C}^n$ and $Z = \mathbb{C}^n \otimes$ (tautological bundle) we have $\mathrm{Inj}(U,Z) = PU(n)$ (using $S^1 \to U(n) \to PU(n) \to \mathbb{C}P^\infty$).

Example: When X is a point, $\mathrm{Inj}(U,Z)$ is homotopy equivalent to a Stiefel manifold. Many things are well-understood in this case.

If we have a fully functorial understanding of this special case then we can apply it fibrewise to cover the general case. Unfortunately, the usual methods for analysing the Stiefel manifold $\mathrm{Inj}(U,Z)$ are not equivariant for $\mathrm{Aut}(U) \times \mathrm{Aut}(Z)$ and so not functorial.

The problem

Let X be a space, and let U and Z be complex vector bundles over X. Put

$$\mathrm{Inj}(U,Z) = \{(x,\phi) \mid \phi\colon U_x \to Z_x \text{ is linear and injective } \}.$$

Problem:
Understand the homotopy type and generalised cohomology of $\mathrm{Inj}(U,Z)$.
(This is on the edge of what can be understood by formal group theory.)

Example: When $X = \mathbb{C}P^\infty$ and $U = \mathbb{C}^n$ and $Z = \mathbb{C}^n \otimes$ (tautological bundle) we have $\mathrm{Inj}(U,Z) = PU(n)$ (using $S^1 \to U(n) \to PU(n) \to \mathbb{C}P^\infty$).

Example: When X is a point, $\mathrm{Inj}(U,Z)$ is homotopy equivalent to a Stiefel manifold. Many things are well-understood in this case.

If we have a fully functorial understanding of this special case then we can apply it fibrewise to cover the general case. Unfortunately, the usual methods for analysing the Stiefel manifold $\mathrm{Inj}(U,Z)$ are not equivariant for $\mathrm{Aut}(U) \times \mathrm{Aut}(Z)$ and so not functorial.

For the rest of the talk, U and Z are just vector spaces, but everything is functorial.

Inner products

We may assume that U and Z have hermitian inner products.

Inner products

We may assume that U and Z have hermitian inner products. We put

$$
\begin{aligned}
L(U,Z) &= \{\text{linear isometric embeddings } U \to Z\} \\
&= \{\theta\colon U \to Z \mid \theta^\dagger\theta = 1_U\} \subseteq \mathrm{Inj}(U,Z)
\end{aligned}
$$

Inner products

We may assume that U and Z have hermitian inner products. We put

$$\begin{aligned} L(U,Z) &= \{\text{linear isometric embeddings } U \to Z\} \\ &= \{\theta\colon U \to Z \mid \theta^\dagger\theta = 1_U\} \subseteq \mathrm{Inj}(U,Z) \end{aligned}$$

Lemma: $L(U,Z)$ is a strong deformation retract of $\mathrm{Inj}(U,Z)$.

Inner products

We may assume that U and Z have hermitian inner products. We put

$$\begin{aligned} L(U,Z) &= \{\text{linear isometric embeddings } U \to Z\} \\ &= \{\theta\colon U \to Z \mid \theta^\dagger\theta = 1_U\} \subseteq \mathrm{Inj}(U,Z) \end{aligned}$$

Lemma: $L(U,Z)$ is a strong deformation retract of $\mathrm{Inj}(U,Z)$.

Proof: This is often done using Gram-Schmidt, but that is not functorial.

Inner products

We may assume that U and Z have hermitian inner products. We put

$$\begin{aligned} L(U,Z) &= \{\text{linear isometric embeddings } U \to Z\} \\ &= \{\theta\colon U \to Z \mid \theta^\dagger\theta = 1_U\} \subseteq \mathrm{Inj}(U,Z) \end{aligned}$$

Lemma: $L(U,Z)$ is a strong deformation retract of $\mathrm{Inj}(U,Z)$.

Proof: This is often done using Gram-Schmidt, but that is not functorial.

Instead, use the maps $h_t(\phi) = \phi \circ (\phi^\dagger\phi)^{-t/2}$ for $0 \leq t \leq 1$.

Inner products

We may assume that U and Z have hermitian inner products. We put

$$\begin{aligned} L(U,Z) &= \{\text{linear isometric embeddings } U \to Z\} \\ &= \{\theta\colon U \to Z \mid \theta^\dagger\theta = 1_U\} \subseteq \mathrm{Inj}(U,Z) \end{aligned}$$

Lemma: $L(U,Z)$ is a strong deformation retract of $\mathrm{Inj}(U,Z)$.

Proof: This is often done using Gram-Schmidt, but that is not functorial.

Instead, use the maps $h_t(\phi) = \phi \circ (\phi^\dagger\phi)^{-t/2}$ for $0 \leq t \leq 1$.

Here $\phi \in \mathrm{Inj}(U,Z)$, and the map $\beta = \phi^\dagger\phi$ is positive self-adjoint.

We may assume that U and Z have hermitian inner products. We put

$$\begin{aligned} L(U,Z) &= \{\text{linear isometric embeddings } U \to Z\} \\ &= \{\theta\colon U \to Z \mid \theta^\dagger\theta = 1_U\} \subseteq \mathrm{Inj}(U,Z) \end{aligned}$$

Lemma: $L(U,Z)$ is a strong deformation retract of $\mathrm{Inj}(U,Z)$.

Proof: This is often done using Gram-Schmidt, but that is not functorial.

Instead, use the maps $h_t(\phi) = \phi \circ (\phi^\dagger\phi)^{-t/2}$ for $0 \le t \le 1$.

Here $\phi \in \mathrm{Inj}(U,Z)$, and the map $\beta = \phi^\dagger\phi$ is positive self-adjoint.

We can define β^s for $s \in \mathbb{R}$ by functional calculus.

We may assume that U and Z have hermitian inner products. We put

$$\begin{aligned} L(U,Z) &= \{\text{linear isometric embeddings } U \to Z\} \\ &= \{\theta\colon U \to Z \mid \theta^\dagger\theta = 1_U\} \subseteq \mathrm{Inj}(U,Z) \end{aligned}$$

Lemma: $L(U,Z)$ is a strong deformation retract of $\mathrm{Inj}(U,Z)$.

Proof: This is often done using Gram-Schmidt, but that is not functorial.

Instead, use the maps $h_t(\phi) = \phi \circ (\phi^\dagger\phi)^{-t/2}$ for $0 \leq t \leq 1$.

Here $\phi \in \mathrm{Inj}(U,Z)$, and the map $\beta = \phi^\dagger\phi$ is positive self-adjoint.

We can define β^s for $s \in \mathbb{R}$ by functional calculus.

In more detail: there is an orthonormal basis $\{e_i\}$ for U with $\beta(e_i) = t_i e_i$ for some $t_i > 0$.

Inner products

We may assume that U and Z have hermitian inner products. We put

$$\begin{aligned} L(U,Z) &= \{\text{linear isometric embeddings } U \to Z\} \\ &= \{\theta\colon U \to Z \mid \theta^\dagger\theta = 1_U\} \subseteq \mathrm{Inj}(U,Z) \end{aligned}$$

Lemma: $L(U,Z)$ is a strong deformation retract of $\mathrm{Inj}(U,Z)$.

Proof: This is often done using Gram-Schmidt, but that is not functorial.

Instead, use the maps $h_t(\phi) = \phi \circ (\phi^\dagger\phi)^{-t/2}$ for $0 \leq t \leq 1$.

Here $\phi \in \mathrm{Inj}(U,Z)$, and the map $\beta = \phi^\dagger\phi$ is positive self-adjoint.

We can define β^s for $s \in \mathbb{R}$ by functional calculus.

In more detail: there is an orthonormal basis $\{e_i\}$ for U with $\beta(e_i) = t_i e_i$ for some $t_i > 0$. We define β^s by $\beta^s(e_i) = t_i^s e_i$.

Inner products

We may assume that U and Z have hermitian inner products. We put

$$L(U,Z) = \{\text{linear isometric embeddings } U \to Z\}$$
$$= \{\theta\colon U \to Z \mid \theta^\dagger\theta = 1_U\} \subseteq \mathrm{Inj}(U,Z)$$

Lemma: $L(U,Z)$ is a strong deformation retract of $\mathrm{Inj}(U,Z)$.

Proof: This is often done using Gram-Schmidt, but that is not functorial.

Instead, use the maps $h_t(\phi) = \phi \circ (\phi^\dagger\phi)^{-t/2}$ for $0 \leq t \leq 1$.

Here $\phi \in \mathrm{Inj}(U,Z)$, and the map $\beta = \phi^\dagger\phi$ is positive self-adjoint.

We can define β^s for $s \in \mathbb{R}$ by functional calculus.

In more detail: there is an orthonormal basis $\{e_i\}$ for U with $\beta(e_i) = t_i e_i$ for some $t_i > 0$. We define β^s by $\beta^s(e_i) = t_i^s e_i$. We can choose a polynomial $p(x)$ that is close to x^s on $[0,K]$; then $p(\beta)$ is close to β^s.

Inner products

We may assume that U and Z have hermitian inner products. We put

$$\begin{aligned} L(U,Z) &= \{\text{linear isometric embeddings } U \to Z\} \\ &= \{\theta\colon U \to Z \mid \theta^\dagger\theta = 1_U\} \subseteq \mathrm{Inj}(U,Z) \end{aligned}$$

Lemma: $L(U,Z)$ is a strong deformation retract of $\mathrm{Inj}(U,Z)$.

Proof: This is often done using Gram-Schmidt, but that is not functorial.

Instead, use the maps $h_t(\phi) = \phi \circ (\phi^\dagger\phi)^{-t/2}$ for $0 \leq t \leq 1$.

Here $\phi \in \mathrm{Inj}(U,Z)$, and the map $\beta = \phi^\dagger\phi$ is positive self-adjoint.

We can define β^s for $s \in \mathbb{R}$ by functional calculus.

In more detail: there is an orthonormal basis $\{e_i\}$ for U with $\beta(e_i) = t_i e_i$ for some $t_i > 0$. We define β^s by $\beta^s(e_i) = t_i^s e_i$. We can choose a polynomial $p(x)$ that is close to x^s on $[0,K]$; then $p(\beta)$ is close to β^s. Using this we see that β^s is well-defined and continuous in (s,β). □

Inner products

We may assume that U and Z have hermitian inner products. We put

$$L(U,Z) = \{\text{linear isometric embeddings } U \to Z\}$$
$$= \{\theta\colon U \to Z \mid \theta^\dagger\theta = 1_U\} \subseteq \mathrm{Inj}(U,Z)$$

Lemma: $L(U,Z)$ is a strong deformation retract of $\mathrm{Inj}(U,Z)$.

Proof: This is often done using Gram-Schmidt, but that is not functorial.

Instead, use the maps $h_t(\phi) = \phi \circ (\phi^\dagger\phi)^{-t/2}$ for $0 \leq t \leq 1$.

Here $\phi \in \mathrm{Inj}(U,Z)$, and the map $\beta = \phi^\dagger\phi$ is positive self-adjoint.

We can define β^s for $s \in \mathbb{R}$ by functional calculus.

In more detail: there is an orthonormal basis $\{e_i\}$ for U with $\beta(e_i) = t_i e_i$ for some $t_i > 0$. We define β^s by $\beta^s(e_i) = t_i^s e_i$. We can choose a polynomial $p(x)$ that is close to x^s on $[0,K]$; then $p(\beta)$ is close to β^s. Using this we see that β^s is well-defined and continuous in (s,β). $\square$

From now on we focus on $L(U,Z)$ rather than $\mathrm{Inj}(U,Z)$.

