The Nilpotence Theorem

Neil Strickland

May 18, 2018

Statement of the Theorem

Let R be a finite ring spectrum, and let u be an element of $\pi_*(R)$. Suppose that the image of u in $\pi_*(MU \wedge R)$ is nilpotent. Then u itself is nilpotent.

This was conjectured by Ravenel, and proved by Hopkins, Devinatz and Smith. It is the key foundational result of chromatic homotopy theory.

We will introduce some spectra

$$S^0 = X(1) \to X(2) \to X(3) \to \cdots \to X(\infty) = MU$$

$$0 = X(n,0) \to X(n) = X(n,1) \to X(n,2) \to X(n,3) \to \cdots \to X(n,\infty) = X(n+1)$$

and prove some facts about their properties. The Nilpotence Theorem will follow easily from these.

Three preliminary reductions:

(a) If E is a ring spectrum, then u becomes nilpotent in $\pi_*(E \wedge R)$ iff $E \wedge R[u^{-1}] = 0$. (Note: this depends only on the Bousfield class of E.)

(b) For a sequence of ring spectra $E(i)$ with colimit $E(\infty)$ we have $E(\infty) = 0$ iff $1 = 0$ in $\varinjlim_i \pi_0(E(i))$ iff $E(i) = 0$ for $i \gg 0$.

(c) For the rest of the talk, we will fix a prime p and work p-locally. It is not hard to recover the integral statement from the p-local ones.

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Properties (a) and (b) are easy. Property (c) is moderately hard. The main work is to prove property (d).

Properties of the spectra $X(n, k)$

(a) $MU = X(\infty)$ is the colimit over n of $X(n)$ (and these are ring spectra).

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The Adams resolution property

Let E be a ring spectrum.

- Say $f\colon X \to Y$ has E-filtration at least s if f can be written as a composite of s maps f_i, each with $1_E \wedge f_i = 0$.
- An E-resolution of Y is a tower of spectra

$$Y = Y_0 \xleftarrow{g_1} Y_1 \xleftarrow{g_2} Y_2 \xleftarrow{g_3} \cdots$$

  such that $1_E \wedge g_i = 0$ for all i, and each fibre $F_i = \mathrm{fib}(g_i)$ admits an E-module structure.
- (Subject to some conditions, this will give a spectral sequence $\mathrm{Ext}^{**}_{E_*E}(E_*X, E_*Y) \Longrightarrow [X, L_EY]_*$. But we do not need that.)
- Suppose we have such a resolution, and a map $f\colon X \to Y$ of E-filtration at least s; then f lifts to Y_s. Thus, if the connectivity of Y_s is greater than $\dim(X)$, then $f = 0$.
- Consider the case where $E = X(n+1)$ and $Y = X(n, p^k)$. We will give an explicit construction of a resolution where the connectivity of Y_s is $2np^ks$.
- Suppose that $u \mapsto 0$ in $\pi_*(X(n+1) \wedge R)$, so multiplication by u has $X(n+1)$-filtration at least 1. Fix k with $2np^k > |u|$. For any $a \in \pi_*(X(n,k) \wedge R)$ we find that $\mathrm{conn}(Y_s \wedge R) - \deg(u^s a) > 0$ for $s \gg 0$, so $a \to 0$ in $\pi_*(X(n,k) \wedge R[u^{-1}])$. This gives $X(n,k) \wedge R[u^{-1}] = 0$.

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Construction of $X(n)$

For ease of comparison with formal group theory, we put $P = \bigvee_{n \in \mathbb{Z}} S^{2n}$ and $MP = MU \wedge P$ and $XP(n) = X(n) \wedge P$ and $XP(n,k) = X(n,k) \wedge P$.

Consider an even periodic ring spectrum E, with associated formal group $G = \mathrm{spf}(E^0(\mathbb{C}P^\infty))$ over $S = \mathrm{spec}(E_0)$.

(a) $E_0 MP = E_0[b_0^{\pm 1}, b_1, b_2, \dots]$, and $\mathrm{spec}(E_0 MP)$ is the scheme $\mathrm{Coord}(G)$ of coordinates on G.

(b) MP is the Thom spectrum of the tautological virtual bundle over $\mathbb{Z} \times BU$. So, $E_0(\mathbb{Z} \times BU)$ is isomorphic to $E_0 MP$, but not in a canonical way.

(c) $\mathrm{spec}(E_0(\mathbb{Z} \times BU))$ is the scheme of invertible functions on G. This acts freely and transitively on $\mathrm{Coord}(G)$ by multiplication.

(d) Bott periodicity: $\mathbb{Z} \times BU = \Omega U$. This gives a virtual bundle over $\Omega U(n)$; define $XP(n)$ to be the Thom spectrum. (Use $\Omega SU(n)$ for $X(n)$.)

(e) $E_0 XP(n) = E_0[b_0^{\pm 1}, b_1, \dots, b_{n-1}]$, and $\mathrm{spec}(E_0(XP(n)))$ is the scheme of n-jets of coordinates on G. (But $\pi_* XP(n)$ is not fully known.)

(f) $E_0 XP(n,m)$ will be the free module over $E_0 XP(n)$ generated by $\{b_n^i \mid 0 \leq i < m\}$. This looks like m copies of $XP(n)$, making it plausible that $\langle X(n) \rangle = \langle X(n,m) \rangle$. But there are attaching maps.

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The Bott periodicity map

- Put $A = \mathbb{C}[z]$ and $K = \mathbb{C}[z, z^{-1}]$.
- By interpreting z as a point in $S^1 \subset \mathbb{C}$, we get a map $GL_n(K) \to \text{Map}(S^1, GL_n(\mathbb{C})) \simeq \text{Map}(S^1, U(n))$; this can be shown to be a homotopy equivalence.
- Using $h_t(z) = tz$ we get $GL_n(A) \simeq GL_n(\mathbb{C}) \simeq U(n)$.
- This gives $\Omega U(n) \simeq \text{Map}(S^1, U(n))/U(n) \simeq GL_n(K)/GL_n(A)$.
- A *lattice* in K^n is an A-submodule $L \leq K^n$ with $z^r A^n \leq L \leq z^{-r} A^n$ for $r \gg 0$. The set of lattices is the $GL_n(K)$-orbit of A^n, which has stabiliser $GL_n(A)$; so $\{\text{lattices}\} \simeq \Omega U(n)$.
- For any lattice L we have a virtual vector space $(L/z^r A) - (A/z^r A)$ for $r \gg 0$. This is the bundle over $\Omega U(n)$ whose Thom spectrum is $X(n)$.
- Define $\rho \colon \mathbb{C}P^{n-1} \to \Omega U(n)$ by $\rho(L)(z) = z.1_L \oplus 1_{L^\perp}$.
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- We have $S^{2n+1}=S^{2n}\wedge S^1$ and so can define $\eta\colon S^{2n}\to\Omega S^{2n+1}$ by $\eta(u)(t)=u\wedge t$.
- This extends to give $J(n)\to\Omega S^{2n+1}$, which is a homotopy equivalence.
- $E_0J(n)=E_0[b_n]$, and $E_0J(n,m)=E_0\{b_n^i\mid i<m\}$.
- $\mathrm{spec}(E_0J(n))$ is the scheme of n-jets of invertible functions on G, for which the corresponding $(n-1)$-jet is trivial. This is a group scheme which acts freely on $\mathrm{Coord}_n(G)=\mathrm{spec}(E_0XP(n))$, with orbit space $\mathrm{Coord}_{n-1}(G)=\mathrm{spec}(E_0XP(n-1))$.
- We define $XP(n,m)$ to be the Thom spectrum for $(\Omega\epsilon)^{-1}J(n,m)\subset\Omega U(n+1)$.

What is special about the p-power stages?

- We have defined $J(n,m)$ for all $m \geq 0$, but the cases $m = p^k$ play a special role.
- $H_*J(n) = \mathbb{Z}[b_n]$, and the monoid structure on $J(n)$ makes this a Hopf algebra, with $\psi(b_n) = b_n \otimes 1 + 1 \otimes b_n$.
- Let $x_n^{[k]} \in H^{2nk}J(n)$ be dual to b_n^k. We find that $x_n^{[j]}x_n^{[k]} = \frac{(j+k)!}{j!k!}x_n^{[j+k]}$, so we have a divided power algebra.
- Put $u_k = x_n^{[p^k]} \in H^{2np^k}(J(n);\mathbb{F}_p)$.
  Using standard congruences of binomial coefficients, we find that

$$H^*(J(n);\mathbb{F}_p) = \mathbb{F}_p[u_0, u_1, u_2, \cdots]/(u_0^p, u_1^p, u_2^p, \cdots).$$

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$$\mathbb{F}_p[u_k, u_{k+1}, \ldots]/(u_i^p) \to \mathbb{F}_p[u_0, u_1, \ldots]/(u_i^p).$$

- It is easy to see that $J(n, p^k) \to J(n) \xrightarrow{h_{p^k}} J(np^k)$ is null so we get a map from $J(n, p^k)$ to the homotopy fibre of h_{p^k}. Using the above calculation, one can show that this is an equivalence.

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- We need to prove that $\langle XP(n,p^k)\rangle = \langle XP(n)\rangle$.
- In general, let R be a ring spectrum, and M and R-module. If $R\wedge Z=0$ then $M\wedge R\wedge Z=0$, but M is a retract of $M\wedge R$, so $M\wedge Z=0$. This gives $\langle M\rangle\leq\langle R\rangle$.
- As a special case: $\langle XP(n,m)\rangle\leq\langle XP(n)\rangle$.
- It will now suffice to show that $\langle XP(n,p^k)\rangle\leq\langle XP(n,p^{k+1}\rangle$.
- Here is the general pattern for the proof: Suppose we have $f\colon U\to\Sigma^a U$ and $g\colon\Sigma^b V\to V$, with $\mathrm{fib}(f)\simeq\mathrm{cof}(g)$. Suppose also that $V[g^{-1}]=0$. We claim that $\langle V\rangle\leq\langle U\rangle$, i.e. $U\wedge Z=0\Rightarrow V\wedge Z=0$.
- Indeed, if $U\wedge Z=0$, then

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- So it will suffice to define self maps ξ and r of $XP(n,p^{k+1})$ and $XP(n,p^k)$ with $\mathrm{fib}(\xi)=\mathrm{cof}(r)$ and $XP(n,p^k)[r^{-1}]=0$.

