Recent progress in chromatic homotopy

Neil Strickland

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The Telescope Conjecture

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- History: in 1984 Ravenel made a series of conjectures about spectra.
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  This led to a huge body of results in chromatic homotopy theory.
- It soon became the consensus that TC was probably false, and there was a programme by Mahowald, Ravenel and Schick to disprove it, but they could not complete the argument.
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- There are invariants $K(p,n)_*(X)$ of spectra X (for p prime and $n \geq 0$) called *Morava K-theory*. These play a central rôle in all the conjectures.
- Idea: focus on aspects of the category of spectra that are detected by $K(p,n)$ for a fixed (p,n). The number n is called *height*.
- There are two subtly different versions of this: TC says they are the same.
- This is easy for $n = 0$, true for $n = 1$ and false for $n > 1$.
- Alternative formulation: TC says that if $K(p, \leq n)_*(X) = 0$, then X is a filtered colimit of *finite* spectra X_α with $K(p, \leq n)_*(X_\alpha) = 0$.

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This talk

- The disproof of the Telescope Conjecture rests on a great deal of recent progress in chromatic homotopy theory.
- The aim of this talk is to survey some of those ideas.
- Blueshift: the Tate construction decreases chromatic height.
- Special case: the Tate construction sends $K(p, n)$ and similar things to zero, making other things canonically self-dual (*ambidexterity*).
- Categorification: if R is a commutative ring (spectrum), then Mod_R is a commutative semiring in the category of categories.
- New ∞-categorical foundations make this work smoothly.
- Roughly $K(R)$ is a ring spectrum obtained by adjoining negatives to Mod_R.
- Redshift: $K(-)$ increases height. Several versions and extensive history.
- Problem: extend ideas from ordinary rings to commutative ring spectra.
- Example: Galois theory. In chromatic homotopy we have analogues of the algebraic closure of $\mathbb{Q}$ and the maximal cyclotomic extension.
- Example: Nullstellensatz: many ring maps to algebraically closed fields.
- Example: groups of units, Picard groups, Brauer groups. These are tied together by categorification e.g. $\text{pic}(R) \to K(R)^{\times}$.
- Categorical shift is essential for correct interpretation of cyclotomic extensions. Ambidexterity is needed for construction.

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- Example: Nullstellensatz: many ring maps to algebraically closed fields.
- Example: groups of units, Picard groups, Brauer groups. These are tied together by categorification e.g. $\mathrm{pic}(R) \to K(R)^{\times}$.
- Categorical shift is essential for correct interpretation of cyclotomic extensions. Ambidexterity is needed for construction.

This talk

- The disproof of the Telescope Conjecture rests on a great deal of recent progress in chromatic homotopy theory.
- The aim of this talk is to survey some of those ideas.
- Blueshift: the Tate construction decreases chromatic height.
- Special case: the Tate construction sends $K(p, n)$ and similar things to zero, making other things canonically self-dual (*ambidexterity*).
- Categorification: if R is a commutative ring (spectrum), then Mod_R is a commutative semiring in the category of categories.
- New ∞-categorical foundations make this work smoothly.
- Roughly $K(R)$ is a ring spectrum obtained by adjoining negatives to Mod_R.
- Redshift: $K(-)$ increases height. Several versions and extensive history.
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- Example: Galois theory. In chromatic homotopy we have analogues of the algebraic closure of $\mathbb{Q}$ and the maximal cyclotomic extension.
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This talk

- Trace methods: understand $K(R)$ by comparing with $THH(R)$ and $TC(R)$.
- Newer approach to $TC(R)$ due to Nikolaus and Scholze: the Tate construction plays a central role.
- One particular example of great importance for TC: $THH(F(S^1_+, S^0))$.
- The key counterexample: $K(BP\langle n\rangle^{h\mathbb{Z}})$.