Example: For a nonnegative self-adjoint matrix $\alpha = \begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix} \in M_2(\mathbb{C})$ with trace $\tau = a + c$ and determinant $\delta = ac - |b|^2$ one can check that

$$\sqrt{\alpha} = \frac{\sqrt{\delta} + \alpha}{\sqrt{\tau + 2\sqrt{\delta}}}.$$

Example: For a nonnegative self-adjoint matrix $\alpha = \begin{bmatrix} a & b \\ \overline{b} & c \end{bmatrix} \in M_2(\mathbb{C})$ with trace $\tau = a + c$ and determinant $\delta = ac - |b|^2$ one can check that

$$\sqrt{\alpha} = \frac{\sqrt{\delta} + \alpha}{\sqrt{\tau + 2\sqrt{\delta}}}.$$

Using this, we see that for $\phi = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{C})$ with determinant $\delta = ad - bc$, the matrix $\rho(\phi) = (\phi^\dagger \phi)^{1/2}$ is given by

$$\rho(\phi) = \frac{1}{\sqrt{\|\phi\|_2^2 + 2|\delta|}} \begin{bmatrix} |a|^2 + |c|^2 + |\delta| & \overline{a}b + \overline{c}d \\ a\overline{b} + c\overline{d} & |b|^2 + |d|^2 + |\delta| \end{bmatrix},$$

where

$$\|\phi\|_2^2 = \mathrm{trace}(\phi^\dagger \phi) = |a|^2 + |b|^2 + |c|^2 + |d|^2.$$

Example: For a nonnegative self-adjoint matrix $\alpha = \begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix} \in M_2(\mathbb{C})$ with trace $\tau = a + c$ and determinant $\delta = ac - |b|^2$ one can check that

$$\sqrt{\alpha} = \frac{\sqrt{\delta} + \alpha}{\sqrt{\tau + 2\sqrt{\delta}}}.$$

Using this, we see that for $\phi = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{C})$ with determinant $\delta = ad - bc$, the matrix $\rho(\phi) = (\phi^\dagger \phi)^{1/2}$ is given by

$$\rho(\phi) = \frac{1}{\sqrt{\|\phi\|_2^2 + 2|\delta|}} \begin{bmatrix} |a|^2 + |c|^2 + |\delta| & \bar{a}b + \bar{c}d \\ a\bar{b} + c\bar{d} & |b|^2 + |d|^2 + |\delta| \end{bmatrix},$$

where

$$\|\phi\|_2^2 = \text{trace}(\phi^\dagger \phi) = |a|^2 + |b|^2 + |c|^2 + |d|^2.$$

The eigenvalues are

$$\left(\sqrt{\|\phi\|_2^2 + 2|\delta|} \pm \sqrt{\|\phi\|_2^2 - 2|\delta|} \right) /2$$

The Miller splitting

Suppose that $U \leq Z$ (and use methods functorial in that context).

The Miller splitting

Suppose that $U \leq Z$ (and use methods functorial in that context). Put

$$F_k(U, Z) = \{\theta \in L(U, Z) \mid \mathrm{rank}(\theta - \mathrm{inc}) \leq k\}$$

The Miller splitting

Suppose that $U \leq Z$ (and use methods functorial in that context). Put

$$F_k(U,Z) = \{\theta \in L(U,Z) \mid \mathrm{rank}(\theta - \mathrm{inc}) \leq k\}, \qquad \text{so}$$

$$\{inc\} = F_0(U,Z) \subseteq F_1(U,Z) \subseteq \cdots \subseteq F_{\dim(U)}(U,Z) = L(U,Z).$$

The Miller splitting

Suppose that $U \leq Z$ (and use methods functorial in that context). Put

$$F_k(U,Z) = \{\theta \in L(U,Z) \mid \mathrm{rank}(\theta - \mathrm{inc}) \leq k\}, \qquad \text{so}$$

$$\{inc\} = F_0(U,Z) \subseteq F_1(U,Z) \subseteq \cdots \subseteq F_{\dim(U)}(U,Z) = L(U,Z).$$

Put

$$G_k(U) = \{k - \text{planes in } U\}$$

The Miller splitting

Suppose that $U \leq Z$ (and use methods functorial in that context). Put

$$F_k(U,Z) = \{\theta \in L(U,Z) \mid \operatorname{rank}(\theta - \operatorname{inc}) \leq k\}, \qquad \text{so}$$

$$\{inc\} = F_0(U,Z) \subseteq F_1(U,Z) \subseteq \cdots \subseteq F_{\dim(U)}(U,Z) = L(U,Z).$$

Put

$$G_k(U) = \{k - \text{planes in } U\}$$
$$T = \text{ tautological bundle over } G_k(U)$$

The Miller splitting

Suppose that $U \leq Z$ (and use methods functorial in that context). Put

$$F_k(U,Z) = \{\theta \in L(U,Z) \mid \text{rank}(\theta - \text{inc}) \leq k\}, \qquad \text{so}$$

$$\{inc\} = F_0(U,Z) \subseteq F_1(U,Z) \subseteq \cdots \subseteq F_{\dim(U)}(U,Z) = L(U,Z).$$

Put

$$\begin{aligned} G_k(U) &= \{k - \text{planes in } U\} \\ T &= \text{ tautological bundle over } G_k(U) \\ s(T) &= \text{ associated bundle of self-adjoint endomorphisms} \end{aligned}$$

The Miller splitting

Suppose that $U \leq Z$ (and use methods functorial in that context). Put

$$F_k(U,Z) = \{\theta \in L(U,Z) \mid \text{rank}(\theta - \text{inc}) \leq k\}, \qquad \text{so}$$

$$\{inc\} = F_0(U,Z) \subseteq F_1(U,Z) \subseteq \cdots \subseteq F_{\dim(U)}(U,Z) = L(U,Z).$$

Put

$$\begin{aligned} G_k(U) &= \{k - \text{planes in } U\} \\ T &= \text{ tautological bundle over } G_k(U) \\ s(T) &= \text{ associated bundle of self-adjoint endomorphisms} \\ Q_k(U,Z) &= G_k(U)^{s(T)\oplus \text{Hom}(T, Z\ominus U)} \qquad \text{(Thom space).} \end{aligned}$$

The Miller splitting

Suppose that $U \leq Z$ (and use methods functorial in that context). Put

$$F_k(U,Z) = \{\theta \in L(U,Z) \mid \text{rank}(\theta - \text{inc}) \leq k\}, \qquad \text{so}$$

$$\{inc\} = F_0(U,Z) \subseteq F_1(U,Z) \subseteq \cdots \subseteq F_{\dim(U)}(U,Z) = L(U,Z).$$

Put

$$\begin{aligned} G_k(U) &= \{k - \text{planes in } U\} \\ T &= \text{ tautological bundle over } G_k(U) \\ s(T) &= \text{ associated bundle of self-adjoint endomorphisms} \\ Q_k(U,Z) &= G_k(U)^{s(T)\oplus \text{Hom}(T, Z\ominus U)} \qquad \text{(Thom space).} \end{aligned}$$

Theorem (Miller): There are natural homeomorphisms

$$\frac{F_k(U,Z)_+}{F_{k-1}(U,Z)_+} \simeq Q_k(U,Z)$$

The Miller splitting

Suppose that $U \leq Z$ (and use methods functorial in that context). Put

$$F_k(U,Z) = \{\theta \in L(U,Z) \mid \text{rank}(\theta - \text{inc}) \leq k\}, \qquad \text{so}$$

$$\{inc\} = F_0(U,Z) \subseteq F_1(U,Z) \subseteq \cdots \subseteq F_{\dim(U)}(U,Z) = L(U,Z).$$

Put

$$\begin{aligned} G_k(U) &= \{k - \text{planes in } U\} \\ T &= \text{ tautological bundle over } G_k(U) \\ s(T) &= \text{ associated bundle of self-adjoint endomorphisms} \\ Q_k(U,Z) &= G_k(U)^{s(T)\oplus \text{Hom}(T, Z\ominus U)} \qquad \text{(Thom space).} \end{aligned}$$

Theorem (Miller): There are natural homeomorphisms

$$\frac{F_k(U,Z)_+}{F_{k-1}(U,Z)_+} \simeq Q_k(U,Z)$$

and stable splittings

$$F_k(U,Z)_+ \simeq \bigvee_{j=0}^{k} Q_j(U,Z).$$

Theorem: Even if $U \not\leq Z$ we have a natural tower of finite spectra:

$$
\begin{array}{ccc}
L(U,Z)_+ = X_n(U,Z) & \longleftarrow & Q_n(U,Z) \\
\downarrow & \nearrow\!\circ & \\
X_{n-1}(U,Z) & \longleftarrow & Q_{n-1}(U,Z) \\
\downarrow & \nearrow\!\circ & \\
X_{n-2}(U,Z) & \longleftarrow & Q_{n-2}(U,Z) \\
\vdots & \cdots\cdots\cdots & \\
X_1(U,Z) & \longleftarrow & Q_1(U,Z) \\
\downarrow & \nearrow\!\circ & \\
S^0 = X_0(U,Z) & &
\end{array}
$$

Here $n = \dim(U)$ and $Q_k(U,Z) = G_k(U)^{s(T)+\mathrm{Hom}(T,Z)-\mathrm{Hom}(T,U)}$ (the Thom spectrum of a virtual bundle), and $X_k(U,Z)$ is yet to be defined. The triangles are distinguished.

The bottom connecting map

The tangent bundle to $PU = G_1(U)$ is $\mathrm{Hom}(T, U) - \mathrm{Hom}(T, T) = \mathrm{Hom}(T, U) - \mathbb{C}$, so we have a Gysin map

$$X_0(U, Z) = S^0 \to \Sigma^2 G_1(U)^{-\mathrm{Hom}(T,U)}$$

The bottom connecting map

The tangent bundle to $PU = G_1(U)$ is
$\mathrm{Hom}(T, U) - \mathrm{Hom}(T, T) = \mathrm{Hom}(T, U) - \mathbb{C}$, so we have a Gysin map

$$X_0(U, Z) = S^0 \to \Sigma^2 G_1(U)^{-\mathrm{Hom}(T,U)} \subseteq \Sigma^2 G_1(U)^{\mathrm{Hom}(T,Z-U)}$$

The bottom connecting map

The tangent bundle to $PU = G_1(U)$ is $\mathrm{Hom}(T, U) - \mathrm{Hom}(T, T) = \mathrm{Hom}(T, U) - \mathbb{C}$, so we have a Gysin map

$$X_0(U, Z) = S^0 \to \Sigma^2 G_1(U)^{-\mathrm{Hom}(T,U)} \subseteq \Sigma^2 G_1(U)^{\mathrm{Hom}(T,Z-U)} = \Sigma Q_1(U, Z).$$

The bottom connecting map

The tangent bundle to $PU = G_1(U)$ is
$\mathrm{Hom}(T, U) - \mathrm{Hom}(T, T) = \mathrm{Hom}(T, U) - \mathbb{C}$, so we have a Gysin map

$$X_0(U, Z) = S^0 \to \Sigma^2 G_1(U)^{-\,\mathrm{Hom}(T,U)} \subseteq \Sigma^2 G_1(U)^{\mathrm{Hom}(T,Z-U)} = \Sigma Q_1(U, Z).$$

Proposition: This is the bottom connecting map in the tower $\square$.

The bottom connecting map

The tangent bundle to $PU = G_1(U)$ is $\mathrm{Hom}(T, U) - \mathrm{Hom}(T, T) = \mathrm{Hom}(T, U) - \mathbb{C}$, so we have a Gysin map

$$X_0(U, Z) = S^0 \to \Sigma^2 G_1(U)^{-\mathrm{Hom}(T,U)} \subseteq \Sigma^2 G_1(U)^{\mathrm{Hom}(T,Z-U)} = \Sigma Q_1(U, Z).$$

Proposition: This is the bottom connecting map in the tower □.

Let E be an even periodic cohomology theory with formal group G.

