A MATHEMATICA REPRESENTATION OF SOME UNSTABLE HOMOTOPY GROUPS OF SPHERES

N. P. STRICKLAND

1. Introduction

This note documents a Mathematica program (called `Toda.m`) that embodies information about the 2-local homotopy groups $\pi_{n+k}S^n$ for $0 \leq k \leq 19$. Most of the information is taken from Toda's book "Composition methods in homotopy groups of spheres"; a few additional facts are proved in this note or quoted from elsewhere. The program does not do any serious calculations for itself, it is merely a convenient way of organizing and accessing results obtained by traditional methods. One can also do some automated consistency checking to detect any errors. I have tried to use a fairly general framework so that other computations in unstable homotopy can be included later if people are interested.

Here are some examples of things one can type, and the program's response:

```
SpherePi[14,2]
```

$$\mathbb{Z}_2\eta_5 \circ \epsilon_6 \oplus \mathbb{Z}_2\mu_5 \oplus \mathbb{Z}_2\nu_5 \circ \nu_8 \circ \nu_{11}$$

```
HomotopySet[Sphere[7],Sphere[4]]
```

$$\mathbb{Z}\nu_4 \oplus \mathbb{Z}_4\nu'$$

```
GroupType[SpherePi[19,10]]
```

$$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$$

```
ShowMap[Hopf,8,4]
```

$$\begin{array}{ccc} \pi_8^4 & \xrightarrow{\texttt{Hopf}} & \pi_8^7 \\ \mathbb{Z}_2\nu_4 \circ \eta_7 \oplus \mathbb{Z}_2(\Sigma\nu') \circ \eta_7 & & \mathbb{Z}_2\eta_7 \\ \nu_4 \circ \eta_7 & \mapsto & \eta_7 \\ (\Sigma\nu') \circ \eta_7 & \mapsto & 0 \end{array}$$

```
ShowMap[Sigma,To[sigmaprime]]
```

$$\begin{array}{ccc} \pi_{13}^6 & \xrightarrow{\Sigma} & \pi_{14}^7 \\ \mathbb{Z}_4\sigma'' & & \mathbb{Z}_8\sigma' \\ \sigma'' & \mapsto & 2\sigma' \end{array}$$

```
Explain[o[nuprime,nu[7]]]
```

$\nu' \circ \nu_7 \mapsto 0$

This takes place in the trivial group $\pi_{10}S^3$ — see Toda's Proposition 5.15

Remarks:

- I have not yet made a serious attempt to get the signs straight, and I have not been honest about this deficiency either. The program will often report that $x = y$ where Toda has proved only that $x = \pm y$. This should certainly be resolved in future, but that will require a very careful and lengthy analysis of conventions and definitions.
- I have tried to resolve all other sources of indeterminacy, and to evaluate all compositions, Hopf invariants and so on that Toda leaves open, but I have not been completely successful. I would appreciate any help that may be on offer to answer the questions in Section 4.
- I have made rather little use of methods beyond those of Toda. In particular, I have barely mentioned the unstable Adams spectral sequence, and I have not incorporated any of the results in Mahowald's memoir [2].

- I have used the following formula for the Hopf invariant of a composite: for $\alpha \in \pi_n S^m$ and $\beta \in \pi_m S^i$ (with $n > 1$ and $m, i > 0$) we have
$$H(\beta \circ \alpha) = H(\beta) \circ \alpha + (\Sigma^{i-1}\beta) \circ (\Sigma^{m-1}\beta) \circ H(\alpha)$$
This is a special case of [1, Corollary III.6.3]. (Note that his γ_2 is our H, and his $\beta\#\beta$ is our $(\Sigma^{i-1}\beta) \circ (\Sigma^{m-1}\beta)$, whereas Toda's $\beta\#\beta$ is $\beta \wedge \beta = (\Sigma^i\beta) \circ (\Sigma^m\beta)$.) It is apparently well-known to the experts that the proof in [1] is incorrect. However, the formula is known to become true after one suspension, and it is also true for easy reasons if either α or β is a suspension. I do not know whether there are any cases in which the formula itself is incorrect.

2. Mathematica notation

2.1. Abelian groups.

- The expression `Group`$[\{x_1, d_1\}, \dots, \{x_r, d_r\}]$ represents an abelian group that is the direct sum of cyclic groups of order d_i generated by elements x_i. The order d_i may be `Infinity`. Expressions of this form are displayed (by default) in traditional notation, for example `Group[{x,Infinity},{y,24}]` is displayed as $\mathbb{Z}x \oplus \mathbb{Z}_{24}y$. One can type `FullForm[A]` to see the internal representation.
- An isomorphism class of finitely generated abelian groups is represented by an expression of the form `GType`$[d_1, \dots, d_r]$ with $1 < d_1 \leq \dots \leq d_r \leq \infty$; this represents $\bigoplus_i \mathbb{Z}_{d_i}$, where $\mathbb{Z}_\infty$ just means $\mathbb{Z}$.
- If `A` is an expression of the form `Group`$[\dots]$, then `GroupType[A]` represents the isomorphism class, `GroupOrder[A]` gives the order, `GroupRank[A]` gives the number of cyclic summands, `Generators[A]` gives the list of generators of these summands, `GroupExponent[A]` gives the exponent, and `Elements[A]` gives the list of all elements of `A` (if `A` is finite). The boolean function `FiniteQ[A]` is true iff `A` is a finite group.
- The function `ElementQ[a,A]` will return `True` if `a` is visibly an integral linear combination of the generators of `A`.
- The expression `Coset`$[a, \{x_1, \dots, x_r\}]$ refers to the coset of a modulo the subgroup generated by the elements x_i.

2.2. Spaces.

- The expression `Sphere[n]` represents the sphere S^n, which we officially define to be the one-point compactification of $\mathbb{R}^n$. There is no real sign issue in identifying this with $I^n/\partial(I^n)$, or in identifying $S^n \wedge S^m$ with S^{n+m}. There are sign issues in identifying S^n with $\Delta_n/\partial(\Delta_n)$ or with the space $S^n_{\text{round}} := \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$.
- We define $\Sigma X = X \wedge S^1$. This is represented in Mathematica as `Sigma[X]`; the 4-fold suspension of X (for example) can then be entered as `SigmaIterate[4,X]`, but it will be represented internally as `Sigma[Sigma[Sigma[Sigma[X]]]]`.
- The expression `OrthogonalGroup[n]` represents the orthogonal group O_n.

2.3. Homotopy sets.

- The expression `HomotopySet[X,Y]` represents the set of homotopy classes of based maps from X to Y. This will generally have at most one reasonable group structure. At present we have no support for nonabelian groups, and no support for multiple group structures.
- The expression `HomotopyGroup[n,X]` refers to $\pi_n X$. It is automatically converted to `HomotopySet[Sphere[n],X]`.
- The expression `SpherePi[n,m]` refers to Toda's π_n^m, which is a naturally defined subgroup of $\pi_n S^m$ such that $\pi_n S^m = \pi_n^m \oplus (\text{odd torsion})$. We will probably redefine it later to refer to $\pi_n S^m$ itself. There is ambiguity about this in the code at the moment.
- The expression `OrthogonalPi[n,m]` refers to $\pi_n O_m$.
- Given an element $a \in [X, Y]$, we have `Source[a]=X` and `Target[a]=Y`. If $X = S^n$ we have `SourceSphere[a]=n`. Similarly, if $Y = S^m$ we have `TargetSphere[a]=m`. If $X = S^n$

and $Y = S^m$ we have `Stem[a]=n-m` and `ST[a]={n,m}`. We do not currently distinguish internally between different zero maps, so `Source[0]` and so on are undefined.
- For general $a \in [X, Y]$, the function `Home[a]` represents the set $[X, Y]$. The function `ShowHome[a]` displays this in a nicely readable form.
- Given $a \in [X, Y]$ and $b \in [Y, Z]$, the composite $b \circ a$ is represented by the expression `o[b,a]`.

2.4. **Functions.**

- The suspension functor is denoted by `Sigma`. We use the definition $\Sigma X = X \wedge S^1$. We are primarily interested in Σ as a homomorphism $\pi_k S^n \to \pi_{k+1} S^{n+1}$.
- The iterated suspension $\Sigma^n X$ is represented by `SigmaIterate[n,X]`.
- The Hopf invariant $H\colon \pi_k S^n \to \pi_k S^{2n-1}$ is represented by `Hopf`. We use the James-Hopf definition with the left lexicographic ordering.
- The connecting map $P\colon \pi_k S^{2n+1} \to \pi_{k-2} S^n$ in the EHP sequence is represented by `WhiteheadP`. I am not sure if we have the sign pinned down.
- The unstable J-homomorphism $\pi_k O_n \to \pi_{k+n} S^n$ is represented by `JHomomorphism`.
- The evident map $\pi_k O_n \to \pi_k O_{n+1}$ is represented by `OrthogonalSigma`.
- The evaluation map $\pi_k O_n \to \pi_k S^{n-1}$ is represented by `OrthogonalHopf` (because it is compatible with `Hopf` via the J-homomorphism). Note that there is an implicit identification of the round sphere in $\mathbb{R}^n$ with our $S^{n-1} = \mathbb{R}^{n-1} \cup \{\infty\}$, so we need to fix the sign.
- The connecting map $\pi_k S^{n-1} \to \pi_{k-1} O_{n-1}$ in the orthogonal sequence is represented by `OrthogonalP`. It comes from an map $\Omega S^{n-1} = JS^{n-2} \to O_{n-1}$ extending the reduced reflection map $S^{n-2} \to SO_{n-1}$, at least up to sign.