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Height

- Fix a prime $p > 2$ and write $K(n)$ for Morava K-theory $K(p, n)$.
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- $K(n)^*(BC_{p^k}) = K(n)^*[x]/x^{p^{nk}} = K(n)^*\{x^i \mid i < p^{nk}\}$.
- For many other theories E^*: ring structure of $E^*(BC_{p^k})$ is complicated, but it is still free of rank p^{nk} over E^*. The integer n is the *height*.
- $H^*(BC_{p^k}; \mathbb{Q}) = \mathbb{Q}$; height zero is the same as rational.
- Complex K-theory has height one.
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- There exists a finite complex $U(n)$ with $\widetilde{K(i)}^* U(n) = 0$ iff $i < n$ and a map $v \colon \Sigma^{2k} U(n) \to U(n)$ for some $k > 0$ such that $\widetilde{K(n)}^*(v)$ is multiplication by u^k. (This uses big theorems.)
- In spectra we can form the telescope $T(n) = \mathrm{End}(U(n))[v^{-1}]$.
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Tate blueshift

- Let G be a finite group acting on M. Put
  $M^G = H^0(G; M) = \{m \in M \mid gm = m \text{ for all } g \in G\}$
  $M_G = H_0(G; M) = M / \sum_g \{gm - m \mid m \in M\}$.
- Define $\tau\colon M_G \to M^G$ by $\tau([m]) = \sum_g gm$ and
  $\widehat{H}^0(G; M) = \text{cok}(\tau)$ and $\widehat{H}^{-1}(G; M) = \ker(\tau)$.
- These fit into a $\mathbb{Z}$-graded group; a short exact sequence $L \rightarrowtail M \twoheadrightarrow N$ gives a long exact sequence $\widehat{H}^i(G; L) \to \widehat{H}^i(G; M) \to \widehat{H}^i(G; N) \to \widehat{H}^{i+1}(G; L)$.
- $\widehat{H}^*(G; \mathbb{Z}[G] \otimes M) = 0$.
- $H^*(C_p; \mathbb{Z}) = \mathbb{Z}[x]/px$ and $\widehat{H}^*(C_p; \mathbb{Z}) = (\mathbb{Z}/p)[x, x^{-1}]$ (with $|x| = 2$)
- For a spectrum X with action of G, there is a parallel construction of a spectrum X^{tG}. If $\pi_i(X) = 0$ for $i \neq 0$ then $\pi_i(X^{tG}) = \widehat{H}^{-i}(G; \pi_0(X))$.
- For E of height n: $E^*(BC_p) = E^*[\![x]\!]/g(x)$ with g monic of degree p^n.
  Then $\pi_*(E^{tC_p}) = (E^*[\![a]\!]/g(a))[a^{-1}]$ which is zero or has height $n-1$.
- For $E = K(n)$ we have $g(x) = x^{p^n}$ and $K(n)^{tC_p} = 0$.
  Various other statements $X^{tG} = 0$ or $K(n)_*(X^{tG}) = 0$ can be deduced.
- By a more complicated argument: Kuhn proved $T(n)^{tG} = 0$.
- There are various related statements about how the Tate construction lowers height.

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Nullstellensatz and commutative redshift

- **Theorem** (Hahn): if $R \neq 0$ is commutative then there exists $n \in [0, \infty]$ such that $K(i) \wedge R = 0$ iff $i > n$. We call n the *height* of R.
- (For a ring spectrum R, we have $K(i) \wedge R = 0$ iff $T(i) \wedge R = 0$.)
- **Theorem** (Burklund, Schlank, Yuan): $\mathrm{ht}(K(R)) = \mathrm{ht}(R) + 1$.
- Say that an ordinary ring R is 0-satzian iff $R \neq 0$, and every finitely-presented R-algebra $A = R[x_1, \ldots, x_n]/(r_1, \ldots, r_m)$ has an R-algebra map to R.
- Hilbert's Nullstellensatz: R is 0-satzian iff it is an algebraically closed field.
- Thus: any nontrivial ring has many maps to 0-satzian rings.
- There is a similar definition of 0-satzian objects in the category of height n commutative ring spectra.
- **Theorem** (BSY): These are just the algebraically closed Morava theories.
- In more detail: suppose $\pi_1(E) = 0$ and $\pi_2(E)$ contains a unit. Then $E^*BS^1 = E^*[\![x]\!]$ and $z \mapsto z^p$ on S^1 induces $x \mapsto \sum_k a_k x^k$. Put $u_k = a_{p^k}$ so $u_0 = p$. Suppose $(u_0, \ldots, u_{n-1})$ is a regular sequence, $\pi_0(E)$ is complete with respect to the corresponding ideal I_n, and $\pi_0(E)/I_n$ is an algebraically closed field in which $u_n \neq 0$. Then E is an algebraically closed Morava theory of height n.
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- **Theorem** (Carmeli, Schlank, Yanovski): there is a similar extension $S_{T(n)}[\mu_{p^\infty}^{(n)}]$. This is again Galois with group $\mathbb{Z}_p^\times$ but is not faithful.

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