Symmetric Powers of Spheres

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Overview of homotopy theory

\[ \pi_* (S^*) \]
Overview of homotopy theory

\[ \pi_\ast(S^\ast) \quad \longrightarrow \quad H_\ast(S^0) \]
Overview of homotopy theory

\[ \pi_*(S^*) \rightarrow \pi_*^S(S^0) \rightarrow MU_*^*(S^0) \rightarrow H_*^*(S^0) \]
Overview of homotopy theory

\[ \pi_\ast(S^\ast) \rightarrow \pi^S_\ast(S^0) \rightarrow MU_\ast(S^0) \rightarrow H_\ast(S^0) \]

Formal groups
Overview of homotopy theory

\[ \pi_{k+1}S^1 \xrightarrow{E} \pi_{k+2}S^2 \xrightarrow{E} \pi_{k+3}S^3 \xrightarrow{E} \pi_{k+4}S^4 \xrightarrow{} \pi_k(QS^0) = \pi_k(S^0) \]

\[ \pi_*(S^*) \xrightarrow{} \pi_*(S^0) \xrightarrow{} MU_*(S^0) \xrightarrow{} H_*(S^0) \]

Formal groups
Overview of homotopy theory

\[ \pi_{k+1}S^1 \rightarrow \pi_{k+2}S^2 \rightarrow \pi_{k+3}S^3 \rightarrow \pi_{k+4}S^4 \rightarrow \pi_k QS^0 \Rightarrow \pi_k S^0 \]

Formal groups

\[ \pi_*(S^*) \rightarrow \pi_* S^0 \rightarrow MU_* S^0 \rightarrow H_* S^0 \]
Overview of homotopy theory

\[ \pi_\ast(S^\ast) \xrightarrow{\text{EHPSS}} \pi^S_\ast(S^0) \rightarrow \text{MU}_\ast(S^0) \rightarrow H_\ast(S^0) \]

Formal groups
Overview of homotopy theory

\[ X(n) = \text{Thom}(\Omega SU(n) \rightarrow \Omega SU = BU) \]

\[ S^0 = X(1) \rightarrow X(2) \rightarrow X(3) \rightarrow X(4) \rightarrow X(\infty) = MU \]

Formal groups

\[ \pi_* (S^*) \rightarrow \pi_* (S^0) \rightarrow MU_* (S^0) \rightarrow H_* (S^0) \]

(Whitehead, Kuhn, Priddy)
Overview of homotopy theory

\[ X(n, k) \text{ from the James filtration on } \Omega(SU(n+1)/SU(n)) = \Omega S^{2n+1} = JS^{2n} \]

\[ X(n) = X(n, 0) \rightarrow X(n, 1) \rightarrow X(n, 2) \rightarrow X(n, 2) \rightarrow X(n, \infty) = X(n+1) \]

\[ S^0 = X(1) \rightarrow X(2) \rightarrow X(3) \rightarrow X(4) \rightarrow X(\infty) = MU \]

\[ \pi_*(S^*) \rightarrow \pi_*(S^0) \rightarrow \pi_*(S^0) \rightarrow MU_*(S^0) \rightarrow H_*(S^0) \]

Formal groups
Overview of homotopy theory

\[
\pi_*(S^*) \xrightarrow{\text{EHPSS}} \pi_*^S(S^0) \xrightarrow{\text{Nilpotence filtration}} \pi_*^S(S^0) \xrightarrow{\text{Adams-Novikov SS}} MU_*(S^0) \xrightarrow{\text{H_*}(S^0)}
\]
Overview of homotopy theory

\[ \pi_*(S^*) \quad \xrightarrow{\text{EHPSS}} \quad \pi_*^S(S^0) \quad \xrightarrow{\text{Nilpotence filtration}} \quad \pi_*^S(S^0) \quad \xrightarrow{\text{Koszul filtration}} \quad MU_*^S(S^0) \quad \xrightarrow{\text{Koszul filtration}} \quad H_*^S(S^0) \]
Overview of homotopy theory

\[ \begin{align*}
S^0 &= SP^1(S^0) 
& \xrightarrow{} SP^2(S^0) 
& \xrightarrow{} SP^3(S^0) 
& \xrightarrow{} SP^4(S^0) 
& \xrightarrow{} SP^{\infty}(S^0) \equiv H
\end{align*} \]

\[ SP^n(S^0) \text{ is a prespectrum with } k' \text{th space } (S^k)^{\times n}/\Sigma_n \]

- **EHPSS**
- **Nilpotence filtration**
- **Koszul filtration**

\[ \begin{align*}
\pi_*(S^*) \xrightarrow{EHPSS} & \pi_*^S(S^0) \\
\pi_*(S^0) \xrightarrow{Nilpotence filtration} & MU_*(S^0) \\
& H_*(S^0)
\end{align*} \]
Overview of homotopy theory

\[ S^0 = SP^1(S^0) \rightarrow SP^2(S^0) \rightarrow SP^3(S^0) \rightarrow SP^\infty(S^0) = H \]

\[ SP^n(S^0) = \text{prespectrum with } k \text{'th space } (S^k)^n / \Sigma_n \]

- **EHPSS**
  \[ \pi_* (S^*) \rightarrow \pi_*^S(S^0) \]

- **Nilpotence filtration**
  \[ \pi_*^S(S^0) \rightarrow MU_* (S^0) \]

- **Koszul filtration**
  \[ MU_* (S^0) \rightarrow H_* (S^0) \]
Overview of homotopy theory

\[ S^0 = \text{SP}^1(S^0) \rightarrow \text{SP}^2(S^0) \rightarrow \text{SP}^3(S^0) \rightarrow \cdots \rightarrow \text{SP}^\infty(S^0) = H \]