The bottom connecting map

The tangent bundle to $PU = G_1(U)$ is
$\mathrm{Hom}(T, U) - \mathrm{Hom}(T, T) = \mathrm{Hom}(T, U) - \mathbb{C}$, so we have a Gysin map

$$X_0(U, Z) = S^0 \to \Sigma^2 G_1(U)^{-\mathrm{Hom}(T,U)} \subseteq \Sigma^2 G_1(U)^{\mathrm{Hom}(T,Z-U)} = \Sigma Q_1(U, Z).$$

Proposition: This is the bottom connecting map in the tower $\square$.

Let E be an even periodic cohomology theory with formal group G.
If U is a complex bundle over X then

$$E^0(PU) = E^0(X)[\![x]\!]/f_U(x)$$

for some monic polynomial f_U of degree equal to $\dim(U)$.

The bottom connecting map

The tangent bundle to $PU = G_1(U)$ is
$\mathrm{Hom}(T,U) - \mathrm{Hom}(T,T) = \mathrm{Hom}(T,U) - \mathbb{C}$, so we have a Gysin map

$$X_0(U,Z) = S^0 \to \Sigma^2 G_1(U)^{-\mathrm{Hom}(T,U)} \subseteq \Sigma^2 G_1(U)^{\mathrm{Hom}(T,Z-U)} = \Sigma Q_1(U,Z).$$

Proposition: This is the bottom connecting map in the tower $\square$.

Let E be an even periodic cohomology theory with formal group G.
If U is a complex bundle over X then

$$E^0(PU) = E^0(X)[\![x]\!]/f_U(x)$$

for some monic polynomial f_U of degree equal to $\dim(U)$.The $E^0(X)$-module $M = E^0(\Sigma Q_1(U,Z))$ is generated by terms like $x^i\, f_Z(x)/f_U(x)\,dx$ and the Gysin map induces the residue map $\mathrm{res}\colon M \to E^0(X)$ (Quillen).

The bottom connecting map

The tangent bundle to $PU = G_1(U)$ is
$\mathrm{Hom}(T, U) - \mathrm{Hom}(T, T) = \mathrm{Hom}(T, U) - \mathbb{C}$, so we have a Gysin map

$$X_0(U, Z) = S^0 \to \Sigma^2 G_1(U)^{-\mathrm{Hom}(T,U)} \subseteq \Sigma^2 G_1(U)^{\mathrm{Hom}(T,Z-U)} = \Sigma Q_1(U, Z).$$

Proposition: This is the bottom connecting map in the tower $\square$.

Let E be an even periodic cohomology theory with formal group G.
If U is a complex bundle over X then

$$E^0(PU) = E^0(X)[\![x]\!]/f_U(x)$$

for some monic polynomial f_U of degree equal to $\dim(U)$.The $E^0(X)$-module $M = E^0(\Sigma Q_1(U, Z))$ is generated by terms like $x^i f_Z(x)/f_U(x)\, dx$ and the Gysin map induces the residue map $\mathrm{res}\colon M \to E^0(X)$ (Quillen).

If $U \leq Z$ then $f_U \mid f_Z$ and $\mathrm{res} = 0$ consistent with the Miller splitting.

The bottom connecting map

The tangent bundle to $PU = G_1(U)$ is $\mathrm{Hom}(T,U) - \mathrm{Hom}(T,T) = \mathrm{Hom}(T,U) - \mathbb{C}$, so we have a Gysin map

$$X_0(U,Z) = S^0 \to \Sigma^2 G_1(U)^{-\mathrm{Hom}(T,U)} \subseteq \Sigma^2 G_1(U)^{\mathrm{Hom}(T,Z-U)} = \Sigma Q_1(U,Z).$$

Proposition: This is the bottom connecting map in the tower $\square$.

Let E be an even periodic cohomology theory with formal group G. If U is a complex bundle over X then

$$E^0(PU) = E^0(X)[\![x]\!]/f_U(x)$$

for some monic polynomial f_U of degree equal to $\dim(U)$.The $E^0(X)$-module $M = E^0(\Sigma Q_1(U,Z))$ is generated by terms like $x^i\, f_Z(x)/f_U(x)\, dx$ and the Gysin map induces the residue map $\mathrm{res}\colon M \to E^0(X)$ (Quillen).

If $U \leq Z$ then $f_U \mid f_Z$ and $\mathrm{res} = 0$ consistent with the Miller splitting.

Conjecture: in general, the chain complex of the tower is $\Lambda^*(M)$ with differential determined by res and the Liebniz rule.

The bottom connecting map

The tangent bundle to $PU = G_1(U)$ is
$\mathrm{Hom}(T, U) - \mathrm{Hom}(T, T) = \mathrm{Hom}(T, U) - \mathbb{C}$, so we have a Gysin map

$$X_0(U, Z) = S^0 \to \Sigma^2 G_1(U)^{-\mathrm{Hom}(T,U)} \subseteq \Sigma^2 G_1(U)^{\mathrm{Hom}(T,Z-U)} = \Sigma Q_1(U, Z).$$

Proposition: This is the bottom connecting map in the tower $\square$.

Let E be an even periodic cohomology theory with formal group G.
If U is a complex bundle over X then

$$E^0(PU) = E^0(X)[\![x]\!]/f_U(x)$$

for some monic polynomial f_U of degree equal to $\dim(U)$.The $E^0(X)$-module $M = E^0(\Sigma Q_1(U, Z))$ is generated by terms like $x^i\, f_Z(x)/f_U(x)\, dx$ and the Gysin map induces the residue map $\mathrm{res}\colon M \to E^0(X)$ (Quillen).

If $U \leq Z$ then $f_U \mid f_Z$ and $\mathrm{res} = 0$ consistent with the Miller splitting.

Conjecture: in general, the chain complex of the tower is $\Lambda^*(M)$ with differential determined by res and the Liebniz rule.

This is bold, as we have not constructed any multiplicative structure in the non-split case.

Relationship with the Miller splitting

Conjecture:

Any choice of inclusion $U \to Z$ gives a canonical nullhomotopy of the connecting maps in the tower, and also canonical data proving that the composites

$$F_k(U,Z)_+ \to L(U,Z)_+ = X_n(U,Z) \to X_k(U,Z)$$

are stable homotopy equivalences.

Relationship with the Miller splitting

Conjecture:

Any choice of inclusion $U \to Z$ gives a canonical nullhomotopy of the connecting maps in the tower, and also canonical data proving that the composites

$$F_k(U, Z)_+ \to L(U, Z)_+ = X_n(U, Z) \to X_k(U, Z)$$

are stable homotopy equivalences.

The resulting splittings

$$F_k(U, Z)_+ \simeq \bigvee_{j=0}^{k} Q_k(U, Z)$$

should be the same as those obtained by Miller.

Relationship with the Miller splitting

Conjecture:

Any choice of inclusion $U \to Z$ gives a canonical nullhomotopy of the connecting maps in the tower, and also canonical data proving that the composites

$$F_k(U,Z)_+ \to L(U,Z)_+ = X_n(U,Z) \to X_k(U,Z)$$

are stable homotopy equivalences.

The resulting splittings

$$F_k(U,Z)_+ \simeq \bigvee_{j=0}^{k} Q_k(U,Z)$$

should be the same as those obtained by Miller.

Using abstract methods of equivariant stable homotopy theory, we can prove a slightly weaker statement. However, we have not yet succeeded in giving an explicit construction of the required maps and homotopies.

Change of notation

The notation used so far is inconvenient for actual constructions and proofs. Instead:

- The target space Z is fixed throughout and is not displayed.
- We will define spaces $X_k^*(U) \subseteq X_k(U)$ and put $\widehat{X}_k(U) = X_k(U)/X_k^*(U)$; the spectrum $X_k(U, Z)$ discussed earlier is $S^{-s(U)} \wedge \widehat{X}_k(U)$.
- We will define spaces $Q_k^*(U) \subseteq Q_k(U)$ and put $\widehat{Q}_k(U) = Q_k(U)/Q_k^*(U)$; the spectrum $Q_k(U, Z)$ discussed earlier is $S^{-s(U)} \wedge \widehat{Q}_k(U)$.
- Parallel notation with stars and hats will be used for various other spaces.

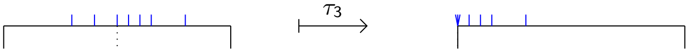
Simplices

Define $\Delta_n = \{t \in \mathbb{R}^n \mid 0 \leq t_1 \leq \cdots \leq t_n \leq 1\}$.
Conventions: $t_0 = 0,\ t_{n+1} = 1,\ \Delta_0 = \{\emptyset\},\ \|t\| = t_n$.

Simplices

Define $\Delta_n = \{t \in \mathbb{R}^n \mid 0 \leq t_1 \leq \cdots \leq t_n \leq 1\}$.
Conventions: $t_0 = 0$, $t_{n+1} = 1$, $\Delta_0 = \{\emptyset\}$, $\|t\| = t_n$.

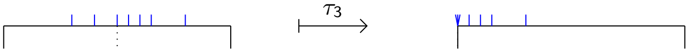
Truncation: $\tau_k : \Delta_n \to \Delta_n$ by $\tau_k(t)_i = \max(t_i - t_k, 0)$.



Simplices

Define $\Delta_n = \{t \in \mathbb{R}^n \mid 0 \leq t_1 \leq \cdots \leq t_n \leq 1\}$.
Conventions: $t_0 = 0$, $t_{n+1} = 1$, $\Delta_0 = \{\emptyset\}$, $\|t\| = t_n$.

Truncation: $\tau_k \colon \Delta_n \to \Delta_n$ by $\tau_k(t)_i = \max(t_i - t_k, 0)$.



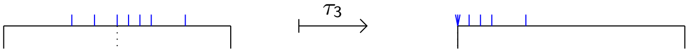
A commutative and associative operation on $I = [0, 1]$:

$$s\#t = s + t - st = 1 - (1-s)(1-t) = s + (1-s)t \geq \max(s, t)$$
$$s\#(t_1, \ldots, t_n) = (s\#t_1, \ldots, s\#t_n).$$

Simplices

Define $\Delta_n = \{t \in \mathbb{R}^n \mid 0 \le t_1 \le \cdots \le t_n \le 1\}$.
Conventions: $t_0 = 0,\ t_{n+1} = 1,\ \Delta_0 = \{\emptyset\},\ \|t\| = t_n$.

Truncation: $\tau_k \colon \Delta_n \to \Delta_n$ by $\tau_k(t)_i = \max(t_i - t_k, 0)$.

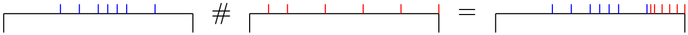


A commutative and associative operation on $I = [0,1]$:

$$s\#t = s + t - st = 1 - (1-s)(1-t) = s + (1-s)t \ge \max(s,t)$$
$$s\#(t_1, \ldots, t_n) = (s\#t_1, \ldots, s\#t_n).$$

For $t \in \Delta_n$ and $u \in \Delta_m$ put

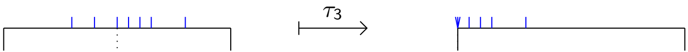
$$t * u = (t, \|t\|\#u) = (t_1, \ldots, t_n, t_n\#u_1, \ldots, t_n\#u_m) \in \Delta_{n+m}$$



Simplices

Define $\Delta_n = \{t \in \mathbb{R}^n \mid 0 \leq t_1 \leq \cdots \leq t_n \leq 1\}$.
Conventions: $t_0 = 0,\ t_{n+1} = 1,\ \Delta_0 = \{\emptyset\},\ \|t\| = t_n$.

Truncation: $\tau_k \colon \Delta_n \to \Delta_n$ by $\tau_k(t)_i = \max(t_i - t_k, 0)$.

