Rational global representation theory

Neil Strickland
(with Luca Pol; thanks to the Hausdorff Institute)

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- Let $\mathcal{G}$ be the category of finite groups and conjugacy classes of surjective homomorphisms.
- Let $\mathcal{U}$ be a full subcategory of $\mathcal{G}$, such as finite abelian p-groups.
- Put $\mathcal{AU} = [\mathcal{U}^{\mathrm{op}}, \mathrm{Vect}]$; then $D(\mathcal{AU})$ is equivalent to the category of globally equivariant spectra with rational homotopy groups, via $X \mapsto (G \mapsto \pi_*(\phi^G(X)))$.
- For each $G \in \mathcal{U}$ we have a projective generator $e_G(K) = \mathbb{Q}[\mathcal{U}(K, G)]$ (which is 0 if $|K| < |G|$).
- $\mathcal{AU}$ is closed symmetric monoidal with $(X \otimes Y)(G) = X(G) \otimes Y(G)$ and $\mathbb{1} = e_1 = (G \mapsto \mathbb{Q})$ and $\underline{\mathrm{Hom}}(X, Y)(G) = \mathcal{AU}(e_G \otimes X, Y)$.
- Goal: study $\mathcal{AU}$, derived category $D(\mathcal{AU})$, Balmer spectrum of compact objects in $D(\mathcal{AU})$.
- Problem: projective generators are not strongly dualisable in $\mathcal{AU}$; compact objects are not strongly dualisable in $D(\mathcal{AU})$.
  This blocks most known approaches to the Balmer spectrum.
- Although duality phenomena are unfamiliar, they are still concrete and tractable.
- Another unusual feature: all projectives are injective, but not conversely.
- For abelian p-groups, or elementary abelian p-groups, or cyclic groups: the category $\mathcal{AU}$ is locally noetherian, i.e. subobjects of finitely generated objects are finitely generated.

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Tensor structure

- Assume for simplicity that groups in $\mathcal{U}$ are abelian.
- Say $W \leq G \times H$ is *wide* if projections to G and H are both surjective iff there exists $N \leq G$, $M \leq H$, $\alpha\colon G/N \xrightarrow{\simeq} H/M$ with $W = \{(g,h) \mid \alpha(gN) = hM\}$.
- Example: $G \times H$ is wide in $G \times H$, diagonal $\Delta \leq G \times G$ is also wide.
- There is an easy isomorphism $e_G \otimes e_H = \bigoplus_W e_W$.
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- Thus e_G is not strongly dualisable unless $G = 1$.
  In fact X is only strongly dualisable if it is constant and finite-dimensional.
- If $|C| = p$ then $\underline{\mathrm{Hom}}(e_C, e_C) = e_{C^2} \oplus (2p-1)e_C \oplus (p-1)\mathbb{1}$ but $D(e_C) \otimes e_C = e_{C^2} \oplus pe_C$.
- For the nonabelian case, we need to pass to conjugacy classes in the right way, but the details are fiddly.

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Tensor structure

- Assume for simplicity that groups in $\mathcal{U}$ are abelian.
- Say $W \leq G \times H$ is *wide* if projections to G and H are both surjective iff there exists $N \leq G$, $M \leq H$, $\alpha\colon G/N \xrightarrow{\simeq} H/M$ with $W = \{(g,h) \mid \alpha(gN) = hM\}$.
- Example: $G \times H$ is wide in $G \times H$, diagonal $\Delta \leq G \times G$ is also wide.
- There is an easy isomorphism $e_G \otimes e_H = \bigoplus_W e_W$.
- Put $DX = \underline{\mathrm{Hom}}(X, \mathbb{1})$ so X is strongly dualisable iff $DX \otimes X \to \underline{\mathrm{Hom}}(X,X)$ is iso.
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More about duals and internal homs

- For $N \leq G$ we define

$$\theta_N \colon (e_{G/N} \otimes e_G)(K) = \mathbb{Q}[\mathcal{U}(K, G/N) \times \mathcal{U}(K, G)] \to \mathbb{1}(K) = \mathbb{Q}$$

  by $\theta_N(K \xrightarrow{\alpha} G/N, K \xrightarrow{\beta} G) = 1$ if $\alpha = \pi \circ \beta$, and $\theta_N(\alpha, \beta) = 0$ otherwise. Adjoint to $\theta_N \colon e_{G/N} \otimes e_G \to \mathbb{1}$ we have $\theta_N^\# \colon e_{G/N} \to \underline{\mathrm{Hom}}(e_G, \mathbb{1}) = D(e_G)$.
- Fact: these maps give $\bigoplus_N e_{G/N} \xrightarrow{\simeq} D(e_G)$.
- We will show later that $\mathbb{1}$ is injective.
  Also any X is flat, and it follows that DX is injective.
  As e_G is a retract of $D(e_G)$, it is also injective.
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- However, $t_G(K) = \mathrm{Map}(\mathcal{U}(G, K), \mathbb{Q})$ is injective but not projective.
- A *virtual homomorphism* from G to H is a pair (A, A') where
  $A' \leq A \leq G \times H$ and A is wide and $A' \cap (1 \times H) = 1$ and $A/A' \in \mathcal{U}$.
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Asymptotic freedom