2.5. **Homotopy elements.**

- $\iota_n : S^n \to S^n$ is the identity map. Mathematica notation is `iota[n]`.
- `w[n]` refers to the Whitehead product $w_n = [\iota_n, \iota_n]\colon S^{2n-1} \to S^n$.
- $\eta_2 : S^3 \to S^2$ is the complex Hopf map. For $k \geq 2$ we put $\eta_k = \Sigma^{k-2}\eta_2$. Mathematica notation is `eta[k]`.
- $\nu_4 : S^7 \to S^4$ is defined by the properties $H(\nu_4) = \iota_7$ and $2\Sigma(\nu_4) = \Sigma^2\nu'$ (Lemma 5.4). I think this only defines it modulo $2\nu'$. However, I think we pin it down precisely by requiring that ν_4 should be equal to the quaternionic Hopf map plus an odd torsion class, and that $\Sigma\nu_4$ should be 2-torsion. For $n \geq 4$ we put $\nu_n = \Sigma^{n-4}\nu_4$. Mathematica notation is `nu[n]`.
- $\sigma_8 : S^{15} \to S^8$ is the octonionic Hopf map plus an odd torsion class, chosen so that $\Sigma\sigma_8$ is 2-torsion. Toda defines it (Lemma 5.14) by the property that $H(\sigma_8) = \iota_{15}$. For $n \geq 8$ we put $\sigma_n = \Sigma^{n-8}\sigma_8$. Mathematica notation is `sigma[n]`.
- $\epsilon_3 : S^{11} \to S^3$ is the unique element of the Toda bracket $\langle \eta_3, \Sigma\nu', \nu_7\rangle_1$ (see the beginning of Toda's Chapter VI). For $k \geq 3$ we put $\epsilon_k = \Sigma^{k-3}\epsilon_3$. Mathematica notation is `epsilon[k]`.
- In Toda's Section VI(ii) he defines $\overline{\nu}_6 : S^{14} \to S^6$ to be an element of the Toda bracket $\langle \nu_6, \eta_9, \nu_{10}\rangle_1$. We will let $\overline{\nu}_6$denote the unique element of this Toda bracket that satisfies $H(\overline{\nu}_6) = \nu_{11}$. This definition is validated in Section 3. We put $\overline{\nu}_n = \Sigma^{n-6}\overline{\nu}_6$ for $n \geq 6$. Mathematica notation is `nubar[n]`.
- The map $\mu_3 : S^{12} \to S^3$ is defined in Toda's Section VI(iii) by a procedure of several steps, involving the cofibre of the map $\nu' : S^6/8 \to S^3$. It turns out that the indeterminacy is $\{0, \eta_3 \circ \epsilon_4\}$ (see the discussion preceeding Lemma 6.5, together with Theorem 7.1). We also have $\pi_{12}S^3 = \mathbb{Z}_2\mu_3 \oplus \mathbb{Z}_2(\eta_3 \circ \epsilon_4)$. The element $\eta_3 \circ \epsilon_4$ has Adams filtration 4, and we make the unique choice of μ_3 that has Adams filtration 5. We write $\mu_k = \Sigma^{k-3}\mu_3$ for $k \geq 3$. Mathematica notation is `mu[k]`.
- We define $\zeta_5 : S^{16} \to S^5$ to be the unique element of the Toda bracket $\langle \nu_5, 8\iota_8, \Sigma\sigma'\rangle_1$ that satisfies $H(\zeta_5) = 8\sigma_9$ and $P(\zeta_5) = \pm\eta_2 \circ \mu'$. This is validated in Section 3, based on Toda's Section VI(v). For $k \geq 5$ we put $\zeta_k = \Sigma^{k-5}\zeta_5$. Mathematica notation is `zeta[k]`.

- The element $\kappa_7 : S^{21} \to S^7$ is defined in Toda's Section X(i) as an element of a Toda bracket involving the Moore space $S^9/2$. The indeterminacy is not given explicitly. For $k \geq 7$ we put $\kappa_k = \Sigma^{k-7}\kappa_7$. Mathematica notation is `kappa[k]`.
- $\overline{\epsilon}_3 : S^{18} \to S^3$ is the unique element of the Toda bracket $\langle \epsilon_3, 2\iota_5, \nu_5 \circ \nu_8 \rangle_6$, and the unique nonzero element of π^3_{18}. See Toda's Section X(i) and Theorem 10.5. For $k \geq 3$ we put $\overline{\epsilon}_k = \Sigma^{k-3}\overline{\epsilon}_3$. Mathematica notation is `epsilonbar[k]`.
- $\rho_{13} : S^{28} \to S^{13}$ is defined in Toda's Lemma 10.9 by the properties $2\rho_{13} = \Sigma^4\rho'$ and $\Sigma^\infty\rho_{13} \in \langle \sigma, 2\sigma, 8\iota \rangle$. For $k \geq 13$ we put $\rho_k = \Sigma^{k-13}\rho_{13}$. Mathematica notation is `rho[k]`.
- $\overline{\zeta}_5 : S^{24} \to S^5$ is an element of the Toda bracket $\langle \zeta_5, 8\iota_{16}, 2\sigma_{16} \rangle_1$. See the preamble to Toda's Lemma 12.4. For $k \geq 5$ we put $\overline{\zeta}_k = \Sigma^{k-5}\overline{\zeta}_5$. Mathematica notation is `zetabar[k]`.
- $\overline{\sigma}_6 : S^{25} \to S^6$ is an element of the Toda bracket $\langle \nu_6, \epsilon_9 + \overline{\nu}_9, \sigma_{16} \rangle_1$. See the preamble to Toda's Lemma 12.5. For $k \geq 6$ we put $\overline{\sigma}_k = \Sigma^{k-6}\overline{\sigma}_6$. Mathematica notation is `sigmabar[k]`.
- $\omega_{14} : S^{30} \to S^{14}$ is defined in Toda's Lemma 12.15(i) by the requirement that $H(\omega_{14}) = \nu_{27}$. The indeterminacy is $\{0, \sigma_{14} \circ \mu_{21}\}$. For $k \geq 14$ we put $\omega_k = \Sigma^{k-14}\omega_{14}$. Mathematica notation is `omega[k]`.
- $\eta^*_{16} : S^{32} \to S^{16}$ is an element of the Toda bracket $\langle \sigma_{16}, 2\sigma_{23}, \eta_{30} \rangle_1$. See Toda's Section XII(iii). For $k \geq 16$ we put $\eta^*_k = \Sigma^{k-16}\eta^*_{16}$. For $k \geq 18$ we have $\eta^*_k = \omega_k$ modulo $\sigma_k \circ \mu_{k+7}$, so either η^*_k or ω_k can be used as a generator of π^k_{k+16}. Mathematica notation is `etastar[k]`.
- $\epsilon^*_{12} : S^{29} \to S^{12}$ is defined in Toda's Lemma 12.15(ii) by the properties $H(\epsilon^*_{12}) = \nu_{23} \circ \nu_{27}$ and $\Sigma^2\epsilon^*_{12} = \omega_{14} \circ \eta_{30}$. This fixes ϵ^*_{12} as a function of ω_{14}. However, the indeterminacy of $\sigma_{14} \circ \mu_{21}$ in ω_{14} creates an indeterminacy of $\sigma_{12} \circ \eta_{19} \circ \mu_{20}$ in ϵ^*_{12}. For $k \geq 12$ we put $\epsilon^*_k = \Sigma^{k-12}\epsilon^*_{12}$. Mathematica notation is `epsilonstar[k]`.
- $\overline{\mu}_3 : S^{20} \to S^3$ is defined in the preamble to Toda's Lemma 12.2 to be an element of the Toda bracket $\langle \mu_3, 2\iota_{12}, 8\sigma_{12} \rangle_1$. I have not checked the indeterminacy. For $k \geq 3$ we put $\overline{\mu}_k = \Sigma^{k-3}\overline{\mu}_3$. Mathematica notation is `mubar[k]`.
- $\nu^*_{16} : S^{34} \to S^{16}$ is defined in Toda's Section XII(iii) to be an element of the Toda bracket $\langle \sigma_{16}, 2\sigma_{23}, \nu_{30} \rangle_1$. I have not checked the indeterminacy. For $k \geq 16$ we put $\nu^*_k = \Sigma^{k-16}\nu^*_{16}$. Mathematica notation is `nustar[k]`.
- $\xi_{12} : S^{30} \to S^{12}$ is defined in Toda's Section XII(iii) to be an element of the Toda bracket $\langle \sigma_{12}, \nu_{19}, \sigma_{22} \rangle_1$. I have not checked the indeterminacy. For $k \geq 12$ we put $\xi_k = \Sigma^{k-12}\xi_{16}$. Mathematica notation is `xi[k]`.
- ν' is an element of the Toda bracket $\langle \nu_3, 2\iota_4, \eta_4 \rangle_1$. I think it is also obtained by applying the unstable J-homomorphism to the generator of $\pi_3 SO(3)$. Toda's definition is indeterminate up to sign. See Toda's Equations 5.3 and 5.4. Mathematica notation is `nuprime`.
- $\sigma''' : S^{12} \to S^5$ is the unique element of the Toda bracket $\langle \nu_5, 8\iota_8, \nu_8 \rangle$, and is the unique nonzero element in $\pi_{12}S^5$. See Toda's Lemma 5.13. Mathematica notation is `sigmathird`.
- $\sigma'' : S^{13} \to S^6$ is characterised by the properties $H(\sigma'') = \eta^2_{11}$ and $\Sigma^3\sigma'' = 4\sigma_9$. It has order 4 and generates $\pi_{13}S^6$. See Toda's Lemma 5.14 (corrected by replacing σ'' by σ' on the second line), and note that $\Sigma^3 : \pi^6_{13} \to \pi^9_{16}$ is injective. Mathematica notation is `sigmasecond`.
- $\sigma' : S^{14} \to S^7$ is characterised by the properties $H(\sigma') = \eta_{13}$ and $\Sigma^2\sigma' = 2\sigma_9$. It has order 8 and generates $\pi_{14}S^7$. See Toda's Lemma 5.14 (corrected by replacing σ'' by $sigma'$ on the second line), and note that $\Sigma^2 : \pi^7_{14} \to \pi^9_{16}$ is injective. Mathematica notation is `sigmaprime`.
- $\epsilon' : S^{13} \to S^3$ is the unique element of the Toda bracket $\langle \nu', 2\nu_6, \nu_9 \rangle_3$. See Toda, Section VI(iv). Mathematica notation is `epsilonprime`.
- The element $\mu' : S^{14} \to S^3$ lies in the Toda bracket $\langle \eta_3, 2\iota_4, \mu_4 \rangle_1$ and satisfies $H(\mu') = \mu_5$ and $2\mu' = \eta_3 \circ \eta_4 \circ \mu_5$; this is explained at the beginning of Toda's Section VII(ii). Toda does not state the indeterminacy and I have not worked it out. Mathematica notation is `muprime`.