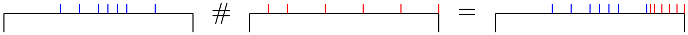


A commutative and associative operation on $I = [0, 1]$:

$$s\#t = s + t - st = 1 - (1-s)(1-t) = s + (1-s)t \geq \max(s, t)$$
$$s\#(t_1, \ldots, t_n) = (s\#t_1, \ldots, s\#t_n).$$

For $t \in \Delta_n$ and $u \in \Delta_m$ put

$$t * u = (t, \|t\|\#u) = (t_1, \ldots, t_n, t_n\#u_1, \ldots, t_n\#u_m) \in \Delta_{n+m}$$



This is associative with $\emptyset * t = t = t * \emptyset$ and $\|t * u\| = \|t\|\#\|u\|$ and $\tau_{n+i}(t * u) = 0 * ((1 - \|t\|)\tau_i(u))$.

Operators

For $\phi\colon U \to V$ put $\|\phi\| = \max\{\|\phi(u)\| \mid u \in B(U)\}$ (the *operator norm*).

Operators

For $\phi \colon U \to V$ put $\|\phi\| = \max\{\|\phi(u)\| \mid u \in B(U)\}$ (the *operator norm*).

As before $\rho(\phi) = \sqrt{\phi^\dagger \phi}$; then $\|\phi\| = \|\rho(\phi)\| =$ largest eigenvalue of $\rho(\phi)$.

Operators

For $\phi\colon U \to V$ put $\|\phi\| = \max\{\|\phi(u)\| \mid u \in B(U)\}$ (the *operator norm*).

As before $\rho(\phi) = \sqrt{\phi^\dagger\phi}$; then $\|\phi\| = \|\rho(\phi)\| =$ largest eigenvalue of $\rho(\phi)$.

$$D(U) = \{\alpha \in \mathrm{End}(U) \mid \alpha^\dagger = \alpha, \text{ eigenvalues in } [0,1]\} \simeq B(s(U)).$$

If $\dim(U) = n$, define $e\colon D(U) \to \Delta_n$ by $e(\alpha) =$ ordered list of eigenvalues .

Operators

For $\phi\colon U \to V$ put $\|\phi\| = \max\{\|\phi(u)\| \mid u \in B(U)\}$ (the *operator norm*).

As before $\rho(\phi) = \sqrt{\phi^\dagger\phi}$; then $\|\phi\| = \|\rho(\phi)\| =$ largest eigenvalue of $\rho(\phi)$.

$$D(U) = \{\alpha \in \text{End}(U) \mid \alpha^\dagger = \alpha, \text{ eigenvalues in } [0,1]\} \simeq B(s(U)).$$

If $\dim(U) = n$, define $e\colon D(U) \to \Delta_n$ by $e(\alpha) =$ ordered list of eigenvalues .

By functional calculus, define $\tau_k\colon D(U) \to D(U)$ with $\tau_k \circ e = e \circ \tau_k$.

Operators

For $\phi\colon U \to V$ put $\|\phi\| = \max\{\|\phi(u)\| \mid u \in B(U)\}$ (the *operator norm*).

As before $\rho(\phi) = \sqrt{\phi^\dagger\phi}$; then $\|\phi\| = \|\rho(\phi)\| =$ largest eigenvalue of $\rho(\phi)$.

$$D(U) = \{\alpha \in \mathrm{End}(U) \mid \alpha^\dagger = \alpha, \text{ eigenvalues in } [0,1]\} \simeq B(s(U)).$$

If $\dim(U) = n$, define $e\colon D(U) \to \Delta_n$ by $e(\alpha) =$ ordered list of eigenvalues .

By functional calculus, define $\tau_k\colon D(U) \to D(U)$ with $\tau_k \circ e = e \circ \tau_k$.

For $s \in I$ and $\alpha \in D(U)$ put $s\#\alpha = s.1_U + (1-s)\alpha$ so $e(s\#\alpha) = s\#e(\alpha)$.

Operators

For $\phi\colon U \to V$ put $\|\phi\| = \max\{\|\phi(u)\| \mid u \in B(U)\}$ (the *operator norm*).

As before $\rho(\phi) = \sqrt{\phi^\dagger\phi}$; then $\|\phi\| = \|\rho(\phi)\| =$ largest eigenvalue of $\rho(\phi)$.

$$D(U) = \{\alpha \in \mathrm{End}(U) \mid \alpha^\dagger = \alpha,\ \text{eigenvalues in } [0,1]\} \simeq B(s(U)).$$

If $\dim(U) = n$, define $e\colon D(U) \to \Delta_n$ by $e(\alpha) =$ ordered list of eigenvalues .

By functional calculus, define $\tau_k\colon D(U) \to D(U)$ with $\tau_k \circ e = e \circ \tau_k$.

For $s \in I$ and $\alpha \in D(U)$ put $s\#\alpha = s.1_U + (1-s)\alpha$ so $e(s\#\alpha) = s\#e(\alpha)$.

For $\alpha \in D(U)$ and $\beta \in D(V)$ put

$$\alpha * \beta = \alpha \oplus (\|\alpha\|\#\beta) \in D(U \oplus V)$$

so $e(\alpha * \beta) = e(\alpha) * e(\beta)$.

Operators

For $\phi\colon U \to V$ put $\|\phi\| = \max\{\|\phi(u)\| \mid u \in B(U)\}$ (the *operator norm*).

As before $\rho(\phi) = \sqrt{\phi^\dagger\phi}$; then $\|\phi\| = \|\rho(\phi)\| =$ largest eigenvalue of $\rho(\phi)$.

$$D(U) = \{\alpha \in \mathrm{End}(U) \mid \alpha^\dagger = \alpha, \text{ eigenvalues in } [0,1]\} \simeq B(s(U)).$$

If $\dim(U) = n$, define $e\colon D(U) \to \Delta_n$ by $e(\alpha) =$ ordered list of eigenvalues .

By functional calculus, define $\tau_k\colon D(U) \to D(U)$ with $\tau_k \circ e = e \circ \tau_k$.

For $s \in I$ and $\alpha \in D(U)$ put $s\#\alpha = s.1_U + (1-s)\alpha$ so $e(s\#\alpha) = s\#e(\alpha)$.

For $\alpha \in D(U)$ and $\beta \in D(V)$ put

$$\alpha * \beta = \alpha \oplus (\|\alpha\|\#\beta) \in D(U \oplus V)$$

so $e(\alpha * \beta) = e(\alpha) * e(\beta)$. We sometimes write $\alpha *_V \beta$.

More operators

$$H(U) = \{\phi\colon U \to Z \mid \|\phi\| \leq 1\} = B(\mathrm{Hom}(U, Z))$$
$$L(U) = \{\phi\colon U \to Z \mid \rho(\phi) = 1_U\} = L(U, Z) \subseteq H(U).$$

More operators

$$H(U) = \{\phi\colon U \to Z \mid \|\phi\| \leq 1\} = B(\mathrm{Hom}(U,Z))$$
$$L(U) = \{\phi\colon U \to Z \mid \rho(\phi) = 1_U\} = L(U,Z) \subseteq H(U).$$

$$L(U) \times D(U) \xrightarrow{\mu} H(U) \xrightarrow{\rho} D(U) \xrightarrow{e} \Delta_n$$

Define $\mu(\theta, \alpha) = \theta\alpha$. Then μ, ρ and e are quotient maps with $\rho\mu(\theta, \alpha) = \theta\alpha$.

More operators

$$H(U) = \{\phi\colon U \to Z \mid \|\phi\| \leq 1\} = B(\mathrm{Hom}(U,Z))$$
$$L(U) = \{\phi\colon U \to Z \mid \rho(\phi) = 1_U\} = L(U,Z) \subseteq H(U).$$

$$\begin{array}{ccccccc}
L(U)\times D(U) & \xrightarrow{\mu} & H(U) & \xrightarrow{\rho} & D(U) & \xrightarrow{e} & \Delta_n \\
 & & & & & & \downarrow m \\
L(U)\times D(U) & \xrightarrow{\mu} & H(U) & \xrightarrow{\rho} & D(U) & \xrightarrow{e} & \Delta_n
\end{array}$$

Define $\mu(\theta, \alpha) = \theta\alpha$. Then μ, ρ and e are quotient maps with $\rho\mu(\theta,\alpha) = \theta\alpha$.
If $m\colon I \to I$ is continuous, and nondecreasing, it gives a self-map of Δ_n

More operators

$$H(U) = \{\phi\colon U \to Z \mid \|\phi\| \leq 1\} = B(\mathrm{Hom}(U,Z))$$
$$L(U) = \{\phi\colon U \to Z \mid \rho(\phi) = 1_U\} = L(U,Z) \subseteq H(U).$$

$$\begin{array}{ccccccc}
L(U)\times D(U) & \xrightarrow{\mu} \!\!\!\!\twoheadrightarrow & H(U) & \xrightarrow{\rho}\!\!\!\!\twoheadrightarrow & D(U) & \xrightarrow{e}\!\!\!\!\twoheadrightarrow & \Delta_n \\
\downarrow{\scriptstyle 1\times m} & & & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} \\
L(U)\times D(U) & \xrightarrow{\mu}\!\!\!\!\twoheadrightarrow & H(U) & \xrightarrow{\rho}\!\!\!\!\twoheadrightarrow & D(U) & \xrightarrow{e}\!\!\!\!\twoheadrightarrow & \Delta_n
\end{array}$$

Define $\mu(\theta,\alpha) = \theta\alpha$. Then μ, ρ and e are quotient maps with $\rho\mu(\theta,\alpha) = \theta\alpha$. If $m\colon I \to I$ is continuous, and nondecreasing, it gives a self-map of Δ_n and (by functional calculus) of $D(U)$.

More operators

$$H(U) = \{\phi\colon U \to Z \mid \|\phi\| \leq 1\} = B(\mathrm{Hom}(U,Z))$$
$$L(U) = \{\phi\colon U \to Z \mid \rho(\phi) = 1_U\} = L(U,Z) \subseteq H(U).$$

$$\begin{array}{ccccccc}
L(U) \times D(U) & \xrightarrow{\mu} & H(U) & \xrightarrow{\rho} & D(U) & \xrightarrow{e} & \Delta_n \\
\downarrow{\scriptstyle 1\times m} & & & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} \\
L(U) \times D(U) & \xrightarrow{\mu} & H(U) & \xrightarrow{\rho} & D(U) & \xrightarrow{e} & \Delta_n
\end{array}$$

Define $\mu(\theta, \alpha) = \theta\alpha$. Then μ, ρ and e are quotient maps with $\rho\mu(\theta,\alpha) = \theta\alpha$. If $m\colon I \to I$ is continuous, and nondecreasing, it gives a self-map of Δ_n and (by functional calculus) of $D(U)$. This depends continuously on m.

More operators

$$H(U) = \{\phi\colon U \to Z \mid \|\phi\| \leq 1\} = B(\mathrm{Hom}(U, Z))$$
$$L(U) = \{\phi\colon U \to Z \mid \rho(\phi) = 1_U\} = L(U, Z) \subseteq H(U).$$

$$\begin{array}{ccccccc}
L(U) \times D(U) & \xrightarrow{\ \mu\ } & H(U) & \xrightarrow{\ \rho\ } & D(U) & \xrightarrow{\ e\ } & \Delta_n \\
\downarrow{\scriptstyle 1\times m} & & \downarrow{\scriptstyle m_\bullet} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} \\
L(U) \times D(U) & \xrightarrow{\ \mu\ } & H(U) & \xrightarrow{\ \rho\ } & D(U) & \xrightarrow{\ e\ } & \Delta_n
\end{array}$$

Define $\mu(\theta, \alpha) = \theta\alpha$. Then μ, ρ and e are quotient maps with $\rho\mu(\theta, \alpha) = \theta\alpha$. If $m\colon I \to I$ is continuous, and nondecreasing, it gives a self-map of Δ_n and (by functional calculus) of $D(U)$. This depends continuously on m. If $m(0) = 0$, then there is a unique $m_\bullet$ making the diagram commute.