EHPSS

\[ \pi_*(S^*) \rightarrow \pi_*^S(S^0) \rightarrow \cdots \rightarrow H_*(S^0) \]

Nilpotence filtration

Koszul filtration

\[ \cdots \rightarrow \pi_*^S(S^0) \rightarrow \text{MU}_*(S^0) \rightarrow H_*(S^0) \]
Overview of homotopy theory

\[ S^0 = S^1(S^0) \to S^p(S^0) \to S^{p^2}(S^0) \to S^{p^3}(S^0) \to \text{SP}\infty(S^0) = H \]

\[ L(0) \to \Sigma L(1) \to \Sigma^2 L(2) \to \Sigma^3 L(3) \]

\( \Omega^\infty L(*) \) is a DGA up to homotopy, chain equivalent to \( \mathbb{Z} \) (Whitehead, Kuhn, Priddy)

**Nilpotence filtration**

**Koszul filtration**

**EHPSS**

\( \pi_*(S^*) \to \pi_*^S(S^0) \to \pi_*^S(S^0) \to \text{MU}_*(S^0) \to \text{H}_*(S^0) \)
Overview of homotopy theory

- **Koszul filtration**
- **Nilpotence filtration**
- **Symmetric power filtration**

\[ \pi_\ast(S^\ast) \rightarrow \pi^\ast(S^0) \rightarrow \pi_\ast(S^0) \rightarrow MU_\ast(S^0) \rightarrow H_\ast(S^0) \]
Overview of homotopy theory

- EHPSS
  - \( \pi_*(S^*) \)
  - \( \pi^S(S^0) \)
  - \( MU_*(S^0) \)

- Nilpotence filtration
  - \( \pi^S(S^0) \)
  - \( MU_*(S^0) \)
  - \( H_*(S^0) \)

- Symmetric power filtration

- Koszul filtration

- \( \overline{MU} \quad \rightarrow \quad S \quad \rightarrow \quad MU \)
Overview of homotopy theory

- EHPSS
- Nilpotence filtration
- Koszul filtration

\[ \pi_*(S^*) \rightarrow \pi_S(S^0) \rightarrow MU_*(S^0) \rightarrow H_*(S^0) \]

\[ \overline{MU}^{(2)} \rightarrow \overline{MU} \rightarrow S \]

- $\Omega SP$ (Goodwillie, Johnson, Arone, Mahowald)
- There is a similar tower for $\Omega$ SP
- It is defined using combinatorics of partitions of $L_\infty$
- Unstable Adams SS, Lambda algebra, central series for simplicial groups

- $S_n$ (0), $S_{n+1}$ (1)
- $\Sigma\pi_*$ (Whitehead, Kuhn, Priddy)
- $\Sigma^3\pi_*$ (2)
- $\Sigma^4\pi_*$ (3)

- $\text{Adams SS}$
- $\text{Lambda algebra}$
- Central series for simplicial groups
Overview of homotopy theory

Symmetric power filtration

Nilpotence filtration

Koszul filtration

$\pi_*(S^*)$ \xrightarrow{EHPSS} $\pi_*(S^0)$ \xrightarrow{\text{Nilpotence filtration}} $\text{MU}_*(S^0)$ \xrightarrow{\text{Koszul filtration}} $H_*(S^0)$

$\overline{\text{MU}}^{(4)} \rightarrow \overline{\text{MU}}^{(3)} \rightarrow \overline{\text{MU}}^{(2)} \rightarrow \overline{\text{MU}} \rightarrow S$

$\text{MU} \wedge \overline{\text{MU}}^{(3)} \rightarrow \text{MU} \wedge \overline{\text{MU}}^{(2)} \rightarrow \text{MU} \wedge \overline{\text{MU}} \rightarrow \text{MU}$
Overview of homotopy theory

$\pi_*(S^*) \xrightarrow{\text{EHPSS}} \pi^S_*(S^0) \xrightarrow{\text{Nilpotence filtration}} \pi^S_*(S^0) \xrightarrow{\text{Adams-Novikov SS}} MU_*(S^0) \xrightarrow{\text{Koszul filtration}} H_*(S^0)$
Overview of homotopy theory

\[ \pi_\ast(S^\ast) \xrightarrow{\text{EHPSS}} \pi_\ast(S^0) \xrightarrow{\text{Adams-Novikov SS}} \mu_\ast(S^0) \xrightarrow{\text{Koszul filtration}} H_\ast(S^0) \]

\[ \text{Symmetric power filtration} \]

\[ \text{Nilpotence filtration} \]

Adams-Novikov SS

Algebraic NSS
Overview of homotopy theory

\[ \pi_* (S^*) \rightarrow \text{EHPSS} \]

\[ \pi_* (S^0) \rightarrow \pi_*^S (S^0) \rightarrow \pi_* (S^0) \]

Symmetric power filtration

Nilpotence filtration

Adams-Novikov SS

Adams SS

Koszul filtration

Algebraic NSS

\[ \text{Adams-Novikov SS} \]

\[ \text{Adams SS} \]

\[ \text{Symmetric power filtration} \]

\[ \text{EHPSS} \]
Overview of homotopy theory

Unstable Adams SS, Lambda algebra, central series for simplicial groups
Overview of homotopy theory

\[ \pi_\ast(S^\ast) \rightarrow \pi^S_\ast(S^0) \rightarrow MU_\ast(S^0) \rightarrow H_\ast(S^0) \]