- Let F_{nm} be the quotient of the free group on n generators by the intersection of all normal subgroups N with quotient in $\mathcal{U}_{\leq m}$. Under mild conditions on $\mathcal{U}$ we have $F_{nm} \in \mathcal{U}$.
- Given morphisms $F_{nm} \xrightarrow{\phi} H \xleftarrow{\alpha} G$ in $\mathcal{U}$ with $|G| \leq \min(n, m)$, we can choose $\psi \colon F_{nm} \to G$ in $\mathcal{U}$ with $\alpha\psi = \phi$. (Some care is needed to ensure that ψ is surjective.)
- We can choose a tower $G_0 \leftarrow G_1 \leftarrow G_2 \leftarrow \cdots$ in $\mathcal{U}$ such that G_n gets rapidly larger and freer as $n \to \infty$.
- We then find that

$$\varinjlim_{G \in \mathcal{U}^{\mathrm{op}}} X(G) = \varinjlim_{n} X(G_n)_{\mathrm{Out}(G_n)},$$

  and this is an exact functor of X (because we work over $\mathbb{Q}$).
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Rates of growth

- If n is large, the proportion of n-tuples in G^n that generate G is close to 1 (theorem of Lynne Butler, 1994).
- Using this plus nearly free groups as on the previous slide: if X is a nontrivial summand of e_G, then an appropriate lim sup of $\dim(X(T))/|G|^{\delta(T)}$ is nonzero and finite, where $\delta(T)$ is the minimal size of a generating set.
- We can define Serre subcategories and then quotient categories using rates of growth. We have not yet exploited this fully.
- This approach show that monomorphisms between projective objects split, even for some $\mathcal{U}$ where projectives are not injective.

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The order filtration

- For a $\mathbb{Q}[\mathrm{Out}(G)]$-module V, put

$$e_{G,V}(K) = V \otimes_{\mathbb{Q}[\mathrm{Out}(G)]} e_G(K)$$

  This is projective. Every indecomposable projective has the form $e_{G,S}$ for some indecomposable $\mathbb{Q}[\mathrm{Out}(G)]$-module S. We define the order of $e_{G,S}$ to be the order of G.
- We say that X is pure of order k if it is isomorphic to a sum of indecomposable projectives of order k.
- The subcategory of such objects is equivalent to the semisimple category $\mathcal{AU}_k = [\mathcal{U}_k^{\mathrm{op}}, \mathrm{Vect}]$.
- If X is pure of order k, and Y is pure of order $m > k$, then $\mathcal{AU}(X, Y) = 0$.
- Let $(L_{\leq m}X)(G)$ be the sum of all $\alpha^*(X(H)) \leq X(G)$ for $H \in \mathcal{U}_{\leq m}$ and $\alpha \in \mathcal{U}(G, H)$.
- Put $L_m X = L_{\leq m}X / L_{<m}X$.
- If P is projective, then $P \simeq \bigoplus_k P_k \simeq \prod_k P_k$, where P_k is pure of order k. It follows that $L_{\leq m}X = \bigoplus_{k \leq m} P_k$ and $L_m X \simeq P_m$ so the filtration splits.

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This is projective. Every indecomposable projective has the form $e_{G,S}$ for some indecomposable $\mathbb{Q}[\mathrm{Out}(G)]$-module S. We define the order of $e_{G,S}$ to be the order of G.
- We say that X is pure of order k if it is isomorphic to a sum of indecomposable projectives of order k.
- The subcategory of such objects is equivalent to the semisimple category $\mathcal{AU}_k = [\mathcal{U}_k^{\mathrm{op}}, \mathrm{Vect}]$.
- If X is pure of order k, and Y is pure of order $m > k$, then $\mathcal{AU}(X, Y) = 0$.
- Let $(L_{\leq m}X)(G)$ be the sum of all $\alpha^*(X(H)) \leq X(G)$ for $H \in \mathcal{U}_{\leq m}$ and $\alpha \in \mathcal{U}(G, H)$.
- Put $L_m X = L_{\leq m}X / L_{<m}X$.
- If P is projective, then $P \simeq \bigoplus_k P_k \simeq \prod_k P_k$, where P_k is pure of order k. It follows that $L_{\leq m}X = \bigoplus_{k \leq m} P_k$ and $L_m X \simeq P_m$ so the filtration splits.

The order filtration

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Noetherian properties

- We prove that $[\mathcal{U}^{\mathrm{op}}, \mathrm{Vect}]$ is locally noetherian when $\mathcal{U}$ is the category of finite abelian p-groups.
  (We also cover a few other cases that are easier and/or already known.)
- By work of Sam and Snowden (2016): $[\mathcal{U}^{\mathrm{op}}, \mathrm{Vect}]$ is locally noetherian if there is a category $\mathcal{C}$ such that
  - $\mathcal{C}$ is close enough to $\mathcal{U}^{\mathrm{op}}$ to allow for transfer of finiteness conditions.
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- One ingredient: a preordered set P is well-quasi-ordered (wqo) if in every sequence $u\colon \mathbb{N} \to P$, there exists $i < j$ with $u(i) \leq u(j)$.
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- $\mathcal{L}_{\dagger}$ is the category of finite, nonempty, totally ordered sets X equipped with $e_X\colon X \to \mathbb{N}$. Morphisms are $\dagger$-monotone surjections $\phi\colon X \to Y$ with $e_Y \circ \phi \leq e_X$.
- The category $\mathcal{L}_{\dagger}^{\mathrm{op}}$ and its slice categories are (nonobviously) wqo.
- There is a functor $\mathcal{L}_{\dagger} \to \mathcal{U}$ sending X to $\bigoplus_x \mathbb{Z}/p^{e_X(x)}$. Using this we prove that $\mathcal{AU}$ is locally noetherian.

Noetherian properties

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Noetherian properties

- We prove that $[\mathcal{U}^{op}, \mathrm{Vect}]$ is locally noetherian when $\mathcal{U}$ is the category of finite abelian p-groups.
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- The category $\mathcal{L}_\dagger^{op}$ and its slice categories are (nonobviously) wqo.
- There is a functor $\mathcal{L}_\dagger \to \mathcal{U}$ sending X to $\bigoplus_x \mathbb{Z}/p^{e_X(x)}$. Using this we prove that $\mathcal{A}\mathcal{U}$ is locally noetherian.

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- There is a functor $\mathcal{L}_{\dagger} \to \mathcal{U}$ sending X to $\bigoplus_x \mathbb{Z}/p^{e_X(x)}$. Using this we prove that $\mathcal{AU}$ is locally noetherian.

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The derived category

- Let $\mathcal{PU}$ be the subcategory of projectives in $\mathcal{AU}$.
- There is an additive functor $P_0 = I_! I^* \colon \mathcal{AU} \to \mathcal{PU}$ with a surjective natural transformation $P_0(X) \to X$, where I is the inclusion $\mathcal{U}^\times \to \mathcal{U}$.
- This extends to give an additive functor $P \colon \mathrm{Ch}(\mathcal{AU}) \to \mathrm{Ch}(\mathcal{PU})$ with a natural surjective quasiisomorphism $P(X) \to X$.
- From this and other results:

  $$\mathrm{Ch}(\mathcal{AU})[\mathrm{we}^{-1}] = \mathrm{hCh}(\mathcal{PU}) := \mathrm{Ch}(\mathcal{PU})/(\text{chain homotopy}).$$

  (For general abelian categories, the story is more subtle.)
- There is a cofibrantly generated proper stable monoidal model structure, in which everything can be defined explicitly using P and one does not need the small object argument.
- If $X, Y \in \mathrm{Ch}(\mathcal{PU})$ then $\underline{\mathrm{Hom}}(X, Y) \in \mathcal{PU}$.
- Say $X \in \mathrm{Ch}(\mathcal{PU})$ is *thin* if for every $m > 0$, the differential on $L_m X$ is 0, i.e. the differential on X involves only maps $e_{G,S} \to e_{H,T}$ with $|H| < |G|$.
- Every homotopy type has an essentially unique thin representative. (But thin $\otimes$ thin and $\underline{\mathrm{Hom}}(\text{thin}, \text{thin})$ need not be thin.)
- A thin complex X is compact iff $\bigoplus_n X_n$ is finitely generated.

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- Given X, Y with $\mathrm{hsupp}(X) \subseteq \mathrm{hsupp}(Y)$, and a large integer $N > 0$, we can show that $\mathrm{thickid}\langle X\rangle \subseteq \mathrm{thickid}\langle \{Y\} \cup \{e_G \mid |G| > N\}\rangle$.
- This work is ongoing.

Supports and thick ideals

- For compact X (represented as a thin complex), several notions of support:
- $\mathrm{hsupp}(X) = \{G \mid H_*(X)(G) \neq 0\}$
- $\mathrm{esupp}(X) = \{G \mid X(G) \neq 0\}$
- $\mathrm{eqsupp}(X) = \{G \mid \text{ some } e_{G,S} \text{ is a retract of some } X_d\}$.
- It is easy to see that $\mathrm{esupp}(X)$ is the upwards closure of $\mathrm{eqsupp}(X)$.
- True but less obvious: $\mathrm{esupp}(X)$ is the upwards closure of $\mathrm{hsupp}(X)$.
- Conjecture: $\mathrm{thickid}\langle X\rangle \subseteq \mathrm{thickid}\langle Y\rangle$ iff $\mathrm{hsupp}(X) \subseteq \mathrm{hsupp}(Y)$.
- There is a very general method that does most of the work of classifying thick ideals, in cases where all compact objects are strongly dualisable. But that is not applicable here.
- The obvious prime ideals are $P_G = \{X \mid H_*(X)(G) = 0\}$.
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