More operators

$$
\begin{aligned}
H(U) &= \{\phi\colon U \to Z \mid \|\phi\| \le 1\} = B(\mathrm{Hom}(U,Z)) \\
L(U) &= \{\phi\colon U \to Z \mid \rho(\phi) = 1_U\} = L(U,Z) \subseteq H(U).
\end{aligned}
$$

$$
\begin{array}{ccccccc}
L(U) \times D(U) & \xrightarrow{\mu} & H(U) & \xrightarrow{\rho} & D(U) & \xrightarrow{e} & \Delta_n \\
\downarrow{\scriptstyle 1\times m} & & \downarrow{\scriptstyle m_\bullet} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} \\
L(U) \times D(U) & \xrightarrow{\mu} & H(U) & \xrightarrow{\rho} & D(U) & \xrightarrow{e} & \Delta_n
\end{array}
$$

Define $\mu(\theta,\alpha) = \theta\alpha$. Then μ, ρ and e are quotient maps with $\rho\mu(\theta,\alpha) = \theta\alpha$. If $m\colon I \to I$ is continuous, and nondecreasing, it gives a self-map of Δ_n and (by functional calculus) of $D(U)$. This depends continuously on m. If $m(0) = 0$, then there is a unique $m_\bullet$ making the diagram commute. Thus:

$$
\begin{array}{ccccc}
L(U) \times D(U) & \xrightarrow{\mu} & H(U) & \xrightarrow{\rho} & D(U) \\
\downarrow{\scriptstyle 1\times \tau_k} & & \downarrow{\scriptstyle \tau_k} & & \downarrow{\scriptstyle \tau_k} \\
L(U) \times D(U) & \xrightarrow[\mu]{} & H(U) & \xrightarrow[\rho]{} & D(U).
\end{array}
$$

Definition of the tower

For $0 \leq k \leq n = \dim(U)$ we put

$$X_k(U) = \{(\alpha, \phi) \in D(U) \times H(U) \mid \tau_{n-k}(\alpha) = \rho(\phi)\}$$

$$Q_k(U) = \{(W, \beta, \psi) \mid W \in G_k(U),\ \beta \in D(W^\perp),\ \psi \in H(W)\}.$$

Definition of the tower

For $0 \leq k \leq n = \dim(U)$ we put

$$X_k(U) = \{(\alpha, \phi) \in D(U) \times H(U) \mid \tau_{n-k}(\alpha) = \rho(\phi)\}$$

$$Q_k(U) = \{(W, \beta, \psi) \mid W \in G_k(U),\ \beta \in D(W^\perp),\ \psi \in H(W)\}.$$

(NB $X_0(U) \simeq Q_0(U) \simeq D(U)$ and $X_n(U) \simeq Q_n(U) \simeq H(U)$.)

Definition of the tower

For $0 \leq k \leq n = \dim(U)$ we put

$$X_k(U) = \{(\alpha, \phi) \in D(U) \times H(U) \mid \tau_{n-k}(\alpha) = \rho(\phi)\}$$
$$Q_k(U) = \{(W, \beta, \psi) \mid W \in G_k(U),\ \beta \in D(W^\perp),\ \psi \in H(W)\}.$$

(NB $X_0(U) \simeq Q_0(U) \simeq D(U)$ and $X_n(U) \simeq Q_n(U) \simeq H(U)$.)

Suppose that $(\alpha, \phi) \in X_k(U)$, where $0 \leq k \leq n = \dim(U) \leq m = \dim(Z)$.

Definition of the tower

For $0 \leq k \leq n = \dim(U)$ we put

$$X_k(U) = \{(\alpha, \phi) \in D(U) \times H(U) \mid \tau_{n-k}(\alpha) = \rho(\phi)\}$$
$$Q_k(U) = \{(W, \beta, \psi) \mid W \in G_k(U),\ \beta \in D(W^\perp),\ \psi \in H(W)\}.$$

(NB $X_0(U) \simeq Q_0(U) \simeq D(U)$ and $X_n(U) \simeq Q_n(U) \simeq H(U)$.)

Suppose that $(\alpha, \phi) \in X_k(U)$, where $0 \leq k \leq n = \dim(U) \leq m = \dim(Z)$.
Put $t = e(\alpha) \in \Delta_n$.

For $0 \leq k \leq n = \dim(U)$ we put

$$X_k(U) = \{(\alpha, \phi) \in D(U) \times H(U) \mid \tau_{n-k}(\alpha) = \rho(\phi)\}$$

$$Q_k(U) = \{(W, \beta, \psi) \mid W \in G_k(U),\ \beta \in D(W^\perp),\ \psi \in H(W)\}.$$

(NB $X_0(U) \simeq Q_0(U) \simeq D(U)$ and $X_n(U) \simeq Q_n(U) \simeq H(U)$.)

Suppose that $(\alpha, \phi) \in X_k(U)$, where $0 \leq k \leq n = \dim(U) \leq m = \dim(Z)$. Put $t = e(\alpha) \in \Delta_n$. Then there is an orthonormal basis $u_1, \ldots, u_n$ for U, and an orthonormal basis $z_1, \ldots, z_m$ for Z, such that $\alpha(u_i) = t_i u_i$ and $\phi(u_i) = 0$ for $1 \leq i \leq n - k$ and $\phi(u_i) = (t_i - t_{n-k})z_i$ for $n - k \leq i \leq n$.

Definition of the tower

For $0 \leq k \leq n = \dim(U)$ we put

$$X_k(U) = \{(\alpha, \phi) \in D(U) \times H(U) \mid \tau_{n-k}(\alpha) = \rho(\phi)\}$$
$$Q_k(U) = \{(W, \beta, \psi) \mid W \in G_k(U),\ \beta \in D(W^\perp),\ \psi \in H(W)\}.$$

(NB $X_0(U) \simeq Q_0(U) \simeq D(U)$ and $X_n(U) \simeq Q_n(U) \simeq H(U)$.)

Suppose that $(\alpha, \phi) \in X_k(U)$, where $0 \leq k \leq n = \dim(U) \leq m = \dim(Z)$. Put $t = e(\alpha) \in \Delta_n$. Then there is an orthonormal basis $u_1, \dots, u_n$ for U, and an orthonormal basis $z_1, \dots, z_m$ for Z, such that $\alpha(u_i) = t_i u_i$ and $\phi(u_i) = 0$ for $1 \leq i \leq n-k$ and $\phi(u_i) = (t_i - t_{n-k}) z_i$ for $n-k \leq i \leq n$.

Also $Q_k(U) \simeq B(s(T^\perp) \oplus \mathrm{Hom}(T, Z))$; using $\mathrm{Hom}(T, T) \simeq s(T) + s(T)$ and $s(U) \simeq s(T^\perp) + s(T) + \mathrm{Hom}(T, U - T)$ we get

$$s(T^\perp) + \mathrm{Hom}(T, Z) \simeq (s(T) + \mathrm{Hom}(T, U - Z)) + s(U).$$

For $0 \leq k \leq n = \dim(U)$ we put

$$X_k(U) = \{(\alpha, \phi) \in D(U) \times H(U) \mid \tau_{n-k}(\alpha) = \rho(\phi)\}$$
$$Q_k(U) = \{(W, \beta, \psi) \mid W \in G_k(U),\ \beta \in D(W^\perp),\ \psi \in H(W)\}.$$

(NB $X_0(U) \simeq Q_0(U) \simeq D(U)$ and $X_n(U) \simeq Q_n(U) \simeq H(U)$.)

Suppose that $(\alpha, \phi) \in X_k(U)$, where $0 \leq k \leq n = \dim(U) \leq m = \dim(Z)$. Put $t = e(\alpha) \in \Delta_n$. Then there is an orthonormal basis $u_1, \ldots, u_n$ for U, and an orthonormal basis $z_1, \ldots, z_m$ for Z, such that $\alpha(u_i) = t_i u_i$ and $\phi(u_i) = 0$ for $1 \leq i \leq n-k$ and $\phi(u_i) = (t_i - t_{n-k}) z_i$ for $n-k \leq i \leq n$.

Also $Q_k(U) \simeq B(s(T^\perp) \oplus \mathrm{Hom}(T, Z))$; using $\mathrm{Hom}(T, T) \simeq s(T) + s(T)$ and $s(U) \simeq s(T^\perp) + s(T) + \mathrm{Hom}(T, U - T)$ we get

$$s(T^\perp) + \mathrm{Hom}(T, Z) \simeq (s(T) + \mathrm{Hom}(T, U - Z)) + s(U).$$

We define maps
$$Q_k(U) \xrightarrow{f_k} X_k(U) \xrightarrow{p_k} X_{k-1}(U)$$

by
$$f_k(W, \beta, \psi) = (\beta *_W \rho(\psi), (1 - \|\beta\|)\psi\pi_W)$$
$$p_k(\alpha, \phi) = (\alpha, \tau_{n-k+1}(\phi)).$$

Collapsing the boundary

We put

$$\begin{aligned}
\Delta_n^* &= \{t \in \Delta_n \mid t_1 = 0 \text{ or } \|t\| = 1\} & \widehat{\Delta}_n &= \Delta_n/\Delta_n^* \\
D^*(U) &= \{\alpha \in D(U) \mid e_1(\alpha) = 0 \text{ or } \|\alpha\| = 1\} & \widehat{D}(U) &= D(U)/D^*(U) \\
H^*(U) &= \{\phi \in H(U) \mid e_1(\rho(\phi)) = 0 \text{ or } \|\phi\| = 1\} & \widehat{H}(U) &= H(U)/H^*(U) \\
X_k^*(U) &= \{(\alpha, \phi) \in X_k(U) \mid \alpha \in D^*(U)\} & \widehat{X}_k(U) &= X_k(U)/X_k^*(U) \\
Q_k^*(U) &= \{(W, \beta, \psi) \in Q_k(U) \mid \beta \in D^*(W^\perp) \text{ or } \|\psi\| = 1\} & \widehat{Q}_k(U) &= Q_k(U)/Q_k^*(U).
\end{aligned}$$

Collapsing the boundary

We put

$$\Delta_n^* = \{t \in \Delta_n \mid t_1 = 0 \text{ or } \|t\| = 1\} \qquad \widehat{\Delta}_n = \Delta_n/\Delta_n^*$$

$$D^*(U) = \{\alpha \in D(U) \mid e_1(\alpha) = 0 \text{ or } \|\alpha\| = 1\} \qquad \widehat{D}(U) = D(U)/D^*(U)$$

$$H^*(U) = \{\phi \in H(U) \mid e_1(\rho(\phi)) = 0 \text{ or } \|\phi\| = 1\} \qquad \widehat{H}(U) = H(U)/H^*(U)$$

$$X_k^*(U) = \{(\alpha, \phi) \in X_k(U) \mid \alpha \in D^*(U)\} \qquad \widehat{X}_k(U) = X_k(U)/X_k^*(U)$$

$$Q_k^*(U) = \{(W, \beta, \psi) \in Q_k(U) \mid \beta \in D^*(W^\perp) \text{ or } \|\psi\| = 1\} \qquad \widehat{Q}_k(U) = Q_k(U)/Q_k^*(U).$$

Note:

$$\widehat{X}_0(U) \simeq \widehat{Q}_0(U) \simeq \widehat{D}(U) \simeq S^{s(U)}$$

$$X_n(U) \simeq Q_n(U) \simeq H(U)$$

$$Q_n^*(U) = \{\phi \in H(U) \mid \|\phi\| = 1\}$$

$$X_n^*(U) = \{\phi \in H(U) \mid \|\phi\| = 1 \text{ or } \phi \text{ is not injective }\}.$$