- $\theta' : S^{23} \to S^{11}$ is the unique element of the Toda bracket $\langle \sigma_{11}, 2\nu_{18}, \eta_{21}\rangle_1$ and the unique nonzero element of π^{11}_{23}. See Toda's Lemma 7.5. Mathematica notation is `thetaprime`.
- $\theta : S^{24} \to S^{12}$ lies in the Toda bracket $\langle \sigma_{12}, \nu_{19}, \eta_{22}\rangle_1$. It is not clear whether the indeterminacy is zero. Mathematica notation is `theta`.
- $\rho^{(4)} : S^{20} \to S^5$ is defined in Toda's Section X(iii) to be an element of the Toda bracket $\langle \sigma^{(3)}, 2\iota_{12}, 8\sigma_{12}\rangle_1$. It is proved in Section 3 that the indeterminacy is zero. Mathematica notation is `rhofourth`.
- $\rho^{(3)} : S^{21} \to S^6$ is defined in Toda's Section X(iii) to be an element of the Toda bracket $\langle \sigma'', 4\iota_{13}, 4\sigma_{13}\rangle_1$. Mathematica notation is `rhothird`.
- $\rho'' : S^{22} \to S^7$ is defined in Toda's Section X(iii) to be an element of the Toda bracket $\langle \sigma', 8\iota_{14}, 2\sigma_{14}\rangle_1$. We will take it to be the unique element for which $H(\rho'') = \mu_{13}$. This is validated in Proposition 30. Mathematica notation is `rhosecond`.
- $\rho' : S^{24} \to S^9$ is defined in Toda's Section X(iii) to be an element of the Toda bracket $\langle \sigma_9, 16\iota_{16}, \sigma_{16}\rangle_1$. Mathematica notation is `rhoprime`.
- $\zeta' : S^{22} \to S^6$ is the unique element such that $H(\zeta') = \zeta_{11} mod 2\zeta_{11}$ and $\Sigma\zeta' = \sigma' \circ \eta_{14} \circ \epsilon_{15}$. This is validated in Proposition 32, which refines Toda's Lemma 12.1. Mathematica notation is `zetaprime`.
- $\overline{\epsilon}' : S^{20} \to S^3$ is defined in Toda's Lemma 12.3 by the properties $2\overline{\epsilon}' = \eta_3 \circ \eta_4 \circ \overline{\epsilon}_5$ and $\Sigma\overline{\epsilon}' = (\Sigma\nu') \circ \kappa_7$. The map Σ here is injective by Toda's Theorem 12.7, so this characterises $\overline{\epsilon}'$ uniquely. Mathematica notation is `epsilonbarprime`.
- $\overline{\mu}' : S^{22} \to S^3$ is an element of the Toda bracket $\langle \mu', 4\iota_{14}, 4\sigma_{14}\rangle_1$. See the preamble to Toda's Lemma 12.4. Mathematica notation is `mubarprime`.
- $\eta^{*\prime} : S^{31} \to S^{15}$ is an element of the Toda bracket $\langle \sigma_{15}, 4\sigma_{22}, \eta_{29}\rangle_1$. (There seems to be a filtration shift so this is not seen in Ext.) See Toda's Section XII(iii). Mathematica notation is `etastarprime`.
- $\lambda'' : S^{28} \to S^{10}$ is defined in Toda's Lemma 12.19 by the properties $\Sigma\lambda'' = 2\lambda'$ and $H(\lambda'') = \eta_{19} \circ \epsilon_{20} mod (\eta_{19} \circ \epsilon_{20} + \nu^3_{19})$. I have not checked the indeterminacy. Mathematica notation is `lambdasecond`.
- $\xi'' : S^{28} \to S^{10}$ is defined in Toda's Lemma 12.19 by the properties $\Sigma\xi'' = 2\xi'$ and $H(\xi'') = \nu_{19} \circ \nu_{22} \circ \nu_{25} + \eta_{13} \circ \epsilon_{20}$. I have not checked the indeterminacy. Mathematica notation is `xisecond`.
- $\lambda' : S^{29} \to S^{11}$ is defined in Toda's Lemma 12.19 by the properties $\Sigma^2\lambda' = 2\lambda$ and $H(\lambda') = \epsilon_{21} mod (\overline{\nu}_{21} + \epsilon_{21})$. I have not checked the indeterminacy. Mathematica notation is `lambdaprime`.
- $\xi' : S^{29} \to S^{11}$ is defined in Toda's Lemma 12.19 by the properties $\Sigma\xi' = 2\xi_{12} \pm w_{12} \circ \sigma_{23}$ and $H(\xi') = \overline{\nu}_{21} + \epsilon_{21}$. I have not checked the indeterminacy. Mathematica notation is `xiprime`.
- $\lambda : S^{31} \to S^{13}$ is defined in Toda's Lemma 12.18 by the properties $\Sigma^3\lambda = 2\nu^*_{16} \pm w_{16} \circ \nu_{31}$ and $H(\lambda) = \nu_{25} \circ \nu_{28}$. I have not checked the indeterminacy. Mathematica notation is `lambda`.
- $\omega' : S^{31} \to S^{12}$ is defined in Toda's Lemma 12.21 by the properties $\Sigma^2\omega' = 2\omega_{14} \circ \nu_{30}$ and $H(\omega') = \epsilon_{23} mod (\overline{\nu}_{23} + \epsilon_{23})$. I have not checked the indeterminacy. Mathematica notation is `omegaprime`.

The additive order of one of these elements a is `AdditiveOrder[a]`.

2.6. **Genealogy.** Recursive application of the EHP spectral sequence gives a filtration of the two-torsion in $\pi_{n+k}S^n$ by subgroups F^n_I, where I runs over sequences $(i_1, \dots, i_r)$ with $i_1 < n$ and $i_{t+1} \leq 2i_t$ and $\sum_t i_t = k$. We have $F^n_I \leq F^n_J$ if $I \leq J$ in lexicographic order, so we have a linear filtration with quotients Q^n_I say, and one can check that Q^n_I is always either zero or $\mathbb{Z}_2$. For each (n, I) such that $Q^n_I \neq 0$ we choose an element $x^n_I \in F^n_I$ mapping to the generator. The sequence I is called the genealogy of x^n_I (the first element is one less than the sphere on which x^n_I is "born", and the rest of the sequence is the genealogy of the Hopf invariant of a maximal desuspension). The program specifies choices for the elements x^n_I up to the 13-stem. The Mathematica notation for $x^6_{4,3,2}$ (for example) is `x[6,{4,3,2}]`. The functions `TodaName` and `GenealogyName` can be used to convert between Toda's names and the genealogy names for elements in π_*S^*.