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- There is an evaluation map $\Sigma\Omega S^{2n+1} \to S^{2n+1}$ given by $t \wedge u \mapsto u(t)$. Desuspending gives a stable map $\omega\colon J(n) \to S^{2n}$. Put

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Relating $X(n,p^k)$ to $X(n,p^{k+1})$

- There is an evaluation map $\Sigma\Omega S^{2n+1} \to S^{2n+1}$ given by $t \wedge u \mapsto u(t)$. Desuspending gives a stable map $\omega\colon J(n) \to S^{2n}$. Put

$$\xi = (XP(n+1) \xrightarrow{\zeta} XP(n+1) \wedge J(np^k) \xrightarrow{1\wedge\omega} XP(n+1) \wedge S^{2np^k}).$$

- On $E_0XP(n+1)$ we get $\xi_*(u) = (p^k!)^{-1}\partial^{p^k}u/\partial b_n^{p^k}$.
- One can check that ξ restricts to give a map $\xi\colon XP(n,p^{k+1}) \to XP(n,p^{k+1}) \wedge S^{2np^k}$, with fibre F say.
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Why is $\mathcal{E}(m)$ an H-algebra?

- The dual Steenrod algebra is H_*H; this is a Hopf algebra.
- For any spectrum Z, there is a coaction map $H_*Z \to H_*H \otimes H_*Z$.
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- All the relevant rings have a polynomial part tensored with an exterior part.
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Doubly looped spheres

- Put $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, so $\Delta_\infty \simeq S^2$, so $\Omega^2\Sigma^2 Y = F(\Delta_\infty, \Delta_\infty \wedge Y)$.
- Let $C(q)$ be the space of lists $f = (f_1, \dots, f_q)$, where $f_i \colon \Delta \to \Delta$ has the form $f_i(z) = a_i + \epsilon_i z$ ($\epsilon_i > 0$), and the images of the f_i are disjoint.
- Say $g \in \Omega^2\Sigma^2 Y$ is *simple* if there is $f \in C(q)$, and $y \in Y^q$, such that
  - $g(f_i(z)) = z \wedge y_i$ for all i and all $z \in \Delta$
  - Outside the images of the maps f_i, we have $f(w) = \text{basepoint}$.

  Let $F(q; Y)$ be the set of such g.
- We can add an extra f_i with $y_i = *$; so $F(q-1; Y) \subseteq F(q; Y)$. Put $\overline{F}(q; Y) = F(q; Y)/F(q-1; Y) = C(q)_+ \wedge_{\Sigma_q} Y^{(q)}$.
- It is a theorem of May that the space of simple maps is homotopy equivalent to all of $\Omega^2\Sigma^2 Y$. (Similar to $JY \simeq \Omega\Sigma Y$.) Taking $Y = S^{2m-1}$, we get $\Omega^2 S^{2m+1} = \Omega J(m)$.
- Note that if $Q \subset \mathbb{C}$ with $|Q| = q$ then $\mathbb{C}[t]_{<q}$ is independent of Q and maps isomorphically to $\text{Map}(Q, \mathbb{C})$. This untwists some Σ_q-actions.
- Put $D(q) = C(q)_+ \wedge_{\Sigma_q} S^{-q}$. We find that $\overline{F}(q; S^{2m-1}) = S^{2mq} \wedge D(q)$ and $\Omega J(m) = \bigvee_q S^{2mq} \wedge D(q)$ (Snaith splitting).
- This circle of ideas gives a map $C(p) \times_{\Sigma_p} (\Omega J(m))^p \to \Omega J(m)$, which gives an operation $\xi \colon H_{2i-1}(\Omega J(m)) \to H_{2pi-1}(\Omega J(m))$.

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$$A = E[u_0, u_1, u_2, \dots] \otimes P[v_1, v_2, \dots]$$

- There is a fibration $\Omega J(m) \to PJ(m) \to J(m)$ with $PJ(m)$ contractible. This gives a Serre spectral sequence

$$H_* J(m) \otimes H_* \Omega J(m) = P[b_m] \otimes H_*(\Omega J(m)) \Longrightarrow \mathbb{F}_p.$$

  This can only work out if the map $A \to H_* \Omega J(m)$ is an isomorphism, and $b_m^{p^j}$ hits u_j, and $b_m^{(p-1)p^j} u_j$ hits v_{j+1}.
- The operation ξ interacts with the Steenrod coaction in a known way, so we can check that the coaction for $\Omega J(m)$ is as on the previous slide, so the coaction on $\mathcal{E}(m)$ is cofree, so $\mathcal{E}(m)$ is an H-module.

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