Collapsing the boundary

We put

$$\begin{aligned}
\Delta_n^* &= \{t \in \Delta_n \mid t_1 = 0 \text{ or } \|t\| = 1\} & \widehat{\Delta}_n &= \Delta_n/\Delta_n^* \\
D^*(U) &= \{\alpha \in D(U) \mid e_1(\alpha) = 0 \text{ or } \|\alpha\| = 1\} & \widehat{D}(U) &= D(U)/D^*(U) \\
H^*(U) &= \{\phi \in H(U) \mid e_1(\rho(\phi)) = 0 \text{ or } \|\phi\| = 1\} & \widehat{H}(U) &= H(U)/H^*(U) \\
X_k^*(U) &= \{(\alpha,\phi) \in X_k(U) \mid \alpha \in D^*(U)\} & \widehat{X}_k(U) &= X_k(U)/X_k^*(U) \\
Q_k^*(U) &= \{(W,\beta,\psi) \in Q_k(U) \mid \beta \in D^*(W^\perp) \text{ or } \|\psi\| = 1\} & \widehat{Q}_k(U) &= Q_k(U)/Q_k^*(U).
\end{aligned}$$

Note:

$$\begin{aligned}
\widehat{X}_0(U) &\simeq \widehat{Q}_0(U) \simeq \widehat{D}(U) \simeq S^{s(U)} \\
X_n(U) &\simeq Q_n(U) \simeq H(U) \\
Q_n^*(U) &= \{\phi \in H(U) \mid \|\phi\| = 1\} \\
X_n^*(U) &= \{\phi \in H(U) \mid \|\phi\| = 1 \text{ or } \phi \text{ is not injective } \}.
\end{aligned}$$

Fact: The maps f_k and p_k preserve starred subspaces and so induce

$$\widehat{Q}_k(U) \xrightarrow{f_k} \widehat{X}_k(U) \xrightarrow{p_k} \widehat{X}_{k-1}(U).$$

$p_k f_k \colon \widehat{Q}_k(U) \to \widehat{X}_{k-1}(U)$ has a natural nullhomotopy

For $0 \le s \le 1$ we define $F_s \colon Q_k(U) \to X_{k-1}(U)$ (with $F_1 = p_k f_k$) by

$F_s(W, \beta, \psi) = (\beta \oplus \gamma,\ s(1 - \|\beta\|)\tau_1(\psi)\pi_W)$ where $\gamma = (1-s)\#\|\beta\|\#\rho(\psi)$.

(Recall: $\beta \in D(W^\perp) \subseteq s(W^\perp)$ and $\psi \in H(W) \subseteq \mathrm{Hom}(W, Z)$.)

$p_k f_k \colon \widehat{Q}_k(U) \to \widehat{X}_{k-1}(U)$ has a natural nullhomotopy

For $0 \le s \le 1$ we define $F_s \colon Q_k(U) \to X_{k-1}(U)$ (with $F_1 = p_k f_k$) by

$F_s(W, \beta, \psi) = (\beta \oplus \gamma,\ s(1 - \|\beta\|)\tau_1(\psi)\pi_W)$ where $\gamma = (1-s)\#\|\beta\|\#\rho(\psi)$.

(Recall: $\beta \in D(W^\perp) \subseteq s(W^\perp)$ and $\psi \in H(W) \subseteq \mathrm{Hom}(W, Z)$.)

One can check that that

$$\tau_{n-k+1}(\beta \oplus \gamma) = 0 \oplus (\gamma - e_1(\gamma)) = 0 \oplus s(1 - \|\beta\|)\tau_1(\rho(\psi))$$
$$\tau_{n-k}(\beta \oplus \gamma) = 0 \oplus (\gamma - \|\beta\|) = 0 \oplus ((1-s)(1 - \|\beta\|) + s(1 - \|\beta\|)\rho(\psi)).$$

$p_k f_k \colon \widehat{Q}_k(U) \to \widehat{X}_{k-1}(U)$ has a natural nullhomotopy

For $0 \le s \le 1$ we define $F_s \colon Q_k(U) \to X_{k-1}(U)$ (with $F_1 = p_k f_k$) by

$F_s(W, \beta, \psi) = (\beta \oplus \gamma,\ s(1 - \|\beta\|)\tau_1(\psi)\pi_W)$ where $\gamma = (1-s)\#\|\beta\|\#\rho(\psi)$.

(Recall: $\beta \in D(W^\perp) \subseteq s(W^\perp)$ and $\psi \in H(W) \subseteq \mathrm{Hom}(W, Z)$.)

One can check that that

$$\tau_{n-k+1}(\beta \oplus \gamma) = 0 \oplus (\gamma - e_1(\gamma)) = 0 \oplus s(1 - \|\beta\|)\tau_1(\rho(\psi))$$
$$\tau_{n-k}(\beta \oplus \gamma) = 0 \oplus (\gamma - \|\beta\|) = 0 \oplus ((1-s)(1 - \|\beta\|) + s(1 - \|\beta\|)\rho(\psi)).$$

We also have

$$F_s(Q_k^*(U)) \subseteq X_{k-1}^*(U)$$
$$F_0(Q_k(U)) \subseteq X_{k-1}^*(U).$$

It follows that F gives a nullhomotopy of the composite

$$\widehat{Q}_k(U) \xrightarrow{f_k} \widehat{X}_k(U) \xrightarrow{p_k} \widehat{X}_{k-1}(U).$$

The connecting maps $g_k \colon \widehat{X}_{k-1}(U) \to \Sigma\widehat{Q}_k(U)$

The connecting maps $g_k \colon \widehat{X}_{k-1}(U) \to \Sigma\widehat{Q}_k(U)$

We will construct a commutative diagram

$$\begin{array}{ccccc} EG_k(U) & \xrightarrow{\ \omega\ } & X_{k-1}(U) & \xrightarrow{\ c\ } & \widehat{X}_{k-1}(U) \\ \Big\downarrow{\scriptstyle \widetilde{g}_k} & & & & \Big\downarrow{\scriptstyle g_k} \\ I \times Q_k(U) & & \xrightarrow[\ c\ ]{\qquad\qquad} & & \Sigma\widehat{Q}_k(U). \end{array}$$

where

- c and c are collapse maps, and ω is a quotient map.
- $\widetilde{g}_k$ is visibly continuous.
- g_k is visibly well-defined but not obviously continuous.

The connecting maps $g_k \colon \widehat{X}_{k-1}(U) \to \Sigma\widehat{Q}_k(U)$

We will construct a commutative diagram

$$\begin{array}{ccccc}
EG_k(U) & \overset{\omega}{\twoheadrightarrow} & X_{k-1}(U) & \overset{c}{\twoheadrightarrow} & \widehat{X}_{k-1}(U) \\
\Big\downarrow{\scriptstyle \widetilde{g}_k} & & & & \Big\downarrow{\scriptstyle g_k} \\
I \times Q_k(U) & & \underset{c}{\xrightarrow{\qquad\qquad\qquad}} & & \Sigma\widehat{Q}_k(U).
\end{array}$$

where

- c and c are collapse maps, and ω is a quotient map.
- $\widetilde{g}_k$ is visibly continuous.
- g_k is visibly well-defined but not obviously continuous.

Commutativity proves that g_k is in fact continuous.

The connecting maps $g_k\colon \widehat{X}_{k-1}(U) \to \Sigma\widehat{Q}_k(U)$

We will construct a commutative diagram

$$\begin{array}{ccccc}
EG_k(U) & \xrightarrow{\ \omega\ } & X_{k-1}(U) & \xrightarrow{\ c\ } & \widehat{X}_{k-1}(U) \\
\Big\downarrow \widetilde{g}_k & & & & \Big\downarrow g_k \\
I \times Q_k(U) & & \xrightarrow[\ c\ ]{} & & \Sigma\widehat{Q}_k(U).
\end{array}$$

where

- c and c are collapse maps, and ω is a quotient map.
- $\widetilde{g}_k$ is visibly continuous.
- g_k is visibly well-defined but not obviously continuous.

Commutativity proves that g_k is in fact continuous.

$$\begin{aligned}
EG_k(U) &= \{(t, W, \beta, \psi) \in I \times Q_k(U) \mid \psi \text{ is not injective } \} \\
&= \{(t, W, \beta, \psi) \in I \times Q_k(U) \mid e_1(\rho(\psi)) = 0\} \\
\widetilde{g}_k &= \text{ the inclusion } : EG_k(U) \to I \times Q_k(U) \\
\omega_k(t, W, \beta, \psi) &= (\beta *_W (t\#\rho(\psi)), (1-t)(1-\|\beta\|)\psi\pi_W)\,.
\end{aligned}$$

Connecting maps

$$\begin{array}{ccccc} EG_k(U) & \xrightarrow{\omega} & X_{k-1}(U) & \xrightarrow{c} & \widehat{X}_{k-1}(U) \\ \downarrow{\scriptstyle \widetilde{g}_k} & & & & \downarrow{\scriptstyle g_k} \\ I \times Q_k(U) & & \xrightarrow[c]{\hspace{4em}} & & \Sigma \widehat{Q}_k(U). \end{array}$$

For generic $(\alpha, \phi) \in \widehat{X}_{k-1}(U)$: put $p = e_{n-k}(\alpha)$ and $q = e_{n-k+1}(\alpha)$. Let W be the sum of all eigenspaces with eigenvalues $\geq q$. Put $t = (q-p)/(1-p)$ and $\beta = \alpha|_{W^\perp}$ and $\psi = (1-q)^{-1}\phi|_W$. Define $g_k(\alpha, \phi) = (t, W, \beta, \psi)$.

Connecting maps

$$\begin{array}{ccccc} EG_k(U) & \xrightarrow{\omega} & X_{k-1}(U) & \xrightarrow{c} & \widehat{X}_{k-1}(U) \\ \downarrow{\scriptstyle \widetilde{g}_k} & & & & \downarrow{\scriptstyle g_k} \\ I \times Q_k(U) & & \xrightarrow[c]{\qquad\qquad} & & \Sigma\widehat{Q}_k(U). \end{array}$$

For generic $(\alpha, \phi) \in \widehat{X}_{k-1}(U)$: put $p = e_{n-k}(\alpha)$ and $q = e_{n-k+1}(\alpha)$. Let W be the sum of all eigenspaces with eigenvalues $\geq q$. Put $t = (q-p)/(1-p)$ and $\beta = \alpha|_{W^\perp}$ and $\psi = (1-q)^{-1}\phi|_W$. Define $g_k(\alpha, \phi) = (t, W, \beta, \psi)$.

In any cases where something goes wrong, put $g_k(\alpha, \phi) = *$.

$$\begin{array}{ccccc} EG_k(U) & \xrightarrow{\omega} & X_{k-1}(U) & \xrightarrow{c} & \widehat{X}_{k-1}(U) \\ \Big\downarrow{\scriptstyle \widetilde{g}_k} & & & & \Big\downarrow{\scriptstyle g_k} \\ I \times Q_k(U) & & \xrightarrow[c]{\qquad\qquad} & & \Sigma \widehat{Q}_k(U). \end{array}$$

For generic $(\alpha, \phi) \in \widehat{X}_{k-1}(U)$: put $p = e_{n-k}(\alpha)$ and $q = e_{n-k+1}(\alpha)$. Let W be the sum of all eigenspaces with eigenvalues $\geq q$. Put $t = (q-p)/(1-p)$ and $\beta = \alpha|_{W^\perp}$ and $\psi = (1-q)^{-1}\phi|_W$. Define $g_k(\alpha, \phi) = (t, W, \beta, \psi)$.

In any cases where something goes wrong, put $g_k(\alpha, \phi) = *$.

One can check that this makes the diagram commute.