2.7. **The Λ-algebra.** There is a separate program called `LambdaAlgebra.m` that knows a little about the lambda algebra and the Adams spectral sequence. It can calculate the cohomology of the lambda algebra, but it uses a very direct and naive method; I have not yet set up the Curtis algorithm, which would be much more efficient.

- An expression $\lambda_{i_1}\cdots\lambda_{i_r}$ in the Λ-algebra is represented by $\texttt{lambda}[i_1,\dots,i_r]$. The same element, thought of as an element of the subspace $\Lambda(m)$, is represented by $\texttt{ulambda}[m,i_1,\dots,i_r]$.
- The product of elements a and b in the Λ-algebra is represented by `o[a,b]`. The differential on a is represented by `del[a]`.
- The bidegree of an element a is `Bidegree[a]`; for example, `Bidegree[lambda[i,j]]={2,i+j+2}`. If `Bidegree[a]={s,t}` then `AdamsFiltration[a]=s` and `Stem[a]=t-s`.
- For $a\in\Lambda(n)^{s,t}$ we have `TargetSphere[a]=n` and `SourceSphere[a]=n+t-s`.
- The functions `Sigma`, `Hopf` and `WhiteheadP` are defined for elements of $\Lambda(n)^{**}$ as in the algebraic EHP sequence.
- The function `Basis[Lambda[s,t]]` returns the admissible basis for $\Lambda^{s,t}$. Similarly, `Basis[Lambda[n,s,t]]` returns a basis for $\Lambda(n)^{s,t}$.
- Given an element $a\in\pi_kS^n$, the function `LambdaRepresentative[a]` is supposed to return a cocycle in $\Lambda(n)^{**}$ that represents a in the unstable Adams spectral sequence. The program only knows representatives for a limited number of elements, however.

2.8. **Display functions.**

- `ShowHome[a]` gives nicely readable information about the group in which a lives.
- `ShowMap[F,n,m]` (where `F` is one of `Sigma`, `Hopf` or `WhiteheadP`) gives nicely readable information about the map `F` out of π_nS^m, including the structure of the source and target groups and the effect on the generators. One can also use `ShowMap[F,To[n,m]]` to display information about `F` into the group π_nS^m. Moreover, if $a\in\pi_nS^m$ then `ShowMap[F,a]` (or `ShowMap[F,From[a]]`) is equivalent to `ShowMap[F,n,m]`, and similarly for `ShowMap[F,To[a]]`.
- `Zap[a]` rewrites `a` in terms of standard generators of the group in which it lives, using all known relations. Typing `Explain[a]` gives an explanation of the steps used.
- `?a` displays the definition of a.

3. Results

Proposition 1. *There is a unique element $\overline{\nu}_6\in\langle\nu_6,\eta_9,\nu_{10}\rangle\subset\pi^6_{14}$ such that $H(\overline{\nu}_6)=\nu_{11}$.*

Proof. Following [3, Section VI(ii)], we temporarily let $\overline{\nu}_6$ denote an arbitrary element of the indicated Toda bracket. Toda explains that the indeterminacy is generated by $P(\nu_{13})$, and then he shows in his Lemma 6.2 that $P(\nu_{13})=\pm2\overline{\nu}_6$. Thus, the elements of the relevant Toda bracket are the odd multiples of $\overline{\nu}_6$. We also learn from Toda's Lemma 6.2 that $H(\overline{\nu}_6)=k\nu_{11}$ for some odd k. From his Proposition 5.6 and Theorem 7.1 we know that both ν_{11} and $\overline{\nu}_6$ have order 8, so $H\colon\mathbb{Z}_8\overline{\nu}_6\to\mathbb{Z}_8\nu_{11}$ is an isomorphism. If we let j be inverse to k modulo 8, we conclude that $j\overline{\nu}_6$ is the unique element of $\langle\nu_6,\eta_9,\nu_{10}\rangle$ that is sent by H to ν_{11}. ☐

Proposition 2. $\eta_3\circ\Sigma\epsilon'=0$

Proof. Recall that ϵ' is the unique element of the Toda bracket $\langle\nu',2\nu_6,\nu_9\rangle_3\subseteq\pi^3_{13}$ (see [3, Section VI(iv)]). Using [3, Propositions 1.2(iv) and 1.3], and the fact that $-\eta_3=\eta_3$, we deduce that $\eta_3\circ\Sigma\epsilon'\in\langle\eta_3\circ\Sigma\nu',\nu_7,\nu_{10}\rangle_4$. Next, we know from [3, Lemma 5.7] that $\eta_2\circ\nu'=\pm P(\nu_5)$. Suspending this gives $\eta_3\circ\Sigma\nu'=0$. This means that the above Toda bracket is equal to its indeterminacy, which is $\nu^*_{11}\pi^3_{11}$. We know from [3, Theorem 7.1] that $\pi^3_{11}=\mathbb{Z}_2\epsilon_3$, so the indeterminacy is generated by $\epsilon_3\circ\nu_{11}$, which is one of the generators for π^3_{14} listed in [3, Theorem 7.4], giving a summand of order 2. Thus $\eta_3\circ\Sigma\epsilon'\in\{0,\epsilon_3\circ\nu_{11}\}$. However, [3, Equations 7.8] say that $H(\epsilon_3\circ\nu_{11})\neq0$, whereas $H(\eta_3\circ\Sigma\epsilon')=H\Sigma(\eta_2\circ\epsilon')=0$ (because $H\Sigma=0$). We must therefore have $\eta_3\circ\Sigma\epsilon'=0$ as claimed. ☐

Corollary 3. $P(\nu_5 \circ \sigma_8) = \pm\eta_2 \circ \epsilon'$.

Proof. We have an exact sequence

$$\pi^3_{15} \xrightarrow{H} \pi^5_{15} \xrightarrow{P} \pi^2_{13} \xrightarrow{\Sigma} \pi^3_{14}.$$

We know from [3, Theorems 7.6, 7.3 and 7.4] that

$$\begin{aligned}
\pi^3_{15} &= \mathbb{Z}_2(\nu' \circ \mu_6) \oplus \mathbb{Z}_2(\nu' \circ \eta_6 \circ \epsilon_7) \\
\pi^5_{15} &= \mathbb{Z}_2(\eta_5 \circ \mu_6) \oplus \mathbb{Z}_8(\nu_5 \circ \sigma_8) \\
\pi^2_{13} &= \mathbb{Z}_4(\eta_2 \circ \epsilon') \oplus \mathbb{Z}_2(\eta_2 \circ \eta_3 \circ \mu_4) \\
\pi^3_{14} &= \mathbb{Z}_4\mu' \oplus \mathbb{Z}_2(\epsilon_3 \circ \nu_{11}) \oplus \mathbb{Z}_2(\nu' \circ \epsilon_6).
\end{aligned}$$

For the maps, we have

$$\begin{aligned}
H(\nu' \circ \mu_6) &= \eta_5 \circ \mu_6 \\
H(\nu' \circ \eta_6 \circ \epsilon_7) &= 4(\nu_5 \circ \sigma_8) \\
\Sigma(\eta_2 \circ \epsilon') &= 0 \\
\Sigma(\eta_2 \circ \eta_3 \circ \mu_4) &= 2\mu'
\end{aligned}$$

(The first two equations are from [3, Bottom of page 75], the third is Proposition 2 above, and the fourth is [3, Equations 7.7].) It follows that

$$\begin{aligned}
\operatorname{cok}(H) &= \mathbb{Z}_4(\nu_5 \circ \sigma_8) \\
\ker(\Sigma) &= \mathbb{Z}_4(\eta_2 \circ \epsilon').
\end{aligned}$$

The EHP sequence tells us that P must induce an isomorphism between these groups, so $P(\nu_5 \circ \sigma_8) = \pm\eta_2 \circ \epsilon'$ as claimed. □

Proposition 4. $P(\sigma') = 0$

Proof. We have $\sigma' \in \pi^7_{14}$, and there is an exact sequence

$$\pi^7_{14} \xrightarrow{P} \pi^3_{12} \xrightarrow{\Sigma} \pi^4_{13}.$$

It will therefore suffice to show that the map Σ is injective. This can be read off from [3, Theorem 7.2]. □

Proposition 5. $H(\eta_2 \circ \epsilon_3) = \epsilon_3$.