- Symmetric power filtration
- Nilpotence filtration
- Koszul filtration

\[ X_2 \rightarrow X_1 = \Omega S^1 = \mathbb{Z} \rightarrow QS^0 \]
Overview of homotopy theory

Koszul filtration

Overview of homotopy theory

Nilpotence filtration

Symmetric power filtration

$\pi_*(S^*) \longrightarrow \pi^S_*(S^0) \longrightarrow MU_*(S^0) \longrightarrow H_*(S^0)$

$X_3 \longrightarrow X_2 \longrightarrow X_1 = \Omega S^1 = \mathbb{Z}$

$\Omega^\infty Q(2)_{h\Sigma_2} \longrightarrow QS^0$
Overview of homotopy theory

Symmetric power filtration

Nilpotence filtration

Koszul filtration

\[ \pi_\ast(S^\ast) \rightarrow \pi_\ast(S^0) \rightarrow MU_\ast(S^0) \rightarrow H_\ast(S^0) \]

\[ X_5 \rightarrow X_4 \rightarrow X_3 \rightarrow X_2 \rightarrow X_1 = \Omega S^1 \cong \mathbb{Z} \]

\[ \Omega^\infty Q(4)_{hS_4} \rightarrow \Omega^\infty Q(3)_{hS_3} \rightarrow \Omega^\infty Q(2)_{hS_2} \rightarrow QS^0 \]
Overview of homotopy theory

![Diagram of homotopy theory]

- **Koszul filtration**
  - \( \pi_*(S^*) \) \( \rightarrow \) \( \pi_*^S(S^0) \) \( \rightarrow \) \( \pi_*^S(\Sigma^1) \) \( \rightarrow \) \( \pi_*^S(\Sigma^2) \) \( \rightarrow \) \( \pi_*^S(\Sigma^3) \) \( \rightarrow \) \( \pi_*^S(\Sigma^4) \) \( \rightarrow \) \( \pi_*^S(\Sigma^5) \)
  - **Nilpotence filtration**
  - \( X_5 \) \( \rightarrow \) \( X_4 \) \( \rightarrow \) \( X_3 \) \( \rightarrow \) \( X_2 \) \( \rightarrow \) \( X_1 = \Omega S^1 = \mathbb{Z} \)
    - \( \Omega^\infty Q(4)_h \Sigma_4 \) \( \rightarrow \) \( \Omega^\infty Q(3)_h \Sigma_3 \) \( \rightarrow \) \( \Omega^\infty Q(2)_h \Sigma_2 \) \( \rightarrow \) \( QS^0 \)
- **Koszul filtration**
  - \( MU_*^S(S^0) \) \( \rightarrow \) \( H_*^S(S^0) \)

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\( Q(n) \) is a certain finite \( \Sigma_n \)-spectrum, with \( H_* Q(n) = H_0 Q(n) = \text{Lie}(n) \).
Overview of homotopy theory

\[ \pi_\ast(S^\ast) \rightarrow \pi_\ast^S(S^0) \rightarrow \pi_\ast^S(S^0) \rightarrow MU_\ast(S^0) \rightarrow H_\ast(S^0) \]

Symmetric power filtration

Nilpotence filtration

Koszul filtration

\[ X_5 \rightarrow X_4 \rightarrow X_3 \rightarrow X_2 \rightarrow X_1 = \Omega S^1 = \mathbb{Z} \]

\[ \Omega^\infty Q(4)_{h\Sigma_4} \rightarrow \Omega^\infty Q(3)_{h\Sigma_3} \rightarrow \Omega^\infty Q(2)_{h\Sigma_2} \rightarrow QS^0 \]

\[ Q(n) \] is a certain finite \( \Sigma_n \)-spectrum, with \( H_\ast Q(n) = H_0 Q(n) = \text{Lie}(n) \).

It is defined using combinatorics of partitions of \( n \) points, and is related to \( \text{SP}^n(S^0)/\text{SP}^{n-1}(S^0) \).
Overview of homotopy theory

Nilpotence filtration

\[ \pi_\ast(S^\ast) \longrightarrow \pi_\ast^S(S^0) \longrightarrow \pi_\ast^S(MU^\ast(S^0)) \longrightarrow \pi_\ast^S(H^\ast(S^0)) \]

Koszul filtration

\[ \Omega^\infty Q(4)_{h\Sigma_4} \longrightarrow \Omega^\infty Q(3)_{h\Sigma_3} \longrightarrow \Omega^\infty Q(2)_{h\Sigma_2} \longrightarrow QS^0 \]

\[ X_5 \longrightarrow X_4 \longrightarrow X_3 \longrightarrow X_2 \longrightarrow X_1 = \Omega S^1 = \mathbb{Z} \]

Symmetric power filtration

Q(n) is a certain finite \(\Sigma_n\)-spectrum, with \(H_\ast Q(n) = H_0 Q(n) = \text{Lie}(n)\).

It is defined using combinatorics of partitions of \(n\) points, and is related to \(SP^n(S^0)/SP^{n-1}(S^0)\).

There is a similar tower for \(\Omega S^{k+1}\), with fibres \(\Omega^\infty (S^{nk} \wedge Q(n))_{h\Sigma_n}\).
There is a similar tower for $\Omega S^n$.

It is defined using combinatorics of partitions of $n$ points, and is related to $\text{SP}^n(S^0)/\text{SP}^{n-1}(S^0)$.

There is a similar tower for $\Omega S^{k+1}$, with fibres $\Omega^\infty(S^{nk} \wedge Q(n))_{h\Sigma_n}$.

(Goodwillie, Johnson, Arone, Mahowald)
Overview of homotopy theory

Unstable Adams SS, Lambda algebra, central series for simplicial groups
Symmetric power filtration

Goodwillie tower

Adams-Novikov SS

Koszul filtration

EHPSS

Nilpotence filtration

MU*(S0)

KU*(S0)

Formal groups

MU* (S0)

H* (S0)
Overview of homotopy theory

EHPSS

$\pi_*(S^*)$

Goodwillie tower

$\pi_*^S(S^0)$

Adams-Novikov SS

Nilpotence filtration

$\pi_*^S(S^0)$

$MU_*(S^0)$

Koszul filtration

$H_*(S^0)$

Symmetric power filtration

$KU_*(S^0)$

$Ell_*(S^0)$
Overview of homotopy theory

- EHPSS
  - $\pi_*(S^*)$
  - Goodwillie tower

- Nilpotence filtration
  - $\pi^S_*(S^0)$
  - Adams-Novikov SS

- Koszul filtration
  - $MU_*(S^0)$
  - $H_*(S^0)$

- Symmetric power filtration
  - $KU_*(S^0)$
  - $K(n)_*(S^0)$
  - $Ell_*(S^0)$
Overview of homotopy theory

- **EHPSS**
- **Goodwillie tower**
- **Adams-Novikov SS**
- **Koszul filtration**
- **Nilpotence filtration**

**Symmetric power filtration**

- $\pi_*(S^*)$
- $\pi_*^S(S^0)$
- $\pi_*(L_K(n)S^0)$
- $K(n)_*(S^0)$
- $Ell_*^S(S^0)$

- $\pi_*(S^0)$
- $MU_*^S(S^0)$
- $K(n)_*(S^0)$
- $KU_*^S(S^0)$
- $H_*^a(S^0)$
Overview of homotopy theory

- **EHPSS**: \( \pi_*(S^*) \) → \( \pi^S_*(S^0) \) → \( MU_*(S^0) \) → \( H_*(S^0) \)
- **Goodwillie tower**: \( v_n^{-1} \pi_*(S^*) \) → \( \pi_*(L_K(n)S^0) \) → \( K(n)_*(S^0) \)
- **Adams-Novikov SS**: \( \pi_*(S^0) \) → \( MU_*(S^0) \) → \( K(n)_*(S^0) \)
- **Nilpotence filtration**: \( \pi_*(S^0) \) → \( MU_*(S^0) \) → \( K(n)_*(S^0) \)
- **Koszul filtration**: \( MU_*(S^0) \) → \( K(n)_*(S^0) \) → \( Ell_*(S^0) \)

Symmetric power filtration
Overview of homotopy theory

\[ \pi_* (S^*) \xrightarrow{\nu_n^{-1}} \pi_* (S^*) \quad \xrightarrow{\text{Goodwillie tower}} \quad \pi_* (S^0) \xrightarrow{\text{EHPSS}} \pi_* (S^0) \]

\[ \pi_* (S^*) \xrightarrow{\text{Bousfield-Kuhn}} \]

\[ \phi_n : \text{Spaces} \rightarrow \text{Spectra} \quad \phi_n (\Omega^\infty X) = L_{K(n)}(X) \]
Symmetric powers of unstable spheres

\[ SP^n(S^V) = (S^V \times \ldots \times S^V) / \Sigma n = (S^V)^n / \Sigma n = S^R_n \otimes V / \Sigma n = (\text{diagonal copy of } R^n) \oplus W^n \]

\[ SP^n(S^0) = \lim_{\to V} \Sigma - V SP^n(S^V) \]

\[ SP^n(S^1) = S^1 SP^n(S^2) = P^n \]

There are natural product maps

\[ SP^n(S^V) \times SP^m(S^W) \to SP^{nm}(S^V \oplus W) \]

and

\[ SP^n(S^V) \wedge SP^m(S^W) \to SP^{nm}(S^V \oplus W) \].
Symmetric powers of unstable spheres

$SP^n(S^V) = (S^V \times \ldots \times S^V)/\Sigma_n = (S^V)^n/\Sigma_n$
Symmetric powers of unstable spheres

$$\text{SP}^n(S^V) = (S^V \times \ldots \times S^V)/\Sigma_n = (S^V)^n/\Sigma_n$$

$$\overline{\text{SP}}^n(S^V) = \text{SP}^n(S^V)/\text{SP}^{n-1}(S^V) = (S^V)^{(n)}/\Sigma_n = S^{nv}/\Sigma_n = S^{\mathbb{R}^n}\otimes V/\Sigma_n$$
Symmetric powers of unstable spheres

\[ \text{SP}^n(S^V) = (S^V \times \ldots \times S^V)/\Sigma_n = (S^V)^n/\Sigma_n \]
\[ \overline{\text{SP}}^n(S^V) = \text{SP}^n(S^V)/\text{SP}^{n-1}(S^V) = (S^V)^{(n)}/\Sigma_n = S^{nV}/\Sigma_n = S^{n \otimes V}/\Sigma_n \]
\[ \mathbb{R}^n = (\text{diagonal copy of } \mathbb{R}) \oplus W_n \]
Symmetric powers of unstable spheres

\[ SP^n(S^V) = (S^V \times \ldots \times S^V) / \Sigma_n = (S^V)^n / \Sigma_n \]

\[ \overline{SP}^n(S^V) = SP^n(S^V) / SP^{n-1}(S^V) = (S^V)^{(n)} / \Sigma_n = S^{nV} / \Sigma_n = S^{\mathbb{R}^n \otimes V} / \Sigma_n \]

\[ \mathbb{R}^n = ( \text{diagonal copy of } \mathbb{R}) \oplus W_n \]

\[ \overline{SP}^n(S^V) = S^{V \oplus (W_n \otimes V)} / \Sigma_n = \Sigma^V (S^{W_n \otimes V} / \Sigma_n) \]
Symmetric powers of unstable spheres

\[ SP^n(S^V) = (S^V \times \ldots \times S^V)/\Sigma_n = (S^V)^n/\Sigma_n \]

\[ \overline{SP^n}(S^V) = SP^n(S^V)/SP^{n-1}(S^V) = (S^V)^{(n)}/\Sigma_n = S^{nV}/\Sigma_n = S^{\mathbb{R}^n \otimes V}/\Sigma_n \]

\[ \mathbb{R}^n = (\text{diagonal copy of } \mathbb{R}) \oplus W_n \]

\[ \overline{SP^n}(S^V) = S^V \oplus (W_n \otimes V)/\Sigma_n = \Sigma^V (S^{W_n \otimes V}/\Sigma_n) \]

\[ SP^n(S^0) = \lim_{\longrightarrow_V} \Sigma^{-V} SP^n(S^V) \]
\[ \text{Symmetric powers of unstable spheres} \]

\[ \text{SP}^n(S^V) = (S^V \times \ldots \times S^V)/\Sigma_n = (S^V)^\times n/\Sigma_n \]

\[ \text{SP}^n(S^V) = S^V/\Sigma_n = S^V/\Sigma_n = S^{\mathbb{R}^n \otimes V}/\Sigma_n \]

\[ \mathbb{R}^n = (\text{diagonal copy of } \mathbb{R}) \oplus W_n \]

\[ \text{SP}^n(S^V) = S^V \oplus (W_n \otimes V)/\Sigma_n = S^V(W_n \otimes V)/\Sigma_n \]

\[ \text{SP}^n(S^0) = \lim_{\rightarrow V} \Sigma^{-V} \text{SP}^n(S^V) \]

\[ \text{SP}^n(S^0) = \lim_{\rightarrow V} S^W_n \otimes V/\Sigma_n = S^\infty W_n/\Sigma_n = \tilde{\Sigma}(S(\infty W_n)/\Sigma_n). \]
Symmetric powers of unstable spheres

\[ \text{SP}^n(S^V) = (S^V \times \ldots \times S^V)/\Sigma_n = (S^V)^n/\Sigma_n \]
\[ \overline{\text{SP}}^n(S^V) = \text{SP}^n(S^V)/\text{SP}^{n-1}(S^V) = (S^V)^{(n)}/\Sigma_n = S^{nV}/\Sigma_n = S^{\mathbb{R}^n \otimes V}/\Sigma_n \]
\[ \mathbb{R}^n = (\text{diagonal copy of } \mathbb{R}) \oplus W_n \]
\[ \overline{\text{SP}}^n(S^V) = S^{V \oplus (W_n \otimes V)}/\Sigma_n = \Sigma^V (S^{W_n \otimes V}/\Sigma_n) \]
\[ \text{SP}^n(S^0) = \lim_{\longrightarrow V} \Sigma^{-V} \text{SP}^n(S^V) \]
\[ \overline{\text{SP}}^n(S^0) = \lim_{\longrightarrow V} S^{W_n \otimes V}/\Sigma_n = S^{\infty W_n}/\Sigma_n = \widetilde{\Sigma}(S(\infty W_n)/\Sigma_n). \]

\[ \text{SP}^n(S^1) = S^1 \quad \text{SP}^n(S^2) = P^n \]
\[ \overline{\text{SP}}^n(S^1) = 0 \quad \overline{\text{SP}}^n(S^2) = S^{2n} \]
Symmetric powers of unstable spheres

\[ SP^n(S^V) = (S^V \times \ldots \times S^V)/\Sigma_n = (S^V)^n/\Sigma_n \]

\[ \overline{SP}^n(S^V) = SP^n(S^V)/SP^{n-1}(S^V) = (S^V)^n/\Sigma_n = S^{nV}/\Sigma_n = S^{\mathbb{R}^n \otimes V}/\Sigma_n \]

\[ \mathbb{R}^n = (\text{diagonal copy of } \mathbb{R}) \oplus W_n \]

\[ \overline{SP}^n(S^V) = S^{V \oplus (W_n \otimes V)}/\Sigma_n = \Sigma^V (S^{W_n \otimes V}/\Sigma_n) \]

\[ SP^n(S^0) = \lim_{\rightarrow V} \Sigma^{-V} SP^n(S^V) \]

\[ \overline{SP}^n(S^0) = \lim_{\rightarrow V} S^{W_n \otimes V}/\Sigma_n = S^{\infty W_n}/\Sigma_n = \tilde{\Sigma}(S^{(\infty W_n)}/\Sigma_n). \]

\[ SP^n(S^1) = S^1 \quad SP^n(S^2) = \mathcal{P}^n \]

\[ \overline{SP}^n(S^1) = 0 \quad \overline{SP}^n(S^2) = S^{2n} \]

There are natural product maps \( SP^n(S^V) \times SP^m(S^W) \rightarrow SP^{nm}(S^{V \oplus W}) \) and \( \overline{SP}^n(S^V) \wedge \overline{SP}^m(S^W) \rightarrow \overline{SP}^{nm}(S^{V \oplus W}). \)
Nontransitive subgroups

Let $F$ be a family of subgroups of a finite group $G$, closed under subconjugacy. Then there is a $G$-space $E_F$ with $E_F H = \{\text{contractible if } H \in F, \emptyset \text{ if } H \not\in F\}$. We put $B_F = E_F / G$.

Take $P_n = \{\text{nontransitive subgroups of } \Sigma_n\}$; then $E_{P_n} = S(W_n)$ and so $S \Sigma^{SP_n}(S_0) = \tilde{\Sigma} B_{P_n}$.
Nontransitive subgroups

Let $\mathcal{F}$ be a family of subgroups of a finite group $G$, closed under subconjugacy. Then there is a $G$-space $E\mathcal{F}$ with

$$E\mathcal{F}^H = \begin{cases} \text{contractible} & \text{if } H \in \mathcal{F} \\ \emptyset & \text{if } H \notin \mathcal{F}. \end{cases}$$
Let $\mathcal{F}$ be a family of subgroups of a finite group $G$, closed under subconjugacy. Then there is a $G$-space $E\mathcal{F}$ with

$$E\mathcal{F}^H = \begin{cases} \text{contractible} & \text{if } H \in \mathcal{F} \\ \emptyset & \text{if } H \notin \mathcal{F}. \end{cases}$$

We put $B\mathcal{F} = E\mathcal{F}/G$. 
Let $\mathcal{F}$ be a family of subgroups of a finite group $G$, closed under subconjugacy. Then there is a $G$-space $E\mathcal{F}$ with

$$E\mathcal{F}^H = \begin{cases} 
\text{contractible} & \text{if } H \in \mathcal{F} \\
\emptyset & \text{if } H \notin \mathcal{F}.
\end{cases}$$

We put $B\mathcal{F} = E\mathcal{F}/G$.

Take $\mathcal{P}_n = \{\text{nontransitive subgroups of } \Sigma_n\}$; then $E\mathcal{P}_n = S(\infty W_n)$ and so $\overline{SP}^n(S^0) = \tilde{\Sigma}B\mathcal{P}_n$. 

A multiset is a finite set with multiplicities. Morphisms are functions, bijective up to multiplicity. 

\[ M = \{ \text{multisets} \} \] is symmetric bimonoidal under \( \oplus \) and \( \times \), so \( K(M) \) is a ring spectrum. In fact \( K(M) = H \).

- \( M_n \): maximum multiplicity \( \leq n \);
- \( M_k \): total multiplicity \( k \);

\[ M_k^n = M_n \cap M_k \]

**Theorem (Lesh):**

\[ K(M_n) = \text{SP}_n(S^0) \text{ and } B_{M_n}^{n-1} = B_P^n \text{ and } K(M_n) / K(M_{n-1}) = \tilde{\Sigma} B_{M_n}^{n-1}. \]

\[ \text{Free}(M_{n-1}) \to M_{n-1} \to \Sigma^\infty + B_{M_n}^{n-1} \to K(M_{n-1}) \to \text{Free}(M_n) \to M_n \to S^0 \to K(M_n) \]
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$K$-theory of multisets

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Theorem (Lesh): $K(\mathcal{M}_n) = \text{SP}^n(S^0)$ and $B\mathcal{M}^n_{n-1} = B\mathcal{P}_n$ and $K(\mathcal{M}_n)/K(\mathcal{M}_{n-1}) = \tilde{\text{SP}}^n(S^0) = \tilde{\Sigma}B\mathcal{M}^n_{n-1}$.

\[
\begin{align*}
\text{Free}(\mathcal{M}^n_{n-1}) &\longrightarrow \mathcal{M}_{n-1} & \Sigma^\infty B\mathcal{M}^n_{n-1} &\longrightarrow K(\mathcal{M}_{n-1}) \\
\downarrow & & \downarrow & \downarrow \\
\text{Free}(\mathcal{M}^n_n) &\longrightarrow \mathcal{M}_n & S^0 &\longrightarrow K(\mathcal{M}_n)
\end{align*}
\]
Mod $p$ (co)homology
The filtration of $H = H\mathbb{Z}$ by the spectra $H(k) = \text{SP}^p(S^0)$ gives rise to a filtration of $\overline{H} = H\mathbb{Z}/p$ by spectra $\overline{H}(k)$.
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Theorem (Nakaoka): $\overline{H}^*\overline{H} = \mathcal{A}^* = \text{Steenrod algebra}$; $\overline{H}^*\overline{H}(k) = \mathcal{A}^*/(\text{admissibles of length } > k)$. 
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Operations of length $k$ are related to $\overline{H}^*(B\Sigma_p k)$ and to $\overline{H}^*(B(\mathbb{Z}/p)^k)^{GL_k(\mathbb{Z}/p)}$ by the extended power construction.
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Operations of length $k$ are related to $\overline{H}^*(B\Sigma_p^k)$ and to $\overline{H}^*(B(\mathbb{Z}/p)^k)^{GL_k(\mathbb{Z}/p)}$ by the extended power construction.

There are still some open questions about how all this fits together, and how it dualises.
Partitions

\[ \pi, \pi', \perp, \top, P_A = \{ \text{partitions of } A \}; \]

\[ \partial P_A = \text{union of simplices not containing } \{ \perp, \top \}; \]

\[ \hat{P}_A = P_A / \partial P_A; \]

\[ A = P_A \cong S^2 \cup \text{(equatorial disc)} \cong S^2 \lor S^2. \]
Partitions

\[ \pi \quad \quad \quad \quad \quad \quad \quad \] 

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\[ \partial \pi = \text{union of simplices not containing } \{ \bot, \top \} \]

\[ \hat{\pi} = \pi / \partial \pi \]

\[ A = \hat{\pi}(A) \cong S^2 \cup (\text{equatorial disc}) \cong S^2 \vee S^2. \]
Partitions

\[ A_{\pi} \prec A_{\pi'} \]

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Partitions

\[ \pi < \perp \]
Partitions

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\[ \partial P \mathcal{A} = \text{union of simplices not containing } \{ \bot, \top \}; \quad \hat{P} \mathcal{A} = P \mathcal{A}/\partial P \mathcal{A} \]
\( \mathcal{P}A = \{ \text{partitions of } A \}; \quad PA = \text{geometric realisation of } \mathcal{P}A = |\mathcal{P}A|.
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\[ A = \triangle \quad \mathcal{P}A = \begin{array}{c}
\triangle \\
\triangle \\
\triangle
\end{array} \quad PA = \begin{array}{c}
\text{tetrahedron}
\end{array} \]
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\hat{P}(A) \cong S^2 \cup (\text{equatorial disc}) \cong S^2 \lor S^2.
Products of partitions
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\( \mathcal{P}(A) \) is a lattice with \( \pi \lor \pi' = \{ B \cap B' \mid B \in \pi, \ B' \in \pi', \ B \cap B' \neq \emptyset \} \).
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Put \( \overline{P}(A) = P(A)/(\text{simplices not containing } \bot) \). There is an induced map \( \mu : \overline{P}(A) \land \overline{P}(A) \to \overline{P}(A) \), making \( \Sigma^\infty \overline{P}(A) \) a (contractible) ring spectrum.
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There is a filtration of $\overline{\mathcal{P}}(A)$ by ranks of partitions, with associated graded $\lor \hat{\mathcal{P}}(\pi)$. The homology of this is thus a DGA, probably chain contractible.
Products of partitions

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There is a filtration of \( \overline{\mathcal{P}}(A) \) by ranks of partitions, with associated graded \( \bigvee_{\pi} \hat{\mathcal{P}}(\pi) \). The homology of this is thus a DGA, probably chain contractible.

We have not yet understood the structure of this.
Partitions

\[ \text{Partitions} \]

\[
\text{C}(A) = \{ \text{nonempty subsets of } A \}
\]

\[ |C(A)| = \{ x : A \to [0, 1] | \max(x) = 1 \} \cong B(WA) \]

\[ s_{C}(A) = \{ \text{chains in } C(A) \}; |s_{C}(A)| = |C(A)| \text{ by barycentric subdivision.} \]

\[
\text{We can define } \phi : s_{C}(A) \to P(A) \text{ by }
\]

\[ \phi(B_{0} \subset \cdots \subset B_{r}) = \{ B_{0}, B_{1} \setminus B_{0}, \ldots, B_{r} \setminus B_{r-1}, A \setminus B_{r} \} \]

\[ \text{This gives } B(WA) \to P(A) \text{ and } S_{WA} = B(WA)/\partial B(WA) \to \hat{P}(A). \]

\[ \text{More generally, we can use the monoid structure on } P(A) \text{ to get } B(WA) \to P(A) \text{ and } S_{NA} \to \hat{P}(A). \]
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\( C(A) = \{ \text{nonempty subsets of } A \} \)
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More generally, we can use the monoid structure on \( PA \) to get
\( B(WA)^N \to P(A) \) and \( S^{NWA} \to \hat{P}(A) \).
Height functions

A height function on $A$ is a map $h : \mathcal{A} = \{\text{nonempty subsets of } A\} \to [0, 1]$ with $h(\{a\}) = 0$, and $h(U \cup V) = \max(h(U), h(V))$ whenever $U \cap V \neq \emptyset$.

A partition $\pi$ gives a height function $h_\pi$ with $h_\pi(U) = 0$ if $U$ is contained in a block of $\pi$, and 1 otherwise.

The space $H(A)$ of height functions is homeomorphic to $P(A) = |P(A)|$.

Say that a set $U$ is $h$-critical if every strict superset $V$ has $h(V) > h(U)$. These sets form a tree. This gives a cell structure on $H(A) = P(A)$ indexed by trees.

By grafting trees, we make the spaces $P(n) = P(\{1, \ldots, n\})$ into an operad. The operad structure maps are nearly embeddings. By a Pontrjagin-Thom construction, we make the spaces $\hat{P}(n)$ into a based cooperad (a theorem of Ching).
A height function on $A$ is a map $h: CA = \{ \text{nonempty subsets of } A \} \to [0, 1]$ with $h(\{a\}) = 0$, and $h(U \cup V) = \max(h(U), h(V))$ whenever $U \cap V \neq \emptyset$. 
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Put \( \text{Inj}_0(\ast, \mathbb{R}^n) = \{ (x_1, \ldots, x_k) \in (\mathbb{R}^n)^k \mid \sum x_i = 0, x_i \neq x_j \} \subseteq W_k \otimes \mathbb{R}^n \subset S W_k \otimes \mathbb{R}^n \).

These spaces form an operad up to homotopy, as they are homotopy equivalent to the Fulton-MacPherson spaces (cf Singh).

It is well-known that \( H^\ast \text{Inj}_0(\ast, \mathbb{R}^n) \) is the operad for Poisson algebras, which are graded commutative rings with a compatible Lie bracket.

The based spaces \( S W_k \) form a (co)operad whose structure maps are homeomorphisms.

The spectra \( \Sigma^{-n} W_k \text{Inj}_0(k, \mathbb{R}^n) \) form an operad with \( H^0 = \text{Lie} \), and \( H^k = 0 \) for \( k > 0 \).

There is a natural map \( S W_k / \text{Inj}_0(k, \mathbb{R}^n) \to \hat{P}(k) \), and by duality we get a map \( Q(k) = F(\hat{P}(k), S W_k) \to \Sigma^{-n} W_k \text{Inj}_0(k, \mathbb{R}^n) \). This gives \( H^\ast Q = \text{Lie} \).

Theorem (Arone-Dwyer): \( SP_n(S_0) = (S W_n \wedge \hat{P}(n)) h \Sigma n \).

Theorem (Johnson, Arone-Mahowald): \( Q(n) \) controls the layers in the Goodwillie tower.
Put
\[ \text{Inj}_0(k, \mathbb{R}^n) = \{(x_1, \ldots, x_k) \in (\mathbb{R}^n)^k \mid \sum x_i = 0, \ x_i \neq x_j \} \subseteq W_k \otimes \mathbb{R}^n \subset S^{W_k \otimes \mathbb{R}^n}. \]
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Configuration space

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Theorem (Johnson, Arone-Mahowald): \( Q(n) \) controls the layers in the Goodwillie tower.
Given \( A \cong \left( \mathbb{Z} / p \mathbb{Z} \right)^d \), put \( T(A) = \{ \text{subgroups of } A \} \) and \( \hat{T}(A) = |T(A)| \).

It is well-known that this is homotopy equivalent to a wedge of spheres of dimension \( d \), the number of spheres being \( p \left( \frac{d^2 - d}{2} \right) \).

\( \text{St}(A) = H^d \left( \hat{T}(A); \mathbb{Z} / p \mathbb{Z} \right) \) is a projective cyclic module over \( \left( \mathbb{Z} / p \mathbb{Z} \right)[\text{Aut}(A)] \), called the Steinberg module.

It follows that for any \( \text{Aut}(A) \)-spectrum \( X \), the spectrum \( \Sigma^{-d} \hat{T}(A) \wedge X \) is a retract of \( X \), called the Steinberg summand of \( X \).

We can map \( \hat{T}(A) \) to \( \hat{P}(A) \) by sending \( B \in T(A) \) to the corresponding coset partition of \( A \). This respects the actions of \( \text{Aff}(A) \) on \( \hat{T}(A) \) and \( \Sigma A \) on \( \hat{P}(A) \).

Theorem (Arone-Dwyer): \( (\Sigma \text{WA} \hat{T}(A))_{h\text{Aff}(A)} = (\Sigma \text{WA} \hat{P}(A))_{h\Sigma A} = \text{SP}_p d (S^0) = \Sigma d L(d) \), and so \( L(d) \) is the Steinberg summand in \( (\Sigma \text{WA})_{hA} \), which is a Thom spectrum over \( BA \).
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and so $L(d)$ is the Steinberg summand in $(S^{WA})_{hA}$, which is a Thom spectrum over $BA$. 

Mitchell’s complexes

\[ X(A) = (\sum_{Bases(C[A])} - d + \wedge \hat{T}(A)) h \text{Aff}(A). \]

This is the Steinberg summand in \( Bases(C[A])/A \).

Theorem (Mitchell): this has type \( n \), so \( K(m) \ast X(A) \) is nonzero iff \( m \geq n \).

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The Steinberg algebra

Put $T(A) = T(A) / \langle{\text{simplices not containing} 0}\rangle$. This has a natural product and compatible filtration, making the associated graded homology into a graded-commutative DGA.

Put $\text{St}^*(A) = \bigoplus_{B \leq A} \text{St}(B)$; this is easily identified with the above DGA.

One can show that $\text{St}^*(A)$ has a generator $x_L \in \text{St}^1(A)$ for each $L \leq A$ of order $p$, subject to relations $x_L x_M + x_M x_N + x_N x_L = 0$ whenever $|L + M + N| < p^3$. The differential is given by $d(x_L) = -1$ for all $L$. 
Put $\overline{T}(A) = T(A)/(\text{simplices not containing } 0)$. This has a natural product and compatible filtration, making the associated graded homology into a graded-commutative DGA.
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Theorem (Kuhn): $K^*(n) \ast L^*(\ast)$ is a finite, contractible DGA over $K^*(n) \ast$.

Let $E$ be Morava $E$-theory (with formal group $G$) and put $E \vee^0 L^*(d) = \pi^0 L^K(n)(E \wedge L^*(d))$. It works out that $E \vee^0 L^*(\ast)$ is a contractible DGA over $E^0$.

Hopkins-Kuhn-Ravenel introduce the group $\Theta = (\mathbb{Z}/p^\infty)$, and a Galois extension $E'_0$ of $\mathbb{Q} \otimes E^0$, with Galois group $\text{Aut}(\Theta)$. For finite groups $H$, they give a natural isomorphism $E'_0 \otimes E^0 E^0 BH = \text{Map}(\text{Hom}(\Theta^* \otimes H, H) /H, E'_0)$ ("generalised character theory").

Put $\Theta[p] = \{\theta \in \Theta | p^\infty \theta = 0\}$.

Theorem: $E'_0 \otimes E^0 E \vee^0 L^*(\ast) = E'_0 \otimes \mathbb{Z}_{St^\ast} (\Theta[p])$.

It is also possible to define $G[p]$ and $\mathbb{Z}_{St^\ast} (G[p])$, and to show that $E \vee^0 L^*(\ast) = \mathbb{Z}_{St^\ast} (G[p])$.

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