Connecting maps

$$\begin{array}{ccccc} EG_k(U) & \overset{\omega}{\twoheadrightarrow} & X_{k-1}(U) & \overset{c}{\twoheadrightarrow} & \widehat{X}_{k-1}(U) \\ {\scriptstyle \widetilde{g}_k}\downarrow & & & & \downarrow{\scriptstyle g_k} \\ I \times Q_k(U) & & \underset{c}{\twoheadrightarrow} & & \Sigma\widehat{Q}_k(U). \end{array}$$

For generic $(\alpha, \phi) \in \widehat{X}_{k-1}(U)$: put $p = e_{n-k}(\alpha)$ and $q = e_{n-k+1}(\alpha)$. Let W be the sum of all eigenspaces with eigenvalues $\geq q$. Put $t = (q-p)/(1-p)$ and $\beta = \alpha|_{W^\perp}$ and $\psi = (1-q)^{-1}\phi|_W$. Define $g_k(\alpha, \phi) = (t, W, \beta, \psi)$.

In any cases where something goes wrong, put $g_k(\alpha, \phi) = *$.

One can check that this makes the diagram commute.

One can use similar methods to define nullhomotopies of composites in the chain

$$\widehat{Q}_k(U) \xrightarrow{f_k} \widehat{X}_k(U) \xrightarrow{p_k} \widehat{X}_{k-1}(U) \xrightarrow{g_k} \Sigma\widehat{Q}_k(U) \xrightarrow{\Sigma f_k} \Sigma\widehat{X}_k(U).$$

The cofibration property

Suppose we have a diagram $Q \xrightarrow{f} X \xrightarrow{p} Y \xrightarrow{g} \Sigma Q \xrightarrow{\Sigma f} \Sigma X$.
We will say that this is a *cofibre sequence* if there exist maps

$$\begin{array}{ccccccc}
X & \xrightarrow{p} & Y & \xrightarrow{j} & Cp & \xrightarrow{d} & \Sigma X \\
\| & & \| & & r\downarrow\;\uparrow s & & \| \\
X & \xrightarrow[p]{} & Y & \xrightarrow[g]{} & \Sigma Q & \xrightarrow[\Sigma f]{} & \Sigma X
\end{array}$$

such that $rj \simeq g$ and $ds \simeq \Sigma f$ and $rs \simeq 1_{\Sigma Q}$ and $sr \simeq 1_{Cp}$.

The cofibration property

Suppose we have a diagram $Q \xrightarrow{f} X \xrightarrow{p} Y \xrightarrow{g} \Sigma Q \xrightarrow{\Sigma f} \Sigma X$.
We will say that this is a *cofibre sequence* if there exist maps

$$
\begin{array}{ccccccc}
X & \xrightarrow{p} & Y & \xrightarrow{j} & Cp & \xrightarrow{d} & \Sigma X \\
\| & & \| & & r \downarrow \; \uparrow s & & \| \\
X & \xrightarrow[p]{} & Y & \xrightarrow[g]{} & \Sigma Q & \xrightarrow[\Sigma f]{} & \Sigma X
\end{array}
$$

such that $rj \simeq g$ and $ds \simeq \Sigma f$ and $rs \simeq 1_{\Sigma Q}$ and $sr \simeq 1_{Cp}$.
(Stably, this is equivalent to all other definitions; unstably, perhaps not.)

The cofibration property

Suppose we have a diagram $Q \xrightarrow{f} X \xrightarrow{p} Y \xrightarrow{g} \Sigma Q \xrightarrow{\Sigma f} \Sigma X$.
We will say that this is a *cofibre sequence* if there exist maps

$$\begin{array}{ccccccc}
X & \xrightarrow{p} & Y & \xrightarrow{j} & Cp & \xrightarrow{d} & \Sigma X \\
\| & & \| & & r \downarrow \uparrow s & & \| \\
X & \xrightarrow[p]{} & Y & \xrightarrow[g]{} & \Sigma Q & \xrightarrow[\Sigma f]{} & \Sigma X
\end{array}$$

such that $rj \simeq g$ and $ds \simeq \Sigma f$ and $rs \simeq 1_{\Sigma Q}$ and $sr \simeq 1_{Cp}$.
(Stably, this is equivalent to all other definitions; unstably, perhaps not.)

From nullhomotopies of gp and pf there is a functorial construction of maps r and s with $rj = g$ and $ds \simeq \Sigma f$ (but not necessarily $rs \simeq 1$ or $sr \simeq 1$).

The cofibration property

Suppose we have a diagram $Q \xrightarrow{f} X \xrightarrow{p} Y \xrightarrow{g} \Sigma Q \xrightarrow{\Sigma f} \Sigma X$.
We will say that this is a *cofibre sequence* if there exist maps

$$\begin{array}{ccccccc}
X & \xrightarrow{p} & Y & \xrightarrow{j} & Cp & \xrightarrow{d} & \Sigma X \\
\| & & \| & & r\downarrow\uparrow s & & \| \\
X & \xrightarrow[p]{} & Y & \xrightarrow[g]{} & \Sigma Q & \xrightarrow[\Sigma f]{} & \Sigma X
\end{array}$$

such that $rj \simeq g$ and $ds \simeq \Sigma f$ and $rs \simeq 1_{\Sigma Q}$ and $sr \simeq 1_{Cp}$.
(Stably, this is equivalent to all other definitions; unstably, perhaps not.)

From nullhomotopies of gp and pf there is a functorial construction of maps r and s with $rj = g$ and $ds \simeq \Sigma f$ (but not necessarily $rs \simeq 1$ or $sr \simeq 1$).
The composite rs is essentially the Toda bracket $\langle g, p, f \rangle$.

The cofibration property

Suppose we have a diagram $Q \xrightarrow{f} X \xrightarrow{p} Y \xrightarrow{g} \Sigma Q \xrightarrow{\Sigma f} \Sigma X$.
We will say that this is a *cofibre sequence* if there exist maps

$$\begin{array}{ccccccc}
X & \xrightarrow{p} & Y & \xrightarrow{j} & Cp & \xrightarrow{d} & \Sigma X \\
\| & & \| & & r \downarrow \uparrow s & & \| \\
X & \xrightarrow[p]{} & Y & \xrightarrow[g]{} & \Sigma Q & \xrightarrow[\Sigma f]{} & \Sigma X
\end{array}$$

such that $rj \simeq g$ and $ds \simeq \Sigma f$ and $rs \simeq 1_{\Sigma Q}$ and $sr \simeq 1_{Cp}$.
(Stably, this is equivalent to all other definitions; unstably, perhaps not.)

From nullhomotopies of gp and pf there is a functorial construction of maps r and s with $rj = g$ and $ds \simeq \Sigma f$ (but not necessarily $rs \simeq 1$ or $sr \simeq 1$).
The composite rs is essentially the Toda bracket $\langle g, p, f \rangle$.

In the case

$$\widehat{Q}_k(U) \xrightarrow{f_k} \widehat{X}_k(U) \xrightarrow{p_k} \widehat{X}_{k-1}(U) \xrightarrow{g_k} \Sigma\widehat{Q}_k(U) \xrightarrow{\Sigma f_k} \Sigma\widehat{X}_k(U).$$

we can define explicit homotopies giving $rs \simeq 1$ and $sr \simeq 1$.

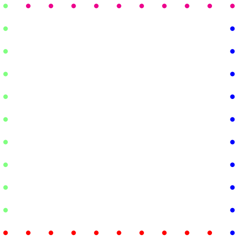
The cofibration property

In the case

$$\widehat{Q}_k(U) \xrightarrow{f_k} \widehat{X}_k(U) \xrightarrow{p_k} \widehat{X}_{k-1}(U) \xrightarrow{g_k} \Sigma\widehat{Q}_k(U) \xrightarrow{\Sigma f_k} \Sigma\widehat{X}_k(U).$$

we can define explicit homotopies giving $rs \simeq 1$ and $sr \simeq 1$.

Many ingredients are similar to earlier constructions. However, we also need a family of maps $w_p \colon I^2 \to I^2$ (for $0 \leq p \leq 1$) with specific properties:

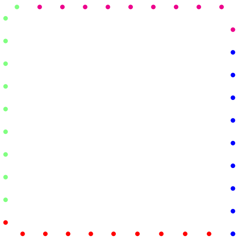


In the case

$$\widehat{Q}_k(U) \xrightarrow{f_k} \widehat{X}_k(U) \xrightarrow{p_k} \widehat{X}_{k-1}(U) \xrightarrow{g_k} \Sigma\widehat{Q}_k(U) \xrightarrow{\Sigma f_k} \Sigma\widehat{X}_k(U).$$

we can define explicit homotopies giving $rs \simeq 1$ and $sr \simeq 1$.

Many ingredients are similar to earlier constructions. However, we also need a family of maps $w_p\colon I^2 \to I^2$ (for $0 \leq p \leq 1$) with specific properties:



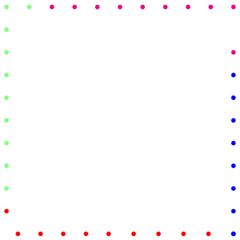
The cofibration property

In the case

$$\widehat{Q}_k(U) \xrightarrow{f_k} \widehat{X}_k(U) \xrightarrow{p_k} \widehat{X}_{k-1}(U) \xrightarrow{g_k} \Sigma\widehat{Q}_k(U) \xrightarrow{\Sigma f_k} \Sigma\widehat{X}_k(U).$$

we can define explicit homotopies giving $rs \simeq 1$ and $sr \simeq 1$.

Many ingredients are similar to earlier constructions. However, we also need a family of maps $w_p \colon I^2 \to I^2$ (for $0 \leq p \leq 1$) with specific properties:



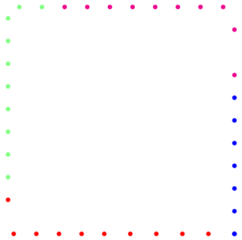
The cofibration property

In the case

$$\widehat{Q}_k(U) \xrightarrow{f_k} \widehat{X}_k(U) \xrightarrow{p_k} \widehat{X}_{k-1}(U) \xrightarrow{g_k} \Sigma\widehat{Q}_k(U) \xrightarrow{\Sigma f_k} \Sigma\widehat{X}_k(U).$$

we can define explicit homotopies giving $rs \simeq 1$ and $sr \simeq 1$.

Many ingredients are similar to earlier constructions. However, we also need a family of maps $w_p \colon I^2 \to I^2$ (for $0 \leq p \leq 1$) with specific properties:



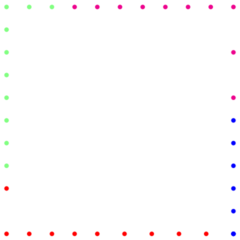
The cofibration property

In the case

$$\widehat{Q}_k(U) \xrightarrow{f_k} \widehat{X}_k(U) \xrightarrow{p_k} \widehat{X}_{k-1}(U) \xrightarrow{g_k} \Sigma\widehat{Q}_k(U) \xrightarrow{\Sigma f_k} \Sigma\widehat{X}_k(U).$$

we can define explicit homotopies giving $rs \simeq 1$ and $sr \simeq 1$.

Many ingredients are similar to earlier constructions. However, we also need a family of maps $w_p \colon I^2 \to I^2$ (for $0 \leq p \leq 1$) with specific properties:



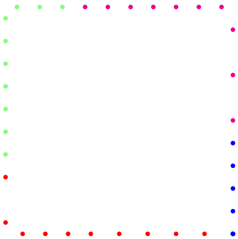
The cofibration property

In the case

$$\widehat{Q}_k(U) \xrightarrow{f_k} \widehat{X}_k(U) \xrightarrow{p_k} \widehat{X}_{k-1}(U) \xrightarrow{g_k} \Sigma\widehat{Q}_k(U) \xrightarrow{\Sigma f_k} \Sigma\widehat{X}_k(U).$$

we can define explicit homotopies giving $rs \simeq 1$ and $sr \simeq 1$.