Proof. We have an exact sequence

$$0 = \pi^1_{10} \xrightarrow{E} \pi^2_{11} \xrightarrow{H} \pi^3_{11} \xrightarrow{P} \pi^1_9 = 0,$$

so the map H is an isomorphism. By [3, Theorems 7.2 and 7.1] we have $\pi^2_{11} = \mathbb{Z}_2(\eta_2 \circ \epsilon_3)$ and $\pi^3_{11} = \mathbb{Z}_2\epsilon_3$. The claim follows. □

Proposition 6. $P(\overline{\nu}_7) = 0$

Proof. It will of course be enough to show that the whole map $P\colon \pi^7_{15} \to \pi^3_{13}$ is zero, or equivalently, that the map $\Sigma\colon \pi^3_{13} \to \pi^4_{14}$ is injective. This can be read off directly from [3, Theorem 7.3]. □

Proposition 7. $P(\sigma_9) = k(\nu_4 \circ \sigma' \pm \Sigma\epsilon')$ *for some odd* k.

Proof. As $\pi^9_{16} = \mathbb{Z}_{16}\sigma_9$, it will suffice to show that $\nu_4 \circ \sigma' \pm \Sigma\epsilon'$ generates the kernel of $\Sigma\colon \pi^4_{14} \to \pi^5_{15}$. The groups are given in [3, Theorem 7.3] as

$$\begin{aligned}
\pi^4_{14} &= \mathbb{Z}_8(\nu_4 \circ \sigma') \oplus \mathbb{Z}_4\Sigma\epsilon' \oplus \mathbb{Z}_2(\eta_4 \circ \mu_5) \\
\pi^5_{15} &= \mathbb{Z}_8(\nu_5 \circ \sigma_8) \oplus \mathbb{Z}_2(\eta_5 \circ \mu_6).
\end{aligned}$$

From the definitions we have $\Sigma(\eta_4 \circ \mu_5) = \eta_5 \circ \mu_6$. We also know from [3, Equations 7.10 and 7.16] that $\Sigma^2\epsilon' = \pm 2\nu_5 \circ \sigma_8 = \Sigma(\nu_4 \circ \sigma')$. The claim follows. □

Proposition 8. $(2\eta_2) \circ \epsilon_3 = 0$ *and* $(2\eta_2) \circ \mu_3 = 0$.

Proof. We have $H(\epsilon') = \epsilon_5$ by [3, Lemma 6.6], so $P(\epsilon_5) = 0$. On the other hand, $P(\epsilon_5) = P(\Sigma^2\epsilon_3) = w_2 \circ \epsilon_3$, and $w_2 = \pm 2\eta_3$ by [3, Proposition 5.1]. The first claim follows, and the second claim follows in the same way from the fact that $H(\mu') = \mu_5$. □

Proposition 9. *$P(\eta_9) = (\Sigma\nu') \circ \eta_7$.*

Proof. As $\eta_9 = \Sigma^2\eta_7$ we have $P(\eta_9) = w_4 \circ \eta_7$. We know from [3, Equations 5.8] that $w_4 = \pm(2\nu_4 - \Sigma\nu')$. As η_7 is a suspension it follows that

$$w_4 \circ \eta_7 = \nu_4 \circ (\pm 2\eta_7) + (\Sigma\nu') \circ (\mp\eta_7).$$

As η_7 has order 2 this just reduces to $(\Sigma\nu') \circ \eta_7$. □

Proposition 10. *$w_6 \circ \nu_{11} = P(\nu_{13}) = 2\overline{\nu}_6$.*

Proof. The first equation holds because $\nu_{13} = \Sigma^2\nu_{11}$. For the second, note that $\pi^3_{16} = \mathbb{Z}_8\nu_{13}$, so $P(\nu_8)$ must generate the kernel of $\Sigma\colon \pi^6_{14} \to \pi^7_{15}$. We read off from [3, Theorem 7.1] the fact that this kernel is $\mathbb{Z}_4.(2\overline{\nu}_6)$, so $P(\nu_{13}) = \pm 2\overline{\nu}_6$. To remove the indeterminacy, recall that $H(\overline{\nu}_6) = \nu_{11}$, which has order 8, so it will suffice to show that $H(P(\nu_{13})) = 2\nu_{11}$. For this we use [1, Corollary 6.6]. This says that for $\alpha\colon \Sigma X \to S^{2k+1}$ (with k odd) we have

$$\gamma_{2n}([j_k, j_k] \circ \alpha) = (2^n j) \circ (\gamma_n(\alpha)).$$

We take $k = 5$ and $X = S^{13}$ and $\alpha = \nu_{11}$. We observe that j_m is Baues's notation for ι_{m+1} and that $[\iota_6, \iota_6] = w_6$, so the left hand side is $\gamma_{2n}(w_6 \circ \nu_{11})$. We then take $n = 2$ and recall that $\gamma_2 = H$ and γ_1 is the identity. The symbol j on the right hand side refers to the identity of the relevant sphere, which is S^{11}. The right hand side is thus $(2\iota_{11}) \circ \nu_{11}$, which is the same as $2\nu_{11}$ because ν_{11} is a suspension. □

Proposition 11. *$w_{10} \circ \eta_{19} = P(\eta_{21}) = 2(\sigma_{10} \circ \nu_{17})$*

Proof. The first equation holds because $\eta_{21} = \Sigma^2\eta_{19}$. For the second, note that $\pi^{21}_{22} = \mathbb{Z}_2\eta_{21}$, so $P(\eta_{21})$ must generate the kernel of $\Sigma\colon \pi^{10}_{20} \to \pi^{11}_{21}$. We read off from [3, Theorem 7.3] the fact that this kernel is $\mathbb{Z}_2.(2\sigma_{10} \circ \nu_{17})$, and the claim follows. □

Lemma 12. *For $\alpha \in \pi^m_n$ and $\beta \in \pi^i_m$ (with $n > 1$ and $m, i > 0$) we have*

$$H(\beta \circ \alpha) = H(\beta) \circ \alpha + (\Sigma^{i-1}\beta) \circ (\Sigma^{m-1}\beta) \circ H(\alpha)$$

Proof. This is a special case of [1, Corollary III.6.3]. (Note that his γ_2 is our H, and his $\beta\#\beta$ is our $(\Sigma^{i-1}\beta) \circ (\Sigma^{m-1}\beta)$, whereas Toda's $\beta\#\beta$ is $\beta \wedge \beta = (\Sigma^i\beta) \circ (\Sigma^m\beta)$. **There are some signs to check here.**) □

Proposition 13. $(\Sigma\nu') \circ \sigma' = 2\Sigma\epsilon'$

Proof. Put

$$x = (\Sigma\nu' \circ \sigma') \in \pi^4_{14} = \mathbb{Z}_2(\eta_4 \circ \mu_5) \oplus \mathbb{Z}_4(\Sigma\epsilon') \oplus \mathbb{Z}_8(\nu_4 \circ \sigma').$$

We can write $x = a(\eta_4 \circ \mu_5) + b\Sigma\epsilon' + c(\nu_4 \circ \sigma')$ for some a, b, c. We now suspend this relation, using the facts that

$$\begin{aligned}
\pi^5_{15} &= \mathbb{Z}_2(\eta_5 \circ \mu_6) \oplus \mathbb{Z}_8(\nu_5 \circ \sigma_8) \\
\Sigma^2\nu' &= 2\nu_5 \\
\nu_5 \circ \Sigma\sigma' &= 2(\nu_5 \circ \sigma_8) \\
\Sigma((\Sigma\nu') \circ \sigma') &= (2\nu_5) \circ \Sigma\sigma' = 4(\nu_5 \circ \sigma_8) \\
\Sigma(\eta_4 \circ \mu_5) &= \eta_5 \circ \mu_6 \\
\Sigma(\Sigma\epsilon') &= 2(\nu_5 \circ \sigma_8) \\
\Sigma(\nu_4 \circ \sigma') &= 2(\nu_5 \circ \sigma_8)
\end{aligned}$$

This gives

$$4(\nu_5 \circ \sigma_8) = a(\eta_5 \circ \mu_6) + 2(b + c)(\nu_5 \circ \sigma_8),$$

so $a = 0$ and $2b + 2c = 4 \pmod 8$, so $b = 2 - c \pmod 4$. As b is really only defined mod 4 we can take $b = 2 - c$, so

$$x = c(\nu_4 \circ \sigma') + (2 - c)\Sigma\epsilon'.$$

We now take Hopf invariants, using the fact that

$$\begin{aligned} \pi^7_{14} &= \mathbb{Z}_8\sigma' \\ H(\Sigma\epsilon') &= 0 \\ H(\nu_4 \circ \sigma') &= \sigma' \\ H((\Sigma\nu') \circ \sigma') &= 0. \end{aligned}$$

We deduce that $c = 0$ and so $x = 2\Sigma\epsilon'$. □

Corollary 14. $\nu' \circ \sigma'' = 0$

Proof. We have $\Sigma\sigma'' = 2\sigma'$ by [3, Lemma 5.14]. Suspending the result of Proposition 13 therefore gives $\Sigma(\nu' \circ \sigma'') = 2((\Sigma\nu') \circ \sigma') = 4\Sigma\epsilon'$, but $4\epsilon' = 0$ by [3, Theorem 7.3] so $\Sigma(\nu' \circ \sigma'') = 0$. However, we have $\nu' \circ \sigma'' \in \pi^3_{13}$ and we can also read off from [3, Theorem 7.3] the fact that $\Sigma\colon \pi^3_{13} \to \pi^4_{14}$ is injective. It follows that $\nu' \circ \sigma'' = 0$. □

Proposition 15. $\mu_3 \circ \eta_{12} = \eta_3 \circ \mu_4$

Proof. We first show that $\mu_5 \circ \eta_{14} = \eta_5 \circ \mu_6$. Indeed, this is essentially the commutativity of the diagram

$$\begin{tikzcd} S^{12} \wedge S^3 \arrow[r, "1\wedge\eta_2"] \arrow[d, "\mu_3\wedge 1"'] & S^{12} \wedge S^2 \arrow[d, "\mu_3\wedge 1"] \\ S^3 \wedge S^3 \arrow[r, "1\wedge\eta_2"'] & S^3 \wedge S^2. \end{tikzcd}$$

There are some coordinate permutations to take care of, so we really get a relation of the form

$$(\pm\iota) \circ \mu_5 \circ (\pm\iota) \circ \eta_{14} \circ (\pm\iota) = (\pm\iota) \circ \eta_5 \circ (\pm\iota) \circ \mu_6 \circ (\pm\iota).$$

As μ_5, η_{14}, η_5 and μ_6 are all suspensions, the signs can be moved around freely. As $-\eta_5 = \eta_5$ and $-\eta_{14} = \eta_{14}$ we conclude that $\mu_5 \circ \eta_{14} = \eta_5 \circ \mu_6$ as claimed. Now consider the double suspension homomorphism

$$\Sigma^2\colon \pi^3_{13} \to \pi^5_{15}.$$

We know from [3, Theorem 7.3] that

$$\begin{aligned} \pi^3_{13} &= \mathbb{Z}_2(\eta_3 \circ \mu_4) \oplus \mathbb{Z}_4\epsilon' \\ \pi^5_{15} &= \mathbb{Z}_2(\eta_5 \circ \mu_6) \oplus \mathbb{Z}_8(\nu_5 \circ \sigma_8), \end{aligned}$$

and from [3, Equations 7.10] that $\Sigma^2\epsilon' = \pm(\nu_5 \circ \sigma_8)$. It follows that Σ^2 is injective. We have seen that

$$\Sigma^2((\mu_3 \circ \eta_{12}) - (\eta_3 \circ \mu_4)) = (\mu_5 \circ \eta_{14}) - (\eta_5 \circ \mu_6) = 0,$$

so $\mu_3 \circ \eta_{12} = \eta_3 \circ \mu_4$ as claimed. □

Proposition 16. $\epsilon' \circ \nu_{13} = 0$.

Proof. We have $\epsilon' \circ \nu_{13} \in \pi^3_{16} = \mathbb{Z}_2(\nu' \circ \eta_6 \circ \mu_7)$. We also have

$$H(\nu' \circ \eta_6 \circ \mu_7) = H(\nu') \circ \eta_6 \circ \mu_7 = \eta_5 \circ \eta_6 \circ \mu_7 = 4\zeta_5,$$

by Toda's Equations 5.3 and the proof of Equations 7.13. Toda's structure theorem for π^5_{16} shows that $4\zeta_5 \neq 0$, so $H\colon \pi^3_{16} \xrightarrow{\zeta}) \pi^5_{16}$ is injective. It will thus suffice to show that $H(\epsilon' \circ \nu_{13}) = 0$. As $H(\epsilon') = \epsilon_5$ (by Toda's lemma 6.6) and ν_{13} is a double suspension, we have $H(\epsilon' \circ \nu_{13}) = \epsilon_5 \circ \nu_{13}$, and this is zero by Equations 7.13, as required. □

Proposition 17. $H(\bar{\epsilon}_3) = \nu_5 \circ \sigma_8 \circ \nu_{15}$.

Proof. We know from [3, Lemma 10.2] that

$$H(\overline{\epsilon}_3) = \nu_5 \circ \sigma_8 \circ \nu_{15} \pmod{\nu_5 \circ \eta_8 \circ \mu_9},$$

and from [3, Theorem 7.7] that

$$\pi^5_{18} = \mathbb{Z}_2(\nu_5 \circ \sigma_8 \circ \nu_{15}) \oplus \mathbb{Z}_2(\nu_5 \circ \eta_8 \circ \mu_9),$$

so

$$H(\overline{\epsilon}_3) \in \{\nu_5 \circ \sigma_8 \circ \nu_{15}, \nu_5 \circ \sigma_8 \circ \nu_{15} + \nu_5 \circ \eta_8 \circ \mu_9\}.$$

We also have $PH = 0$ and

$$\begin{aligned} P(\nu_5 \circ \sigma_8 \circ \nu_{15}) &= P(\nu_5 \circ \sigma_8) \circ \nu_{13} \\ &= \eta_2 \circ \epsilon' \circ \nu_{13} (\text{Corollary 3}) \\ &= 0 (\text{Proposition 16}) \\ P(\nu_5 \circ \eta_8 \circ \mu_9) &= P(\nu_5) \circ \eta_6 \circ \mu_7 \\ &= \eta_2 \circ \nu' \circ \eta_6 \circ \mu_7 [3, \text{Lemma 5.7}] \\ &\neq 0 [3, \text{Theorem 10.3}] \end{aligned}$$

We must therefore have $H(\overline{\epsilon}_3) = \nu_5 \circ \sigma_8 \circ \nu_{15}$ as claimed. □

Proposition 18. *There is a unique element $\zeta_5 \in \langle \nu_5, 8\iota_8, \Sigma\sigma' \rangle_1$ such that $H(\zeta_5) = 8\sigma_9$ and $P(\zeta_5) = \pm\eta_2 \circ \mu'$.*

Proof. Put $B = \langle \nu_5, 8\iota_8, \Sigma\sigma' \rangle_1$ and choose $\xi \in B$ (so ξ is a "choice of ζ_5"). Toda shows in [3, Section VI(v)] that B is a well-defined coset of $\nu_5 \circ \Sigma\pi^7_{15}$ in π^5_{16}. We read off from [3, Theorem 7.1] that the indeterminacy is generated by $\nu_5 \circ \Sigma\sigma' \circ \eta_{15}$, $\nu_5 \circ \overline{\nu}_8$ and $\nu_5 \circ \epsilon_8$. We also see from [3, Equations 7.16] that $\nu_5 \circ \Sigma\sigma' = 2\nu_5 \circ \sigma_8$ which implies $\nu_5 \circ \Sigma\sigma' \circ \eta_{15} = 0$. We also have $H(\xi) = 8\sigma_9$ and $4\xi = \eta_5 \circ \eta_6 \circ \mu_7$ by [3, Lemma 6.7]. (The second equation is stated modulo $\nu_5 \circ 2\Sigma\pi^7_{15}$, but we see from [3, Theorem 7.1] that $2\Sigma\pi^7_{15} = 0$.)

We now consider the exact sequence

$$\pi^3_{16} \xrightarrow{H} \pi^5_{16} \xrightarrow{P} \pi^2_{14}.$$

The groups involved are calculated in Theorems 7.7, 7.4 and 7.6 of [3], giving the following exact sequence:

$$\mathbb{Z}_2(\nu'\eta_6\mu_7) \xrightarrow{H} \mathbb{Z}_2(\nu_5\overline{\nu}_8) \oplus \mathbb{Z}_2(\nu_5\epsilon_8) \oplus \mathbb{Z}_8(\xi) \xrightarrow{P} \mathbb{Z}_2(\eta_2\epsilon_3\nu_{11}) \oplus \mathbb{Z}_2(\eta_2\nu'\epsilon_6) \oplus \mathbb{Z}_4(\eta_2\mu').$$

We know from [3, Equations 5.3] that $H(\nu') = \eta_5$, and $\eta_6 \circ \mu_7$ is a suspension, so

$$H(\nu' \circ \eta_6 \circ \mu_7) = H(\nu') \circ \eta_6 \circ \mu_7 = \eta_5 \circ \eta_6 \circ \mu_7 = 4\xi.$$

Because the sequence is exact, the image of P must have order 16, so P must be surjective. We can thus choose an element $\xi' = x\xi + y(\nu_5 \circ \overline{\nu}_8) + z(\nu_5 \circ \epsilon_8)$ such that $P(\xi') = \eta_2 \circ \mu'$. As this has order 4, we see that x must be odd, so we can put $\zeta_5 = \xi'/x$ and note that this lies in the coset B. As $x = \pm 1 \pmod 4$ we have $P(\zeta_5) = \pm\eta_2 \circ \mu'$. One can work back through the argument to see that ζ_5 is unique. □

Corollary 19. $\nu_5 \circ \sigma_8 \circ \eta_{15} = \nu_5 \circ \epsilon_8 \pmod{4\zeta_5}$.

Proof. We saw in the previous proof that $4\zeta_5$ generates the kernel of $P\colon \pi^5_{16} \to \pi^2_{14}$, so it will suffice to show that $P(\nu_5 \circ \sigma_8 \circ \eta_{15}) = P(\nu_5 \circ \epsilon_8)$. Corollary 3 gives $P(\nu_5 \circ \sigma_8) = \pm\eta_2 \circ \epsilon'$, so the rule $P(\alpha \circ \Sigma^2\beta) = P(\alpha) \circ \beta$ gives $P(\nu_5 \circ \sigma_8 \circ \eta_{15}) = (\pm\eta_2 \circ \epsilon') \circ \eta_{13}$. As η_{13} is still a suspension and has order 2, the sign can be ignored giving $\eta_2 \circ \epsilon' \circ \eta_{13}$. Using [3, Equations 7.12] we can convert this to $\eta_2 \circ \nu' \circ \epsilon_6$. On the other hand, we have $P(\nu_5) = \pm\eta_2 \circ \nu'$ by [3, Lemma 5.7], so $P(\nu_5 \circ \epsilon_8) = (\pm\eta_2 \circ \nu') \circ \epsilon_6 = \eta_2 \circ \nu' \circ \epsilon_6$ as required. □

Proposition 20. *$\nu_6 \circ \nu_9 \circ \nu_{12} \circ \nu_{15} = 0$.*

Proof. We will show that $P(\overline{\nu}_{11}) = \nu_5 \circ \nu_8 \circ \nu_{11} \circ \nu_{14}$; the claim will follow by suspending this. First, we have

$$P(\overline{\nu}_{11}) = P(\iota_{11} \circ \Sigma^2 \overline{\nu}_9) = w_5 \circ \overline{\nu}_9.$$

We know from [3, Equations 5.10] that $w_5 = \nu_5 \circ \eta_8$, and from [3, Lemma 6.3] that $\eta_8 \circ \overline{\nu}_9 = \nu_8 \circ \nu_{11} \circ \nu_{14}$. Putting this together gives the result. □

Proposition 21. $\nu_5 \circ \nu_8 \circ \sigma_{11} = 0$.

Proof. $\nu_5 \circ \nu_8 \circ \sigma_{11}$ is in the common kernel of the maps $H\colon \pi^5_{18} \to \pi^9_{18}$ and $P\colon \pi^5_{18} \to \pi^2_{16}$, which is trivial. **Fill in the details** □

Proposition 22. $\nu_6 \circ \overline{\nu}_9 = 2\overline{\nu}_6 \circ \nu_{14}$

Proof. We know from [3, Equations 7.17] that $P(\sigma_{11}) = \nu_5 \circ \epsilon_8 + \nu_5 \circ \overline{\nu}_8$. Suspending this relation and using [3, Eqations 7.18] gives $\nu_6 \circ \overline{\nu}_9 = -\nu_6 \circ \epsilon_9 = -2\overline{\nu}_6 \circ \nu_{14}$. We can drop the minus sign because $4\overline{\nu}_6 \circ \nu_{14} = 0$ (by [3, Theorem 7.4]). □

Proposition 23. $\overline{\nu}_{10} \circ \nu_{18} = 0$

Proof. We will show that $\overline{\nu}_9 \circ \nu_{17} = P(\nu_{19})$; the claim follows by suspending this.

Using the rule $P(\alpha \circ \Sigma^2\beta) = P(\alpha) \circ \beta$ and [3, Equations 7.1] we have

$$P(\nu_{19}) = P(\iota_{19}) \circ \nu_{17} = \sigma_9 \circ \eta_{16} \circ \nu_{17} + \overline{\nu}_9 \circ \nu_{17} + \epsilon_9 \circ \nu_{17}.$$

Next, we see from [3, Equations 5.9] that $\eta_{16} \circ \nu_{17} = 0$. Moreover, [3, Equations 7.13] gives $\epsilon_4 \circ \nu_{12} = P(\epsilon_9)$, and suspending this five times gives $\epsilon_9 \circ \nu_{17} = 0$. Putting these relations into the displayed equation gives $P(\nu_{19}) = \overline{\nu}_9 \circ \nu_{17}$ as required. □

Proposition 24. $\sigma''' \circ \epsilon_{12} = 0$.

Proof. We have $\Sigma\sigma''' = 2\sigma''$ by [3, Lemma 5.14], and $2\epsilon_{13} = 0$ by [3, Theorem 7.1], so $\Sigma(\sigma''' \circ \epsilon_{12}) = 0$. Moreover, we know from [3, Equations 10.14] that $\Sigma\colon \pi^5_{20} \to \pi^6_{21}$ is injective, so $\sigma''' \circ \epsilon_{12} = 0$ as claimed. □

Proposition 25. $\sigma''' \circ \overline{\nu}_{12} = 0$.

Proof. We have $\Sigma\sigma''' = 2\sigma''$ by [3, Lemma 5.14], and $2\overline{\nu}_{13} = 0$ by [3, Theorem 7.1], so $\Sigma(\sigma''' \circ \overline{\nu}_{12}) = 0$. Moreover, we know from [3, Equations 10.14] that $\Sigma\colon \pi^5_{20} \to \pi^6_{21}$ is injective, so $\sigma''' \circ \overline{\nu}_{12} = 0$ as claimed. □

Corollary 26. *There is no indeterminacy in the definition* $\rho^{(4)} = \langle \sigma''', 2\iota_{12}, 8\sigma_{12}\rangle_1$.

Proof. The general rule in [3, Lemma 1.1] says that the indeterminacy is $\sigma''' \circ \Sigma\pi^{11}_{19} + \pi^5_{13} \circ (8\sigma_{13})$. We see from [3, Theorem 7.1] and Propositions 24 and 25 that $\sigma''' \circ \Sigma\pi^{11}_{19} = 0$. Moreover, $\rho^{(4)}$ lies in π^5_{20}, which has exponent 2 by [3, Theorem 10.5], so $\pi^5_{13} \circ (8\sigma_{13}) = 0$. □

Proposition 27. $\sigma'' \circ \epsilon_{13} = 0$.

Proof. We have $\Sigma\sigma'' = 2\sigma'$ by [3, Lemma 5.14], and $2\epsilon_{14} = 0$ by [3, Theorem 7.1], so $\Sigma(\sigma'' \circ \epsilon_{13}) = 0$. Moreover, we know from [3, Equations 10.14] that $\Sigma\colon \pi^6_{21} \to \pi^7_{22}$ is injective, so $\sigma'' \circ \epsilon_{13} = 0$ as claimed. □

Proposition 28. $\sigma'' \circ \overline{\nu}_{13} = 0$.

Proof. We have $\Sigma\sigma'' = 2\sigma'$ by [3, Lemma 5.14], and $2\overline{\nu}_{14} = 0$ by [3, Theorem 7.1], so $\Sigma(\sigma'' \circ \overline{\nu}_{13}) = 0$. Moreover, we know from [3, Equations 10.14] that $\Sigma\colon \pi^6_{21} \to \pi^7_{22}$ is injective, so $\sigma'' \circ \overline{\nu}_{13} = 0$ as claimed. □

Corollary 29. *There is no indeterminacy in the definition* $\rho''' = \langle \sigma'', 4\iota_{13}, 4\sigma_{13}\rangle_1$.

Proof. The general rule in [3, Lemma 1.1] says that the indeterminacy is $\sigma'' \circ \Sigma\pi^{12}_{20} + \pi^6_{14} \circ (4\sigma_{14})$. We see from [3, Theorem 7.1] and Propositions 24 and 25 that $\sigma'' \circ \Sigma\pi^{12}_{20} = 0$. Moreover, ρ''' lies in π^6_{21}, which has exponent 4 by [3, Theorem 10.5], so $\pi^6_{14} \circ (4\sigma_{14}) = 0$. □

Proposition 30. *There is a unique element $\rho'' \in \langle \sigma', 8\iota_{14}, 2\sigma_{14}\rangle_1$ that satisfies $H(\rho'') = \mu_{13}$.*

Proof. Put $B = \langle \sigma', 8\iota_{14}, 2\sigma_{14}\rangle_1$ and let ξ be any element of B. We know from [3, Equations 10.12] that

$$H(\xi) = \mu_{13} + x\nu_{13} \circ \nu_{16} \circ \nu_{19} + y\eta_{13} \circ \epsilon_{14}$$

for some integers x and y. Next, we know from [3, Lemma 5.14] that $H(\sigma') = \eta_{13}$, so $H(\sigma' \circ \epsilon_{14}) = \eta_{13} \circ \epsilon_{14}$ and $H(\sigma' \circ \overline{\nu}_{14}) = \eta_{13} \circ \overline{\nu}_{14}$, which is the same as $\nu_{13} \circ \nu_{16} \circ \nu_{19}$ by [3, Lemma 6.3]. It follows that the element $\rho'' = \xi - x\sigma' \circ \overline{\nu}_{14} - y\sigma' \circ \epsilon_{14}$ has $H(\rho'') = \mu_{13}$. The indeterminacy of B is $\sigma' \circ \Sigma\pi^{13}_{21} + \pi^7_{15} \circ (2\sigma_{15})$. We see from [3, Theorem 7.1] that π^7_{15} has exponent 2 and $\Sigma\pi^{13}_{21} = \mathbb{Z}_2\epsilon_{14} \oplus \mathbb{Z}_2\overline{\nu}_{14}$; it follows that the indeterminacy in B is generated by $\sigma' \circ \overline{\nu}_{14}$ and $\sigma' \circ \epsilon_{14}$. This shows that $\rho'' \in B$, and that ρ'' is the unique element in B with $H(\rho'') = \mu_{13}$. □

Proposition 31. $H(\overline{\epsilon}') = \overline{\epsilon}_5 \pmod{\rho^{(4)}}$.

Proof. Using [3, Theorem 12.7 and Lemma 12.3] we see that the map $\Sigma\colon \pi^2_{19} \to \pi^3_{20}$ is as follows:

$$\begin{gathered}\mathbb{Z}_2(\eta_2\mu_3\sigma_{12}) \oplus \mathbb{Z}_2(\eta_2\eta_3\overline{\epsilon}_4) \to \mathbb{Z}_2(\eta_3\mu_4\sigma_{13}) \oplus \mathbb{Z}_4(\overline{\epsilon}') \oplus \mathbb{Z}_2(\overline{\mu}_3)\\ \eta_2\mu_3\sigma_{12} \mapsto \eta_3\mu_4\sigma_{13}\\ \eta_2\eta_3\overline{\epsilon}_4 \mapsto 2\overline{\epsilon}'.\end{gathered}$$

As the EHP sequence is exact, we see that H induces a monomorphism from $\mathbb{Z}_2(\overline{\epsilon}') \oplus \mathbb{Z}_2(\overline{\mu}_3)$ to the group $\pi^5_{20} = \mathbb{Z}_2(\overline{\epsilon}_5) \oplus \mathbb{Z}_2(\rho^{(4)})$. We also learn from [3, Lemma 12.2] that $H(\overline{\mu}_3) = \rho^{(4)}$. The proposition follows. □

Proposition 32. *There is a unique element $\zeta' \in \pi^6_{22}$ such that $H(\zeta') = \zeta_{11}$ and $\Sigma\zeta' = \sigma' \circ \eta_{14}$.*

Proof. We know from [3, Lemma 12.1] that there is an element in π^6_{22}, which we temporarily call ξ, such that $H(\xi) = \zeta_{11} \pmod{2\zeta_{11}}$ and $\Sigma\xi = \sigma' \circ \eta_{14} \circ \epsilon_{15} \pmod{2\pi^7_{23}}$. We see from [3, Theorem 12.6] that $2\pi^7_{23} = 0$ and thus that $\Sigma(n\xi) = \sigma' \circ \eta_{14} \circ \epsilon_{15}$ for any odd n. As $H(\xi) = \zeta_{11} \pmod{2\zeta_{11}}$, we can choose n so that $H(n\xi) = \zeta_{11}$, and we then put $\zeta' = n\xi$. Uniqueness is left to the reader. □

Proposition 33. $H(\kappa_7) = \epsilon_{13}$.

Proof. We first note from [3, Theorem 7.1] that $\pi^{13}_{21} = \mathbb{Z}_2\epsilon_{13} \oplus \mathbb{Z}_2\overline{\nu}_{13}$, and the P homomorphism kills both generators by [3, Equations 7.27], so $H\colon \pi^7_{21} \to \pi^{13}_{21}$ must be surjective. We have $\pi^7_{21} = \mathbb{Z}_8\sigma' \circ \sigma_{14} \oplus \mathbb{Z}_4\kappa_7$ by [3, Theorem 10.3], and $H(\sigma' \circ \sigma_{14}) = H(\sigma') \circ \sigma_{14} = \epsilon_{13} + \overline{\nu}_{13}$ by [3, Lemmas 5.14 and 6.4]. It follows that $H(\kappa_7)$ must be ϵ_{13} or $\overline{\nu}_{13}$. I think that Λ-algebra representatives are as follows:

$$\begin{aligned}\overline{\nu}_{13} &\sim \lambda_{53}\\ \epsilon_{13} &\sim \lambda_{233}\\ \kappa_7 &\sim \lambda_{6233} + \lambda_{4721} + \lambda_{3623} + \lambda_{3443}\\ H(\kappa_7) &\sim \lambda_{233}.\end{aligned}$$

In particular, $H(\kappa_7)$ has Adams filtration 3 so it cannot equal $\overline{\nu}_{13}$, and must therefore equal ϵ_{13} instead. □

Proposition 34. $(\Sigma\sigma') \circ \nu_{15} \circ \nu_{18} = \nu_8 \circ \sigma_{11} \circ \nu_{18}$.

Proof. The claimed equation takes place in the group

$$\pi^8_{21} = \mathbb{Z}_2(\nu_8 \circ \sigma_{11} \circ \nu_{18}) \oplus \mathbb{Z}_2(\sigma_8 \circ \nu_{15} \circ \nu_{18}).$$

This has exponent two so we need not worry about signs.

We will show that both sides are equal to $P(\nu_{17} \circ \nu_{20})$. For the right hand side, this is proved by Toda, just above Equations 7.28. For the left hand side, we have $P(\nu_{17} \circ \nu_{20}) = w_8 \circ \nu_{15} \circ \nu_{18}$, and $w_8 = \pm(2\sigma_8 - \Sigma\sigma')$ by [3, Equations 5.16]. As $\nu_{15} \circ \nu_{18}$ is a suspension we can distribute, and as π^8_{21} has exponent 2 we are just left with $(\Sigma\sigma') \circ \nu_{15} \circ \nu_{18}$ as claimed. □

Proposition 35. $\sigma' \circ \nu_{14} = \nu_7 \circ \sigma_{10} \pmod{2\nu_7 \circ \sigma_{10}}$

Proof. This takes place in $\pi_{17}S^7 = \mathbb{Z}_2(\eta_7 \circ \mu_8) \oplus \mathbb{Z}_8(\nu_7 \circ \sigma_{10})$, which maps to $\pi_{10}^S = \mathbb{Z}_2(\eta \circ \mu)$. We know that $\nu\sigma = 0$ stably, so $\sigma\nu = 0$ stably, but $\Sigma^2\sigma' = 2\sigma_9$ so $\sigma'\nu$ also vanishes in the stable group. It follows that $\sigma' \circ \nu_{14} = m\nu_7 \circ \sigma_{10}$ for some integer m. The part of the relevant unstable Adams E_2 term that contributes to $\pi_{17}S^7$ has rank 4, with a single $\mathbb{Z}_2$ in each of filtrations 3, 4, 5 and 6, the first generator corresponding to λ_{433} in the lambda algebra. One checks that σ' is represented by $\lambda_{43} + \lambda_{61}$ and ν_{14} is represented by λ_3 so $\sigma' \circ \nu_{14}$ is represented by $(\lambda_{43} + \lambda_{61})\lambda_3 = \lambda_{433}$. This means that $\sigma' \circ \nu_{14}$ has minimal Adams filtration and so cannot be divided by 2. It follows that m is odd. □

4. Questions

What are the following elements, in terms of Toda's generators for the groups in which they live?

- $\sigma'\nu_{14} \in \pi_{17}S^7$
- $\nu_9\sigma_{12} \in \pi_{19}S^9$
- $\overline{\nu}_7\nu_{15}\nu_{18} \in \pi_{21}S^7$
- $\sigma_{10}\epsilon_{17} \in \pi_{25}S^{10}$
- $\sigma'''\sigma_{12} \in \pi_{19}S^5$
- $H(\overline{\epsilon}') \in \pi_{20}S^5$
- $\eta_{13}\eta_{14}\mu_{15} \in \pi_{24}S^{13}$
- $P(\sigma_9) \in \pi_{14}S^4$
- $\eta_9\overline{\epsilon}_{10} \in \pi_{25}S^9$
- $\nu_5\sigma_8\eta_{15} \in \pi_{16}S^9$

(This is only a sample of the open questions in Toda's range.)

References

[1] H. J. Baues. *Commutator calculus and groups of homotopy classes*, volume 50 of *London Mathematical Society Lecture Notes*. Cambridge University Press, 1981.

[2] M. E. Mahowald. *The Metastable Homotopy of S^n*, volume 72 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 1967.

[3] H. Toda. *Composition Methods in Homotopy Groups of Spheres*. Number 49 in Annals of Mathematics Studies. Princeton University Press, Princeton, 1962.