Many ingredients are similar to earlier constructions. However, we also need a family of maps $w_p \colon I^2 \to I^2$ (for $0 \leq p \leq 1$) with specific properties:



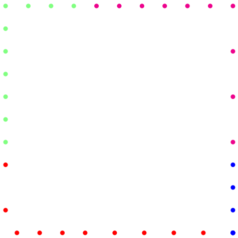
The cofibration property

In the case

$$\widehat{Q}_k(U) \xrightarrow{f_k} \widehat{X}_k(U) \xrightarrow{p_k} \widehat{X}_{k-1}(U) \xrightarrow{g_k} \Sigma\widehat{Q}_k(U) \xrightarrow{\Sigma f_k} \Sigma\widehat{X}_k(U).$$

we can define explicit homotopies giving $rs \simeq 1$ and $sr \simeq 1$.

Many ingredients are similar to earlier constructions. However, we also need a family of maps $w_p\colon I^2 \to I^2$ (for $0 \leq p \leq 1$) with specific properties:



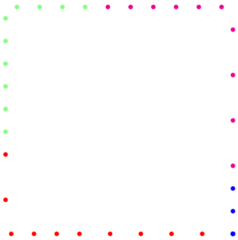
The cofibration property

In the case

$$\widehat{Q}_k(U) \xrightarrow{f_k} \widehat{X}_k(U) \xrightarrow{p_k} \widehat{X}_{k-1}(U) \xrightarrow{g_k} \Sigma\widehat{Q}_k(U) \xrightarrow{\Sigma f_k} \Sigma\widehat{X}_k(U).$$

we can define explicit homotopies giving $rs \simeq 1$ and $sr \simeq 1$.

Many ingredients are similar to earlier constructions. However, we also need a family of maps $w_p \colon I^2 \to I^2$ (for $0 \leq p \leq 1$) with specific properties:



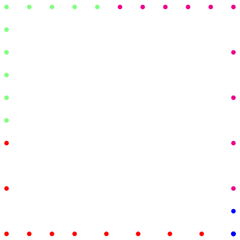
The cofibration property

In the case

$$\widehat{Q}_k(U) \xrightarrow{f_k} \widehat{X}_k(U) \xrightarrow{p_k} \widehat{X}_{k-1}(U) \xrightarrow{g_k} \Sigma\widehat{Q}_k(U) \xrightarrow{\Sigma f_k} \Sigma\widehat{X}_k(U).$$

we can define explicit homotopies giving $rs \simeq 1$ and $sr \simeq 1$.

Many ingredients are similar to earlier constructions. However, we also need a family of maps $w_p\colon I^2 \to I^2$ (for $0 \leq p \leq 1$) with specific properties:



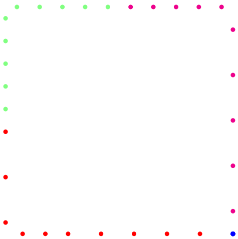
The cofibration property

In the case

$$\widehat{Q}_k(U) \xrightarrow{f_k} \widehat{X}_k(U) \xrightarrow{p_k} \widehat{X}_{k-1}(U) \xrightarrow{g_k} \Sigma\widehat{Q}_k(U) \xrightarrow{\Sigma f_k} \Sigma\widehat{X}_k(U).$$

we can define explicit homotopies giving $rs \simeq 1$ and $sr \simeq 1$.

Many ingredients are similar to earlier constructions. However, we also need a family of maps $w_p \colon I^2 \to I^2$ (for $0 \leq p \leq 1$) with specific properties:



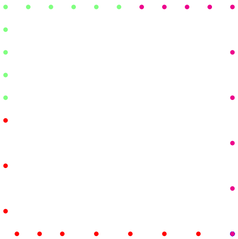
The cofibration property

In the case

$$\widehat{Q}_k(U) \xrightarrow{f_k} \widehat{X}_k(U) \xrightarrow{p_k} \widehat{X}_{k-1}(U) \xrightarrow{g_k} \Sigma\widehat{Q}_k(U) \xrightarrow{\Sigma f_k} \Sigma\widehat{X}_k(U).$$

we can define explicit homotopies giving $rs \simeq 1$ and $sr \simeq 1$.

Many ingredients are similar to earlier constructions. However, we also need a family of maps $w_p \colon I^2 \to I^2$ (for $0 \leq p \leq 1$) with specific properties:



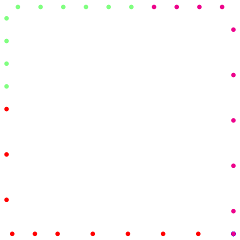
The cofibration property

In the case

$$\widehat{Q}_k(U) \xrightarrow{f_k} \widehat{X}_k(U) \xrightarrow{p_k} \widehat{X}_{k-1}(U) \xrightarrow{g_k} \Sigma\widehat{Q}_k(U) \xrightarrow{\Sigma f_k} \Sigma\widehat{X}_k(U).$$

we can define explicit homotopies giving $rs \simeq 1$ and $sr \simeq 1$.

Many ingredients are similar to earlier constructions. However, we also need a family of maps $w_p \colon I^2 \to I^2$ (for $0 \leq p \leq 1$) with specific properties:



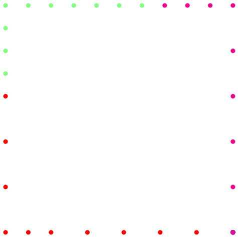
The cofibration property

In the case

$$\widehat{Q}_k(U) \xrightarrow{f_k} \widehat{X}_k(U) \xrightarrow{p_k} \widehat{X}_{k-1}(U) \xrightarrow{g_k} \Sigma\widehat{Q}_k(U) \xrightarrow{\Sigma f_k} \Sigma\widehat{X}_k(U).$$

we can define explicit homotopies giving $rs \simeq 1$ and $sr \simeq 1$.

Many ingredients are similar to earlier constructions. However, we also need a family of maps $w_p \colon I^2 \to I^2$ (for $0 \leq p \leq 1$) with specific properties:

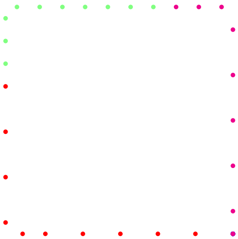


In the case

$$\widehat{Q}_k(U) \xrightarrow{f_k} \widehat{X}_k(U) \xrightarrow{p_k} \widehat{X}_{k-1}(U) \xrightarrow{g_k} \Sigma\widehat{Q}_k(U) \xrightarrow{\Sigma f_k} \Sigma\widehat{X}_k(U).$$

we can define explicit homotopies giving $rs \simeq 1$ and $sr \simeq 1$.

Many ingredients are similar to earlier constructions. However, we also need a family of maps $w_p \colon I^2 \to I^2$ (for $0 \leq p \leq 1$) with specific properties:



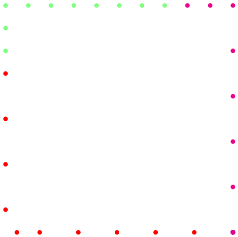
The cofibration property

In the case

$$\widehat{Q}_k(U) \xrightarrow{f_k} \widehat{X}_k(U) \xrightarrow{p_k} \widehat{X}_{k-1}(U) \xrightarrow{g_k} \Sigma\widehat{Q}_k(U) \xrightarrow{\Sigma f_k} \Sigma\widehat{X}_k(U).$$

we can define explicit homotopies giving $rs \simeq 1$ and $sr \simeq 1$.

Many ingredients are similar to earlier constructions. However, we also need a family of maps $w_p \colon I^2 \to I^2$ (for $0 \leq p \leq 1$) with specific properties:



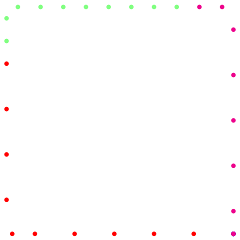
The cofibration property

In the case

$$\widehat{Q}_k(U) \xrightarrow{f_k} \widehat{X}_k(U) \xrightarrow{p_k} \widehat{X}_{k-1}(U) \xrightarrow{g_k} \Sigma\widehat{Q}_k(U) \xrightarrow{\Sigma f_k} \Sigma\widehat{X}_k(U).$$

we can define explicit homotopies giving $rs \simeq 1$ and $sr \simeq 1$.

Many ingredients are similar to earlier constructions. However, we also need a family of maps $w_p \colon I^2 \to I^2$ (for $0 \leq p \leq 1$) with specific properties:



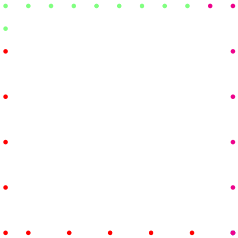
The cofibration property

In the case

$$\widehat{Q}_k(U) \xrightarrow{f_k} \widehat{X}_k(U) \xrightarrow{p_k} \widehat{X}_{k-1}(U) \xrightarrow{g_k} \Sigma\widehat{Q}_k(U) \xrightarrow{\Sigma f_k} \Sigma\widehat{X}_k(U).$$

we can define explicit homotopies giving $rs \simeq 1$ and $sr \simeq 1$.

Many ingredients are similar to earlier constructions. However, we also need a family of maps $w_p\colon I^2 \to I^2$ (for $0 \leq p \leq 1$) with specific properties:



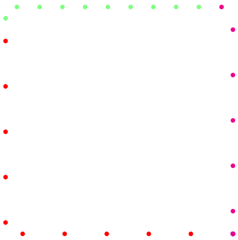
The cofibration property

In the case

$$\widehat{Q}_k(U) \xrightarrow{f_k} \widehat{X}_k(U) \xrightarrow{p_k} \widehat{X}_{k-1}(U) \xrightarrow{g_k} \Sigma\widehat{Q}_k(U) \xrightarrow{\Sigma f_k} \Sigma\widehat{X}_k(U).$$

we can define explicit homotopies giving $rs \simeq 1$ and $sr \simeq 1$.

Many ingredients are similar to earlier constructions. However, we also need a family of maps $w_p \colon I^2 \to I^2$ (for $0 \leq p \leq 1$) with specific properties:



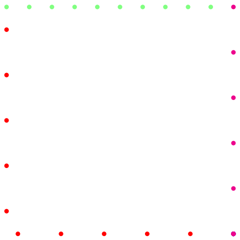
The cofibration property

In the case

$$\widehat{Q}_k(U) \xrightarrow{f_k} \widehat{X}_k(U) \xrightarrow{p_k} \widehat{X}_{k-1}(U) \xrightarrow{g_k} \Sigma\widehat{Q}_k(U) \xrightarrow{\Sigma f_k} \Sigma\widehat{X}_k(U).$$

we can define explicit homotopies giving $rs \simeq 1$ and $sr \simeq 1$.

Many ingredients are similar to earlier constructions. However, we also need a family of maps $w_p \colon I^2 \to I^2$ (for $0 \leq p \leq 1$) with specific properties:

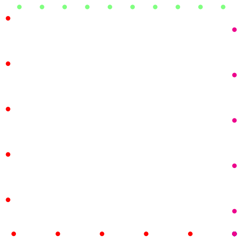


In the case

$$\widehat{Q}_k(U) \xrightarrow{f_k} \widehat{X}_k(U) \xrightarrow{p_k} \widehat{X}_{k-1}(U) \xrightarrow{g_k} \Sigma\widehat{Q}_k(U) \xrightarrow{\Sigma f_k} \Sigma\widehat{X}_k(U).$$

we can define explicit homotopies giving $rs \simeq 1$ and $sr \simeq 1$.

Many ingredients are similar to earlier constructions. However, we also need a family of maps $w_p \colon I^2 \to I^2$ (for $0 \leq p \leq 1$) with specific properties:



The cofibration property

In the case

$$\widehat{Q}_k(U) \xrightarrow{f_k} \widehat{X}_k(U) \xrightarrow{p_k} \widehat{X}_{k-1}(U) \xrightarrow{g_k} \Sigma\widehat{Q}_k(U) \xrightarrow{\Sigma f_k} \Sigma\widehat{X}_k(U).$$

we can define explicit homotopies giving $rs \simeq 1$ and $sr \simeq 1$.

Many ingredients are similar to earlier constructions. However, we also need a family of maps $w_p \colon I^2 \to I^2$ (for $0 \leq p \leq 1$) with specific properties: