Vector fields in polar coordinates
Two dimensions

At any point in the plane, we can define vectors $\mathbf{r}_r$ and $\mathbf{e}_\theta$ as shown:

\[ \mathbf{r}_r = \cos(\theta) \mathbf{i} + \sin(\theta) \mathbf{j} \]
\[ \mathbf{e}_\theta = -\sin(\theta) \mathbf{i} + \cos(\theta) \mathbf{j} \]
\[ \mathbf{i} = \cos(\theta) \mathbf{r}_r - \sin(\theta) \mathbf{e}_\theta \]
\[ \mathbf{j} = \sin(\theta) \mathbf{r}_r + \cos(\theta) \mathbf{e}_\theta \]
At any point in the plane, we can define vectors $\mathbf{r}_r$ and $\mathbf{e}_\theta$ as shown:
Two dimensions

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\[ e_r = \cos(\theta) i + \sin(\theta) j \]
\[ e_\theta = -\sin(\theta) i + \cos(\theta) j \]

\[ i = \cos(\theta) e_r - \sin(\theta) e_\theta \]
\[ j = \sin(\theta) e_r + \cos(\theta) e_\theta \]
At any point in the plane, we can define vectors $\mathbf{r}$ and $\mathbf{e}_{\theta}$ as shown:

In situations with circular symmetry, it is often more natural to describe vector fields in terms of $\mathbf{e}_r$ and $\mathbf{e}_{\theta}$ rather than $\mathbf{i}$ and $\mathbf{j}$. 
Two dimensions

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\[ \mathbf{e}_r = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j} \quad \mathbf{e}_\theta = -\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j} \]

In situations with circular symmetry, it is often more natural to describe vector fields in terms of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ rather than $\mathbf{i}$ and $\mathbf{j}$. One can translate between the two descriptions as follows:

\[ \mathbf{e}_r = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j} \quad \mathbf{e}_\theta = -\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j} \]
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In situations with circular symmetry, it is often more natural to describe vector fields in terms of $\mathbf{r}_r$ and $\mathbf{e}_\theta$ rather than $\mathbf{i}$ and $\mathbf{j}$. One can translate between the two descriptions as follows:

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\mathbf{e}_r = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j} \quad \quad \mathbf{e}_\theta = -\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j}
\]
\[
\mathbf{i} = \cos(\theta)\mathbf{e}_r - \sin(\theta)\mathbf{e}_\theta \quad \quad \mathbf{j} = \sin(\theta)\mathbf{e}_r + \cos(\theta)\mathbf{e}_\theta.
\]
Here are two examples of vector fields described in terms of $\mathbf{e}_r$ and $\mathbf{e}_\theta$:

\[ \mathbf{u} = \sin(\theta)\mathbf{e}_r \]

\[ \mathbf{u} = \sqrt{r}(\mathbf{e}_\theta + \mathbf{e}_r/10) \]
Div, grad and curl in polar coordinates

We will need to express the operators grad, div and curl in terms of polar coordinates.

(a) For any two-dimensional scalar field $f$ (expressed as a function of $r$ and $\theta$) we have

$$\nabla (f) = \text{grad}(f) = f r e_r + r^{-1} f \theta e_\theta.$$

(b) For any 2-dimensional vector field $\mathbf{u} = m e_r + p e_\theta$ (where $m$ and $p$ are expressed as functions of $r$ and $\theta$) we have

$$\text{div}(\mathbf{u}) = r^{-1} m + m r + r^{-1} p \theta = r^{-1} (r m + p \theta).$$

$$\text{curl}(\mathbf{u}) = r^{-1} p + p r - r^{-1} m \theta = r^{-1} (r p - m \theta).$$

Note that the product rule gives $(rm)r = m + rm r$ and $(rp)r = p + rp r$.

(c) For any two-dimensional scalar field $f$ we have

$$\nabla^2 (f) = r^{-1} f r + f_{rr} + r^{-2} f_{\theta\theta} = r^{-1} (rf r) r + r^{-2} f_{\theta\theta}.$$

Note: in the exam, if you need these formulae, they will be provided.
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$$\text{div}(\mathbf{u}) = r^{-1} m + m_r + r^{-1} p_\theta$$

$$\text{curl}(\mathbf{u}) = r^{-1} p + p_r - r^{-1} m_\theta$$

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$$\text{div}(\mathbf{u}) = r^{-1} m + m_r + p \quad \text{and} \quad \text{curl}(\mathbf{u}) = r^{-1} p + p_r - r^{-1} m_\theta$$

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$$\text{div}(\mathbf{u}) = r^{-1} m + m_r + r^{-1} p_\theta = r^{-1} ((rm)_r + p_\theta)$$
$$\text{curl}(\mathbf{u}) = r^{-1} p + p_r - r^{-1} m_\theta = r^{-1} ((rp)_r - m_\theta)$$

Note that the product rule gives $(rm)_r = m + r m_r$ and $(rp)_r = p + r p_r$. 

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(b) For any 2-dimensional vector field $\mathbf{u} = m e_r + p e_\theta$ (where $m$ and $p$ are expressed as functions of $r$ and $\theta$) we have

$$\text{div(}\mathbf{u}\text{)} = r^{-1} m + m_r + r^{-1} p_\theta = r^{-1} ((rm)_r + p_\theta)$$

$$\text{curl(}\mathbf{u}\text{)} = r^{-1} p + p_r - r^{-1} m_\theta = r^{-1} ((rp)_r - m_\theta)$$

$$= \frac{1}{r} \det \begin{bmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} \\ m & rp \end{bmatrix}.$$ 

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$$\text{div}(\mathbf{u}) = r^{-1} m + m_r + r^{-1} p_\theta = r^{-1} ((r m)_r + p_\theta)$$

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(c) For any two-dimensional scalar field $f$ we have

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\nabla(f) = \text{grad}(f) = f_r \, \mathbf{e}_r + r^{-1} f_\theta \, \mathbf{e}_\theta.
\]

(b) For any 2-dimensional vector field \( \mathbf{u} = m \mathbf{e}_r + p \mathbf{e}_\theta \) (where \( m \) and \( p \) are expressed as functions of \( r \) and \( \theta \)) we have

\[
\begin{align*}
\text{div}(\mathbf{u}) &= r^{-1} m + m_r + r^{-1} p_\theta = r^{-1} \left( (rm)_r + p_\theta \right) \\
\text{curl}(\mathbf{u}) &= r^{-1} p + p_r - r^{-1} m_\theta = r^{-1} \left( (rp)_r - m_\theta \right) \\
&= \frac{1}{r} \det \begin{bmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} \\ m & rp \end{bmatrix}.
\end{align*}
\]

Note that the product rule gives \((rm)_r = m + r m_r\) and \((rp)_r = p + r p_r\).

(c) For any two-dimensional scalar field \( f \) we have

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\nabla^2(f) = r^{-1} f_r + f_{rr} + r^{-2} f_{\theta\theta} = r^{-1} (rf_r)_r + r^{-2} f_{\theta\theta}
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$$\text{div}(\mathbf{u}) = r^{-1} m + m_r + \frac{1}{r} p_\theta = r^{-1} (rm)_r + p_\theta$$

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For any two-dimensional scalar field \( f \) (as a function of \( r \) and \( \theta \)) we have

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\nabla(f) = \text{grad}(f) = f_r \mathbf{e}_r + r^{-1} f_\theta \mathbf{e}_\theta.
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Grad in polar coordinates

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\nabla (f) = \text{grad}(f) = f_r e_r + r^{-1} f_\theta e_\theta.
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**Justification:** Consider the field \( u = f_r e_r + r^{-1} f_\theta e_\theta \); we show that this is the same as \( \text{grad}(f) \).

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\[
f_r = f_x x_r + f_y y_r.
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$$x = r \cos(\theta) \quad y = r \sin(\theta)$$
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\begin{align*}
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\[
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x_r &= \cos(\theta) \\
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$$x_r = \cos(\theta) \quad y_r = \sin(\theta)$$

$$x_\theta = -r \sin(\theta) \quad y_\theta = r \cos(\theta), \text{ so}$$

$$f_r = f_x x_r + f_y y_r = \cos(\theta) f_x + \sin(\theta) f_y$$
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$$x_r = \cos(\theta) \quad \quad \quad y_r = \sin(\theta)$$
$$x_\theta = -r \sin(\theta) \quad \quad \quad y_\theta = r \cos(\theta),$$

$$f_r = f_x x_r + f_y y_r = \cos(\theta) f_x + \sin(\theta) f_y$$
$$f_\theta = f_x x_\theta + f_y y_\theta$$
Grad in polar coordinates

For any two-dimensional scalar field \( f \) (as a function of \( r \) and \( \theta \)) we have

\[
\nabla(f) = \text{grad}(f) = f_r \mathbf{e}_r + r^{-1} f_\theta \mathbf{e}_\theta.
\]

**Justification:** Consider the field \( u = f_r \mathbf{e}_r + r^{-1} f_\theta \mathbf{e}_\theta \); we show that this is the same as \( \text{grad}(f) \). Two-variable chain rule: suppose we make a small change \( \delta r \) to \( r \). This causes a change \( \delta x \simeq x_r \delta r \) to \( x \), which in turn causes a change \( \simeq f_x \delta x \simeq f_x x_r \delta r \) to \( f \). At the same time, our change in \( r \) also causes a change \( \delta y \simeq y_r \delta r \) to \( x \), which causes a change \( \simeq f_y \delta y = f_y y_r \delta r \) to \( f \). Altogether, the change in \( f \) is \( \delta f \simeq (f_x x_r + f_y y_r) \delta r \). By passing to the limit \( \delta r \to 0 \), we get \( f_r = f_x x_r + f_y y_r \). Similarly, \( f_\theta = f_x x_\theta + f_y y_\theta \). Moreover, we can differentiate the formulae

\[
\begin{align*}
x &= r \cos(\theta) \\
y &= r \sin(\theta)
\end{align*}
\]

to get

\[
\begin{align*}
x_r &= \cos(\theta) \\
x_\theta &= -r \sin(\theta) \\
y_r &= \sin(\theta) \\
y_\theta &= r \cos(\theta), \text{ so}
\end{align*}
\]

\[
\begin{align*}
f_r &= f_x x_r + f_y y_r = \cos(\theta)f_x + \sin(\theta)f_y \\
f_\theta &= f_x x_\theta + f_y y_\theta = -r \sin(\theta)f_x + r \cos(\theta)f_y
\end{align*}
\]
Grad in polar coordinates

For any two-dimensional scalar field $f$ (as a function of $r$ and $\theta$) we have
\[
\nabla(f) = \text{grad}(f) = f_r \mathbf{e}_r + r^{-1} f_\theta \mathbf{e}_\theta.
\]

**Justification:** Consider the field $u = f_r \mathbf{e}_r + r^{-1} f_\theta \mathbf{e}_\theta$; we show that this is the same as $\text{grad}(f)$. Two-variable chain rule: suppose we make a small change $\delta r$ to $r$. This causes a change $\delta x \simeq x_r \delta r$ to $x$, which in turn causes a change $\simeq f_x \delta x \simeq f_x x_r \delta r$ to $f$. At the same time, our change in $r$ also causes a change $\delta y \simeq y_r \delta r$ to $x$, which causes a change $\simeq f_y \delta y = f_y y_r \delta r$ to $f$. Altogether, the change in $f$ is $\delta f \simeq (f_x x_r + f_y y_r) \delta r$. By passing to the limit $\delta r \to 0$, we get $f_r = f_x x_r + f_y y_r$. Similarly, $f_\theta = f_x x_\theta + f_y y_\theta$. Moreover, we can differentiate the formulae
\[
x = r \cos(\theta) \quad \quad \quad \quad \quad \quad \quad y = r \sin(\theta)
\]
and get
\[
x_r = \cos(\theta) \quad \quad \quad \quad \quad \quad \quad y_r = \sin(\theta)
\]
\[
x_\theta = -r \sin(\theta) \quad \quad \quad \quad \quad \quad \quad y_\theta = r \cos(\theta), \text{ so}
\]
\[
f_r = f_x x_r + f_y y_r = \cos(\theta) f_x + \sin(\theta) f_y
\]
\[
f_\theta = f_x x_\theta + f_y y_\theta = -r \sin(\theta) f_x + r \cos(\theta) f_y
\]
\[
u = f_r \mathbf{e}_r + r^{-1} f_\theta \mathbf{e}_\theta
\]
Grad in polar coordinates

For any two-dimensional scalar field \( f \) (as a function of \( r \) and \( \theta \)) we have

\[
\nabla(f) = \text{grad}(f) = f_r \mathbf{e}_r + r^{-1}f_\theta \mathbf{e}_\theta.
\]

**Justification:** Consider the field \( u = f_r \mathbf{e}_r + r^{-1}f_\theta \mathbf{e}_\theta \); we show that this is the same as \( \text{grad}(f) \). Two-variable chain rule: suppose we make a small change \( \delta r \) to \( r \). This causes a change \( \delta x \simeq x_r \delta r \) to \( x \), which in turn causes a change \( \simeq f_x \delta x \simeq f_x x_r \delta r \) to \( f \). At the same time, our change in \( r \) also causes a change \( \delta y \simeq y_r \delta r \) to \( x \), which causes a change \( \simeq f_y \delta y \simeq f_y y_r \delta r \) to \( f \). Altogether, the change in \( f \) is \( \delta f \simeq (f_x x_r + f_y y_r)\delta r \). By passing to the limit \( \delta r \to 0 \), we get \( f_r = f_x x_r + f_y y_r \). Similarly, \( f_\theta = f_x x_\theta + f_y y_\theta \). Moreover, we can differentiate the formulae

\[
\begin{align*}
x &= r \cos(\theta) \\
y &= r \sin(\theta)
\end{align*}
\]

to get

\[
\begin{align*}
x_r &= \cos(\theta) \\
x_\theta &= -r \sin(\theta) \\
y_r &= \sin(\theta) \\
y_\theta &= r \cos(\theta),
\end{align*}
\]

\[
\begin{align*}
f_r &= f_x x_r + f_y y_r = \cos(\theta) f_x + \sin(\theta) f_y \\
f_\theta &= f_x x_\theta + f_y y_\theta = -r \sin(\theta) f_x + r \cos(\theta) f_y \\
u &= f_r \mathbf{e}_r + r^{-1}f_\theta \mathbf{e}_\theta = f_x \cos(\theta) \mathbf{e}_r + f_y \sin(\theta) \mathbf{e}_r - f_x \sin(\theta) \mathbf{e}_\theta + f_y \cos(\theta) \mathbf{e}_\theta
\end{align*}
\]
For any two-dimensional scalar field \( f \) (as a function of \( r \) and \( \theta \)) we have

\[
\nabla(f) = \text{grad}(f) = f_r e_r + r^{-1} f_\theta e_\theta.
\]

**Justification:** Consider the field \( u = f_r e_r + r^{-1} f_\theta e_\theta \); we show that this is the same as \( \text{grad}(f) \). Two-variable chain rule: suppose we make a small change \( \delta r \) to \( r \). This causes a change \( \delta x \simeq x_r \delta r \) to \( x \), which in turn causes a change \( \sim f_x \delta x \simeq f_x x_r \delta r \) to \( f \). At the same time, our change in \( r \) also causes a change \( \delta y \simeq y_r \delta r \) to \( x \), which causes a change \( \sim f_y \delta y = f_y y_r \delta r \) to \( f \). Altogether, the change in \( f \) is \( \delta f \simeq (f_x x_r + f_y y_r) \delta r \). By passing to the limit \( \delta r \to 0 \), we get

\[
f_r = f_x x_r + f_y y_r. \quad \text{Similarly,} \quad f_\theta = f_x x_\theta + f_y y_\theta. \quad \text{Moreover, we can differentiate the formulae}
\]

\[
x = r \cos(\theta) \quad y = r \sin(\theta)
\]

\[
x_r = \cos(\theta) \quad y_r = \sin(\theta)
\]

\[
x_\theta = -r \sin(\theta) \quad y_\theta = r \cos(\theta), \quad \text{so}
\]

\[
f_r = f_x x_r + f_y y_r = \cos(\theta) f_x + \sin(\theta) f_y
\]

\[
f_\theta = f_x x_\theta + f_y y_\theta = -r \sin(\theta) f_x + r \cos(\theta) f_y
\]

\[
u = f_r e_r + r^{-1} f_\theta e_\theta = f_x \cos(\theta)e_r + f_y \sin(\theta)e_r - f_x \sin(\theta)e_\theta + f_y \cos(\theta)e_\theta
\]

\[
= f_x (\cos(\theta)e_r - \sin(\theta)e_\theta) + f_y (\sin(\theta)e_r + \cos(\theta)e_\theta)
\]
Grad in polar coordinates

For any two-dimensional scalar field $f$ (as a function of $r$ and $\theta$) we have

$$\nabla (f) = \text{grad}(f) = f_r \mathbf{e}_r + r^{-1} f_\theta \mathbf{e}_\theta.$$  

**Justification:** Consider the field $\mathbf{u} = f_r \mathbf{e}_r + r^{-1} f_\theta \mathbf{e}_\theta$; we show that this is the same as $\text{grad}(f)$. Two-variable chain rule: suppose we make a small change $\delta r$ to $r$. This causes a change $\delta x \simeq x_r \delta r$ to $x$, which in turn causes a change $\simeq f_x \delta x \simeq f_x x_r \delta r$ to $f$. At the same time, our change in $r$ also causes a change $\delta y \simeq y_r \delta r$ to $x$, which causes a change $\simeq f_y \delta y = f_y y_r \delta r$ to $f$. Altogether, the change in $f$ is $\delta f \simeq (f_x x_r + f_y y_r) \delta r$. By passing to the limit $\delta r \to 0$, we get $f_r = f_x x_r + f_y y_r$. Similarly, $f_\theta = f_x x_\theta + f_y y_\theta$. Moreover, we can differentiate the formulae

$$x = r \cos(\theta) \quad \quad \quad \quad \quad \quad y = r \sin(\theta)$$

to get

$$x_r = \cos(\theta) \quad \quad \quad \quad \quad \quad y_r = \sin(\theta)$$

$$x_\theta = -r \sin(\theta) \quad \quad \quad \quad \quad \quad y_\theta = r \cos(\theta), \text{ so}$$

$$f_r = f_x x_r + f_y y_r = \cos(\theta) f_x + \sin(\theta) f_y$$
$$f_\theta = f_x x_\theta + f_y y_\theta = -r \sin(\theta) f_x + r \cos(\theta) f_y$$

$$\mathbf{u} = f_r \mathbf{e}_r + r^{-1} f_\theta \mathbf{e}_\theta = f_x \cos(\theta) \mathbf{e}_r + f_y \sin(\theta) \mathbf{e}_r - f_x \sin(\theta) \mathbf{e}_\theta + f_y \cos(\theta) \mathbf{e}_\theta$$
$$= f_x (\cos(\theta) \mathbf{e}_r - \sin(\theta) \mathbf{e}_\theta) + f_y (\sin(\theta) \mathbf{e}_r + \cos(\theta) \mathbf{e}_\theta) = f_x \mathbf{i} + f_y \mathbf{j}$$
For any two-dimensional scalar field $f$ (as a function of $r$ and $\theta$) we have

\[ \nabla(f) = \text{grad}(f) = f_r \mathbf{e}_r + r^{-1} f_\theta \mathbf{e}_\theta. \]

**Justification:** Consider the field $u = f_r \mathbf{e}_r + r^{-1} f_\theta \mathbf{e}_\theta$; we show that this is the same as grad$(f)$. Two-variable chain rule: suppose we make a small change $\delta r$ to $r$. This causes a change $\delta x \simeq x_r \delta r$ to $x$, which in turn causes a change $\simeq f_x \delta x \simeq f_x x_r \delta r$ to $f$. At the same time, our change in $r$ also causes a change $\delta y \simeq y_r \delta r$ to $x$, which causes a change $\simeq f_y \delta y = f_y y_r \delta r$ to $f$. Altogether, the change in $f$ is $\delta f \simeq (f_x x_r + f_y y_r) \delta r$. By passing to the limit $\delta r \to 0$, we get $f_r = f_x x_r + f_y y_r$. Similarly, $f_\theta = f_x x_\theta + f_y y_\theta$. Moreover, we can differentiate the formulae

\[ x = r \cos(\theta) \quad y = r \sin(\theta) \]

to get

\[ x_r = \cos(\theta) \quad y_r = \sin(\theta) \]
\[ x_\theta = -r \sin(\theta) \quad y_\theta = r \cos(\theta), \text{ so} \]

\[ f_r = f_x x_r + f_y y_r = \cos(\theta) f_x + \sin(\theta) f_y \]
\[ f_\theta = f_x x_\theta + f_y y_\theta = -r \sin(\theta) f_x + r \cos(\theta) f_y \]

\[ u = f_r \mathbf{e}_r + r^{-1} f_\theta \mathbf{e}_\theta = f_x \cos(\theta) \mathbf{e}_r + f_y \sin(\theta) \mathbf{e}_r - f_x \sin(\theta) \mathbf{e}_\theta + f_y \cos(\theta) \mathbf{e}_\theta \]
\[ = f_x (\cos(\theta) \mathbf{e}_r - \sin(\theta) \mathbf{e}_\theta) + f_y (\sin(\theta) \mathbf{e}_r + \cos(\theta) \mathbf{e}_\theta) = f_x \mathbf{i} + f_y \mathbf{j} = \text{grad}(f). \]
Examples of polar div, grad and curl

**Example:** Consider \( f = r^n \).

\[
\begin{align*}
\text{Clearly } f_r &= nr^{n-1} \\
\text{and } f_\theta &= 0, \text{ so } \nabla f &= f_re_r + r^{n-1}f_\theta e_\theta \\
\text{Note also that } r &= (x, y) = (r \cos(\theta), r \sin(\theta)) = re_r, \text{ so } e_r &= r/r = 1, \text{ so we can rewrite as } \\
\nabla (r^n) &= nr^{n-1}e_r \\
\text{(Obtained earlier using rectangular coordinates.)}
\end{align*}
\]

**Example:** Consider \( f = \theta \).

\[
\begin{align*}
\text{Clearly } f_r &= 0 \text{ and } f_\theta &= 1, \text{ so } \nabla f &= f_re_r + r^{n-1}f_\theta e_\theta = r^{n-1}e_\theta \\
\text{(Obtained earlier using rectangular coordinates.)}
\end{align*}
\]

**Example:** Consider \( u = \sqrt{r} (e_\theta + e_r/10) \) from the plot above.

\[
\begin{align*}
\text{This is } u &= pe_r + qe_\theta \text{ where } p = r^{1/2}/10 \text{ and } q = r^{1/2}, \text{ so } \\
p_\theta &= q_\theta = 0 \text{ and } p_r &= r^{1/2}/20 \text{ and } q_r = r^{1/2}/2. \text{ It follows that } \\
\n\text{div} (u) &= r^{1/2}p + pr + r^{1/2}q_\theta = 3r^{1/2}/20 \\
\text{curl} (u) &= r^{1/2}q + qr - r^{1/2}p_\theta = 3r^{1/2}/2.
\end{align*}
\]
Example: Consider $f = r^n$. Clearly $f_r = nr^{n-1}$ and $f_\theta = 0$.
Examples of polar div, grad and curl

**Example:** Consider $f = r^n$. Clearly $f_r = nr^{n-1}$ and $f_\theta = 0$, so

\[
\text{grad}(f) = f_r \mathbf{e}_r + r^{-1}f_\theta \mathbf{e}_\theta
\]
Examples of polar div, grad and curl

**Example:** Consider \( f = r^n \). Clearly \( f_r = nr^{n-1} \) and \( f_\theta = 0 \), so

\[
\text{grad}(f) = f_r \mathbf{e}_r + r^{-1}f_\theta \mathbf{e}_\theta = nr^{n-1} \mathbf{e}_r.
\]
Examples of polar div, grad and curl

Example: Consider $f = r^n$. Clearly $f_r = n r^{n-1}$ and $f_\theta = 0$, so

$$\text{grad}(f) = f_r \mathbf{e}_r + r^{-1} f_\theta \mathbf{e}_\theta = n r^{n-1} \mathbf{e}_r.$$ 

Note also that $\mathbf{r} = (x, y) = (r \cos(\theta), r \sin(\theta)) = r \mathbf{e}_r$.
Example: Consider $f = r^n$. Clearly $f_r = nr^{n-1}$ and $f_\theta = 0$, so

$$\text{grad}(f) = f_r \mathbf{e}_r + r^{-1}f_\theta \mathbf{e}_\theta = nr^{n-1}\mathbf{e}_r.$$  

Note also that $\mathbf{r} = (x, y) = (r \cos(\theta), r \sin(\theta)) = r \mathbf{e}_r$, so $\mathbf{e}_r = \mathbf{r}/r$. 


Examples of polar div, grad and curl

**Example:** Consider $f = r^n$. Clearly $f_r = nr^{n-1}$ and $f_\theta = 0$, so

$$\text{grad}(f) = f_r \mathbf{e}_r + r^{-1}f_\theta \mathbf{e}_\theta = nr^{n-1}\mathbf{e}_r.$$  

Note also that $\mathbf{r} = (x, y) = (r \cos(\theta), r \sin(\theta)) = r \mathbf{e}_r$, so $\mathbf{e}_r = \mathbf{r}/r$, so we can rewrite as $\text{grad}(r^n) = nr^{n-2}\mathbf{r}$.  

Examples of polar div, grad and curl

**Example:** Consider \( f = r^n \). Clearly \( f_r = nr^{n-1} \) and \( f_\theta = 0 \), so

\[
\text{grad}(f) = f_r \mathbf{e}_r + f_\theta \mathbf{e}_\theta = nr^{n-1} \mathbf{e}_r.
\]

Note also that \( \mathbf{r} = (x, y) = (r \cos(\theta), r \sin(\theta)) = r \mathbf{e}_r \), so \( \mathbf{e}_r = \mathbf{r}/r \), so we can rewrite as \( \text{grad}(r^n) = nr^{n-2} \mathbf{r} \). (Obtained earlier using rectangular coordinates.)
Examples of polar div, grad and curl

**Example:** Consider \( f = r^n \). Clearly \( f_r = nr^{n-1} \) and \( f_\theta = 0 \), so

\[
\text{grad}(f) = f_r \mathbf{e}_r + r^{-1} f_\theta \mathbf{e}_\theta = nr^{n-1} \mathbf{e}_r.
\]

Note also that \( r = (x, y) = (r \cos(\theta), r \sin(\theta)) = r \mathbf{e}_r \), so \( \mathbf{e}_r = r / r \), so we can rewrite as \( \text{grad}(r^n) = nr^{n-2} r \). (Obtained earlier using rectangular coordinates.)

**Example:** Consider \( f = \theta \).
Examples of polar div, grad and curl

Example: Consider \( f = r^n \). Clearly \( f_r = nr^{n-1} \) and \( f_\theta = 0 \), so
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\text{grad}(f) = f_r \mathbf{e}_r + r^{-1}f_\theta \mathbf{e}_\theta = nr^{n-1}\mathbf{e}_r.
\]

Note also that \( \mathbf{r} = (x, y) = (r \cos(\theta), r \sin(\theta)) = r \mathbf{e}_r \), so \( \mathbf{e}_r = \mathbf{r}/r \), so we can rewrite as \( \text{grad}(r^n) = nr^{n-2}\mathbf{r} \). (Obtained earlier using rectangular coordinates.)

Example: Consider \( f = \theta \). Clearly \( f_r = 0 \) and \( f_\theta = 1 \)
Examples of polar div, grad and curl

**Example:** Consider \( f = r^n \). Clearly \( f_r = nr^{n-1} \) and \( f_\theta = 0 \), so

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Note also that \( r = (x, y) = (r \cos(\theta), r \sin(\theta)) = r e_r \), so \( e_r = r/r \), so we can rewrite as \( \text{grad}(r^n) = nr^{n-2}r \). (Obtained earlier using rectangular coordinates.)

**Example:** Consider \( f = \theta \). Clearly \( f_r = 0 \) and \( f_\theta = 1 \), so

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Examples of polar div, grad and curl

Example: Consider \( f = r^n \). Clearly \( f_r = nr^{n-1} \) and \( f_\theta = 0 \), so

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\]

Note also that \( \mathbf{r} = (x, y) = (r \cos(\theta), r \sin(\theta)) = r \mathbf{e}_r \), so \( \mathbf{e}_r = \mathbf{r}/r \), so we can rewrite as \( \text{grad}(r^n) = nr^{n-2} \mathbf{r} \). (Obtained earlier using rectangular coordinates.)

Example: Consider \( f = \theta \). Clearly \( f_r = 0 \) and \( f_\theta = 1 \), so

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\text{grad}(f) = f_r \mathbf{e}_r + r^{-1} f_\theta \mathbf{e}_\theta = r^{-1} \mathbf{e}_\theta
\]
Examples of polar div, grad and curl

**Example:** Consider \( f = r^n \). Clearly \( f_r = nr^{n-1} \) and \( f_\theta = 0 \), so

\[
\text{grad}(f) = f_r \mathbf{e}_r + r^{-1} f_\theta \mathbf{e}_\theta = nr^{n-1} \mathbf{e}_r.
\]

Note also that \( \mathbf{r} = (x, y) = (r \cos(\theta), r \sin(\theta)) = r \mathbf{e}_r \), so \( \mathbf{e}_r = \mathbf{r}/r \), so we can rewrite as \( \text{grad}(r^n) = nr^{n-2} \mathbf{r} \). (Obtained earlier using rectangular coordinates.)

**Example:** Consider \( f = \theta \). Clearly \( f_r = 0 \) and \( f_\theta = 1 \), so

\[
\text{grad}(f) = f_r \mathbf{e}_r + r^{-1} f_\theta \mathbf{e}_\theta = r^{-1} \mathbf{e}_\theta = r^{-2}(-r \sin(\theta), r \cos(\theta))
\]
Examples of polar div, grad and curl

**Example:** Consider \( f = r^n \). Clearly \( f_r = nr^{n-1} \) and \( f_\theta = 0 \), so

\[
\nabla f = f_r e_r + r^{-1}f_\theta e_\theta = nr^{n-1}e_r.
\]

Note also that \( r = (x, y) = (r \cos(\theta), r \sin(\theta)) = r e_r \), so \( e_r = r/r \), so we can rewrite as \( \nabla (r^n) = nr^{n-2}r \). (Obtained earlier using rectangular coordinates.)

**Example:** Consider \( f = \theta \). Clearly \( f_r = 0 \) and \( f_\theta = 1 \), so

\[
\nabla f = f_r e_r + r^{-1}f_\theta e_\theta = r^{-1}e_\theta = r^{-2}(-r \sin(\theta), r \cos(\theta)) = \left( -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right).
\]
Examples of polar div, grad and curl

**Example:** Consider \( f = r^n \). Clearly \( f_r = nr^{n-1} \) and \( f_\theta = 0 \), so

\[
\nabla(f) = f_r \mathbf{e}_r + r^{-1}f_\theta \mathbf{e}_\theta = nr^{n-1}\mathbf{e}_r.
\]

Note also that \( \mathbf{r} = (x, y) = (r \cos(\theta), r \sin(\theta)) = r \mathbf{e}_r \), so \( \mathbf{e}_r = \mathbf{r}/r \), so we can rewrite as \( \nabla(r^n) = nr^{n-2}r \). (Obtained earlier using rectangular coordinates.)

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\[
\nabla(f) = f_r \mathbf{e}_r + r^{-1}f_\theta \mathbf{e}_\theta = r^{-1}\mathbf{e}_\theta = r^{-2}(-r \sin(\theta), r \cos(\theta)) = \left( \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right).
\]

(Obtained earlier using rectangular coordinates.)
**Examples of polar div, grad and curl**

**Example:** Consider \( f = r^n \). Clearly \( f_r = nr^{n-1} \) and \( f_\theta = 0 \), so
\[
\nabla(f) = f_r \mathbf{e}_r + r^{-1} f_\theta \mathbf{e}_\theta = nr^{n-1} \mathbf{e}_r.
\]

Note also that \( r = (x, y) = (r \cos(\theta), r \sin(\theta)) = r \mathbf{e}_r \), so \( \mathbf{e}_r = r/r \), so we can rewrite as \( \nabla(r^n) = nr^{n-2}r \). (Obtained earlier using rectangular coordinates.)

**Example:** Consider \( f = \theta \). Clearly \( f_r = 0 \) and \( f_\theta = 1 \), so
\[
\nabla(f) = f_r \mathbf{e}_r + r^{-1} f_\theta \mathbf{e}_\theta = r^{-1} \mathbf{e}_\theta = r^{-2}(-r \sin(\theta), r \cos(\theta)) = \left( \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right).
\]
(Obtained earlier using rectangular coordinates.)

**Example:** Consider \( \mathbf{u} = \sqrt{r}(\mathbf{e}_\theta + \mathbf{e}_r/10) \) from the plot above.
Examples of polar div, grad and curl

**Example:** Consider $f = r^n$. Clearly $f_r = nr^{n-1}$ and $f_\theta = 0$, so

$$\text{grad}(f) = f_r \mathbf{e}_r + r^{-1} f_\theta \mathbf{e}_\theta = nr^{n-1} \mathbf{e}_r.$$  

Note also that $r = (x, y) = (r \cos(\theta), r \sin(\theta)) = r \mathbf{e}_r$, so $\mathbf{e}_r = r / r$, so we can rewrite as $\text{grad}(r^n) = nr^{n-2} r$. (Obtained earlier using rectangular coordinates.)

**Example:** Consider $f = \theta$. Clearly $f_r = 0$ and $f_\theta = 1$, so

$$\text{grad}(f) = f_r \mathbf{e}_r + r^{-1} f_\theta \mathbf{e}_\theta = r^{-1} \mathbf{e}_\theta = r^{-2}(-r \sin(\theta), r \cos(\theta)) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right).$$

(Obtained earlier using rectangular coordinates.)

**Example:** Consider $u = \sqrt{r}(\mathbf{e}_\theta + \mathbf{e}_r/10)$ from the plot above. This is $u = p \mathbf{e}_r + q \mathbf{e}_\theta$ where $p = r^{1/2} / 10$ and $q = r^{1/2}$
Examples of polar div, grad and curl

Example: Consider $f = r^n$. Clearly $f_r = nr^{n-1}$ and $f_\theta = 0$, so

$$\text{grad}(f) = f_r \mathbf{e}_r + r^{-1}f_\theta \mathbf{e}_\theta = nr^{n-1}\mathbf{e}_r.$$ 

Note also that $r = (x, y) = (r \cos(\theta), r \sin(\theta)) = r \mathbf{e}_r$, so $\mathbf{e}_r = r/r$, so we can rewrite as $\text{grad}(r^n) = nr^{n-2}r$. (Obtained earlier using rectangular coordinates.)

Example: Consider $f = \theta$. Clearly $f_r = 0$ and $f_\theta = 1$, so

$$\text{grad}(f) = f_r \mathbf{e}_r + r^{-1}f_\theta \mathbf{e}_\theta = r^{-1}\mathbf{e}_\theta = r^{-2}(-r \sin(\theta), r \cos(\theta)) = \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right).$$ 

(Obtained earlier using rectangular coordinates.)

Example: Consider $\mathbf{u} = \sqrt{r}(\mathbf{e}_\theta + \mathbf{e}_r/10)$ from the plot above. This is $\mathbf{u} = p\mathbf{e}_r + q\mathbf{e}_\theta$ where $p = r^{1/2}/10$ and $q = r^{1/2}$, so $p_\theta = q_\theta = 0$ and $p_r = r^{-1/2}/20$ and $q_r = r^{-1/2}/2$. 
Examples of polar div, grad and curl

Example: Consider \( f = r^n \). Clearly \( f_r = nr^{n-1} \) and \( f_\theta = 0 \), so

\[
\text{grad}(f) = f_r \mathbf{e}_r + r^{-1} f_\theta \mathbf{e}_\theta = nr^{n-1} \mathbf{e}_r.
\]

Note also that \( \mathbf{r} = (x, y) = (r \cos(\theta), r \sin(\theta)) = r \mathbf{e}_r \), so \( \mathbf{e}_r = \mathbf{r} / r \), so we can rewrite as \( \text{grad}(r^n) = nr^{n-2} \mathbf{r} \). (Obtained earlier using rectangular coordinates.)

Example: Consider \( f = \theta \). Clearly \( f_r = 0 \) and \( f_\theta = 1 \), so

\[
\text{grad}(f) = f_r \mathbf{e}_r + r^{-1} f_\theta \mathbf{e}_\theta = r^{-1} \mathbf{e}_\theta = r^{-2}(-r \sin(\theta), r \cos(\theta)) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right).
\]

(Obtained earlier using rectangular coordinates.)

Example: Consider \( \mathbf{u} = \sqrt{r} (\mathbf{e}_\theta + \mathbf{e}_r / 10) \) from the plot above. This is \( \mathbf{u} = p \mathbf{e}_r + q \mathbf{e}_\theta \) where \( p = r^{\frac{1}{2}} / 10 \) and \( q = r^{\frac{1}{2}} \), so \( p_\theta = q_\theta = 0 \) and \( p_r = r^{-\frac{1}{2}} / 20 \) and \( q_r = r^{-\frac{1}{2}} / 2 \). It follows that

\[
\text{div}(\mathbf{u}) = r^{-1} p + p_r + r^{-1} q_\theta
\]
Examples of polar div, grad and curl

**Example:** Consider \( f = r^n \). Clearly \( f_r = nr^{n-1} \) and \( f_\theta = 0 \), so

\[
\text{grad}(f) = f_r \mathbf{e}_r + r^{-1}f_\theta \mathbf{e}_\theta = nr^{n-1} \mathbf{e}_r.
\]

Note also that \( r = (x, y) = (r \cos(\theta), r \sin(\theta)) = r \mathbf{e}_r \), so \( \mathbf{e}_r = r/r \), so we can rewrite as \( \text{grad}(r^n) = nr^{n-2}r \). (Obtained earlier using rectangular coordinates.)

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\[
\text{div}(\mathbf{u}) = r^{-1} p + p_r + r^{-1} q_\theta = r^{-1} r^{1/2}/10 + r^{-1/2}/20 + 0
\]
Examples of polar div, grad and curl

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\]

Note also that \( \mathbf{r} = (x, y) = (r \cos(\theta), r \sin(\theta)) = r \mathbf{e}_r \), so \( \mathbf{e}_r = \mathbf{r}/r \), so we can rewrite as \( \text{grad}(r^n) = nr^{n-2} \mathbf{r} \). (Obtained earlier using rectangular coordinates.)

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\]

(Obtained earlier using rectangular coordinates.)

Example: Consider \( \mathbf{u} = \sqrt{r}(\mathbf{e}_\theta + \mathbf{e}_r/10) \) from the plot above. This is \( \mathbf{u} = p \mathbf{e}_r + q \mathbf{e}_\theta \) where \( p = r^{\frac{1}{2}}/10 \) and \( q = r^{\frac{1}{2}} \), so \( p_\theta = q_\theta = 0 \) and \( p_r = r^{-\frac{1}{2}}/20 \) and \( q_r = r^{-\frac{1}{2}}/2 \). It follows that
\[
\text{div}(\mathbf{u}) = r^{-1} p + p_r + r^{-1} q_\theta = r^{-1} r^{\frac{1}{2}}/10 + r^{-\frac{1}{2}}/20 + 0 = 3r^{-\frac{1}{2}}/20
\]
Examples of polar div, grad and curl

**Example:** Consider \( f = r^n \). Clearly \( f_r = nr^{n-1} \) and \( f_\theta = 0 \), so

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\]

(Obtained earlier using rectangular coordinates.)

**Example:** Consider \( u = \sqrt{r}(\hat{e}_\theta + \hat{e}_r / 10) \) from the plot above. This is \( u = p \hat{e}_r + q \hat{e}_\theta \) where \( p = r^{1/2} / 10 \) and \( q = r^{1/2} \), so \( p_\theta = q_\theta = 0 \) and \( p_r = r^{-1/2} / 20 \) and \( q_r = r^{-1/2} / 2 \). It follows that

\[
\nabla(u) = r^{-1} p + p_r + r^{-1} q_\theta = r^{-1} r^{1/2} / 10 + r^{-1/2} / 20 + 0 = 3r^{-1/2} / 20
\]

\[\text{curl}(u)\]
Examples of polar div, grad and curl

Example: Consider \( f = r^n \). Clearly \( f_r = nr^{n-1} \) and \( f_\theta = 0 \), so

\[
\nabla f = f_r \mathbf{e}_r + r^{-1} f_\theta \mathbf{e}_\theta = nr^{n-1} \mathbf{e}_r.
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\]

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\[
\text{div}(\mathbf{u}) = r^{-1}p + p_r + r^{-1}q_\theta = r^{-1}r^{1/2}/10 + r^{-1/2}/20 + 0 = 3r^{-1/2}/20
\]

\[
\text{curl}(\mathbf{u}) = r^{-1}q + q_r - r^{-1}p_\theta
\]
Examples of polar div, grad and curl

Example: Consider \( f = r^n \). Clearly \( f_r = nr^{n-1} \) and \( f_\theta = 0 \), so

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\text{grad}(f) = f_r \mathbf{e}_r + r^{-1} f_\theta \mathbf{e}_\theta = nr^{n-1} \mathbf{e}_r.
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\]

\[
\text{curl}(u) = r^{-1}q + q_r - r^{-1}p_\theta = r^{-1}r^{-1/2} + r^{-1/2}/2 - 0
\]
Examples of polar div, grad and curl

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\[
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\]

\[
\text{curl}(\mathbf{u}) = r^{-1}q + q_r - r^{-1}p_\theta = r^{-1}r^{-1/2} + r^{-1/2}/2 - 0 = 3r^{-1/2}/2.
\]
In cylindrical polar coordinates we use unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$ as shown below:
Cylindrical polar coordinates

In cylindrical polar coordinates we use unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$ as shown below:

Thus, $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are the same as for two-dimensional polar coordinates, and $\mathbf{e}_z$ is just the vertical unit vector $\mathbf{k}$. 
In cylindrical polar coordinates we use unit vectors \( e_r \), \( e_\theta \) and \( e_z \) as shown below:

\[
\begin{align*}
\mathbf{e}_r &= \cos(\theta) \mathbf{i} + \sin(\theta) \mathbf{j} \\
\mathbf{e}_\theta &= -\sin(\theta) \mathbf{i} + \cos(\theta) \mathbf{j} \\
\mathbf{e}_z &= \mathbf{k}
\end{align*}
\]

Thus, \( e_r \) and \( e_\theta \) are the same as for two-dimensional polar coordinates, and \( e_z \) is just the vertical unit vector \( \mathbf{k} \). The equations are:
In cylindrical polar coordinates we use unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$ as shown below:

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\mathbf{e}_r &= \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j} \\
\mathbf{e}_\theta &= -\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j} \\
\mathbf{e}_z &= \mathbf{k} \\
\mathbf{i} &= \cos(\theta)\mathbf{e}_r - \sin(\theta)\mathbf{e}_\theta \\
\mathbf{j} &= \sin(\theta)\mathbf{e}_r + \cos(\theta)\mathbf{e}_\theta \\
\mathbf{k} &= \mathbf{e}_z.
\end{align*}
\]
The rules for div, grad and curl are as follows:

(a) For any three-dimensional scalar field $f$ (expressed as a function of $r$, $\theta$ and $z$) we have
\[
\nabla (f) = \text{grad} (f) = f_r e_r + r^{-1} f_\theta e_\theta + f_z e_z.
\]

(b) For any three-dimensional vector field $u = me_r + pe_\theta + q e_z$ (where $m$, $p$ and $q$ are expressed as functions of $r$, $\theta$ and $z$) we have
\[
\text{div} (u) = r^{-1} m + m r + r^{-1} p_\theta + q_z = r^{-1} (mr).\]
\[
\text{curl} (u) = 1 r \det \begin{pmatrix} e_r & e_\theta & e_z \\
\partial_r & \partial_\theta & \partial_z \\
m & p & q \end{pmatrix}.
\]

(c) For any three-dimensional scalar field $f$ we have
\[
\nabla^2 (f) = r^{-1} f_r + f_{rr} + r^{-2} f_{\theta\theta} + f_{zz} = r^{-1} (rf_r).\]
Div, grad and curl in cylindrical polar coordinates

The rules for div, grad and curl are as follows:

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\]

(b) For any three-dimensional vector field \( \mathbf{u} = m \mathbf{e}_r + p \mathbf{e}_\theta + q \mathbf{e}_z \) (where \( m, p \) and \( q \) are expressed as functions of \( r, \theta \) and \( z \)) we have

\[
\text{div}(\mathbf{u}) = r^{-1} m + m_r + r^{-1} p_\theta + q_z
\]
Div, grad and curl in cylindrical polar coordinates

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$$\text{div}(u) = r^{-1} m + m_r + r^{-1} p_\theta + q_z = r^{-1} (rm)_r + r^{-1} p_\theta + q_z.$$
Div, grad and curl in cylindrical polar coordinates

The rules for div, grad and curl are as follows:

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\[ \text{div}(\mathbf{u}) = r^{-1} m + m_r + r^{-1} p_\theta + q_z = r^{-1} (rm)_r + r^{-1} p_\theta + q_z \]

\[ \text{curl}(\mathbf{u}) = \frac{1}{r} \det \begin{bmatrix} e_r & re_\theta & e_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ m & rp & q \end{bmatrix}. \]
Div, grad and curl in cylindrical polar coordinates

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$$\text{div}(\mathbf{u}) = r^{-1}m + m_r + r^{-1}p_\theta + q_z = r^{-1}(rm)_r + r^{-1}p_\theta + q_z$$

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Div, grad and curl in cylindrical polar coordinates

The rules for div, grad and curl are as follows:

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$$\text{div}(u) = r^{-1} m + m_r + r^{-1}p_\theta + q_z = r^{-1}(rm)_r + r^{-1}p_\theta + q_z$$

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$$\nabla^2(f) = r^{-1}f_r + f_{rr} + r^{-2}f_{\theta\theta} + f_{zz} = r^{-1}(rf)_r + r^{-2}f_{\theta\theta} + f_{zz}.$$
Consider the vector field $\mathbf{u}$ given in cylindrical polar coordinates by
$\mathbf{u} = r(\mathbf{e}_\theta + \mathbf{e}_z)$. 

$$\text{curl}(\mathbf{u}) = \frac{1}{r} \left( \frac{\partial}{\partial \theta} (r) - \frac{\partial}{\partial z} (0) \right) \mathbf{e}_r - \frac{\partial}{\partial r} (r) \mathbf{e}_\theta + \frac{\partial}{\partial z} (r^2) \mathbf{e}_z$$

$$= \frac{1}{r} \left( -r \mathbf{e}_\theta + 2r \mathbf{e}_z \right)$$

$$= 2 \mathbf{e}_z - \mathbf{e}_\theta.$$
Example of curl in cylindrical polar coordinates

Consider the vector field \( \mathbf{u} \) given in cylindrical polar coordinates by
\[ \mathbf{u} = r(\mathbf{e}_\theta + \mathbf{e}_z). \]
This is \( \mathbf{u} = m\mathbf{e}_r + p\mathbf{e}_\theta + q\mathbf{e}_z \), where \( m = 0 \) and \( p = q = r \).
Example of curl in cylindrical polar coordinates

Consider the vector field $u$ given in cylindrical polar coordinates by $u = r(e_\theta + e_z)$. This is $u = me_r + pe_\theta + qe_z$, where $m = 0$ and $p = q = r$, so

$$\text{curl}(u) = \frac{1}{r} \det \begin{bmatrix} e_r & re_\theta & e_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & r^2 & r \end{bmatrix}$$

$$= \frac{1}{r} \begin{bmatrix} 0 & -r & 2r \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$
Example of curl in cylindrical polar coordinates

Consider the vector field \( \mathbf{u} \) given in cylindrical polar coordinates by $u = r(e_\theta + e_z)$. This is $u = me_r + pe_\theta + qe_z$, where $m = 0$ and $p = q = r$, so

curl(\( u \))

$$
\begin{align*}
\mathbf{u} &= \frac{1}{r} \begin{vmatrix}
    e_r & re_\theta & e_z \\
    \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\
    0 & r^2 & r
\end{vmatrix} \\
&= \frac{1}{r} \left( \left( \frac{\partial}{\partial \theta} (r) - \frac{\partial}{\partial z} (r^2) \right) e_r - \left( \frac{\partial}{\partial r} (r) - \frac{\partial}{\partial z} (0) \right) re_\theta + \left( \frac{\partial}{\partial r} (r^2) - \frac{\partial}{\partial \theta} (0) \right) e_z \right)
\end{align*}
$$
Consider the vector field $\mathbf{u}$ given in cylindrical polar coordinates by $\mathbf{u} = r(\mathbf{e}_\theta + \mathbf{e}_z)$. This is $\mathbf{u} = m\mathbf{e}_r + p\mathbf{e}_\theta + q\mathbf{e}_z$, where $m = 0$ and $p = q = r$, so

$$\text{curl}(\mathbf{u}) = \frac{1}{r} \det \begin{bmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & r^2 & r \end{bmatrix}$$

$$= \frac{1}{r} \left( \left( \frac{\partial}{\partial \theta} (r) - \frac{\partial}{\partial z} (r^2) \right) \mathbf{e}_r - \left( \frac{\partial}{\partial r} (r) - \frac{\partial}{\partial z} (0) \right) r \mathbf{e}_\theta + \left( \frac{\partial}{\partial r} (r^2) - \frac{\partial}{\partial \theta} (0) \right) \mathbf{e}_z \right)$$

$$= \frac{1}{r} \left( -r \mathbf{e}_\theta + 2r \mathbf{e}_z \right)$$
Consider the vector field \( \mathbf{u} \) given in cylindrical polar coordinates by
\[
\mathbf{u} = r(e_\theta + e_z).
\]
This is \( \mathbf{u} = me_r + pe_\theta + qe_z \), where \( m = 0 \) and \( p = q = r \), so

\[
\text{curl}(\mathbf{u}) = \frac{1}{r} \begin{vmatrix}
e_r & re_\theta & e_z \\
\frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\
0 & r^2 & r
\end{vmatrix}
\]

\[
= \frac{1}{r} \left( \left( \frac{\partial}{\partial \theta} (r) - \frac{\partial}{\partial z} (r^2) \right) e_r - \left( \frac{\partial}{\partial r} (r) - \frac{\partial}{\partial z} (0) \right) re_\theta + \left( \frac{\partial}{\partial r} (r^2) - \frac{\partial}{\partial \theta} (0) \right) e_z \right)
\]

\[
= \frac{1}{r} \left( -re_\theta + 2re_z \right) = 2e_z - e_\theta.
\]
Spherical polar coordinates

In spherical polar coordinates we use unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ as on the right:

Note that $\mathbf{e}_\theta$ has the same meaning as it did in the cylindrical case, but $\mathbf{e}_r$ has changed. It used to be the unit vector pointing horizontally away from the $z$-axis, but now it points directly away from the origin.

The vectors $\mathbf{e}_r$, $\mathbf{e}_\phi$ and $\mathbf{e}_\theta$ are related to $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ as follows.

$\mathbf{e}_r = \sin(\phi) \cos(\theta) \mathbf{i} + \sin(\phi) \sin(\theta) \mathbf{j} + \cos(\phi) \mathbf{k}$

$\mathbf{e}_\phi = \cos(\phi) \cos(\theta) \mathbf{i} + \cos(\phi) \sin(\theta) \mathbf{j} - \sin(\phi) \mathbf{k}$

$\mathbf{e}_\theta = -\sin(\theta) \mathbf{i} + \cos(\theta) \mathbf{j}$
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Spherical polar coordinates

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In spherical polar coordinates we use unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ as on the right:

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The vectors $\mathbf{e}_r$, $\mathbf{e}_\phi$ and $\mathbf{e}_\theta$ are related to $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ as follows.

\[
\begin{align*}
\mathbf{e}_r &= \sin(\phi) \cos(\theta) \mathbf{i} + \sin(\phi) \sin(\theta) \mathbf{j} + \cos(\phi) \mathbf{k} \\
\mathbf{e}_\phi &= \cos(\phi) \cos(\theta) \mathbf{i} + \cos(\phi) \sin(\theta) \mathbf{j} - \sin(\phi) \mathbf{k} \\
\mathbf{e}_\theta &= -\sin(\theta) \mathbf{i} + \cos(\theta) \mathbf{j}
\end{align*}
\]
Spherical polar coordinates

In spherical polar coordinates we use unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ as on the right:

Note that $\mathbf{e}_\theta$ has the same meaning as it did in the cylindrical case, but $\mathbf{e}_r$ has changed. It used to be the unit vector pointing horizontally away from the $z$-axis, but now it points directly away from the origin.

The vectors $\mathbf{e}_r$, $\mathbf{e}_\phi$ and $\mathbf{e}_\theta$ are related to $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ as follows.

\[
\mathbf{e}_r = \sin(\phi) \cos(\theta) \mathbf{i} + \sin(\phi) \sin(\theta) \mathbf{j} + \cos(\phi) \mathbf{k}
\]
\[
\mathbf{e}_\phi = \cos(\phi) \cos(\theta) \mathbf{i} + \cos(\phi) \sin(\theta) \mathbf{j} - \sin(\phi) \mathbf{k}
\]
\[
\mathbf{e}_\theta = -\sin(\theta) \mathbf{i} + \cos(\theta) \mathbf{j}
\]
\[
\mathbf{i} = \sin(\phi) \cos(\theta) \mathbf{e}_r + \cos(\phi) \cos(\theta) \mathbf{e}_\phi - \sin(\theta) \mathbf{e}_\theta
\]
\[
\mathbf{j} = \sin(\phi) \sin(\theta) \mathbf{e}_r + \cos(\phi) \sin(\theta) \mathbf{e}_\phi + \cos(\theta) \mathbf{e}_\theta
\]
\[
\mathbf{k} = \cos(\phi) \mathbf{e}_r - \sin(\phi) \mathbf{e}_\phi.
\]
Div, grad and curl in spherical polar coordinates

The rules for div, grad and curl in spherical polar coordinates are as follows.
Div, grad and curl in spherical polar coordinates

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(a) For any three-dimensional scalar field $f$ (expressed as a function of $r$, $\phi$ and $\theta$) we have

$$\nabla f = \text{grad}(f) = f_r \mathbf{e}_r + r^{-1} f_\phi \mathbf{e}_\phi + (r \sin(\phi))^{-1} f_\theta \mathbf{e}_\theta.$$
The rules for div, grad and curl in spherical polar coordinates are as follows.

(a) For any three-dimensional scalar field \( f \) (expressed as a function of \( r, \phi \) and \( \theta \)) we have

\[
\nabla(f) = \text{grad}(f) = f_r \mathbf{e}_r + r^{-1} f_\phi \mathbf{e}_\phi + (r \sin(\phi))^{-1} f_\theta \mathbf{e}_\theta.
\]

(b) For any three-dimensional vector field \( \mathbf{u} = m \mathbf{e}_r + p \mathbf{e}_\phi + q \mathbf{e}_\theta \) (where \( m, p \) and \( q \) are expressed as functions of \( r, \phi \) and \( \theta \)) we have

\[
\text{div}(\mathbf{u}) = r^{-2} (r^2 m)_r + (r \sin(\phi))^{-1} (\sin(\phi) p)_\phi + (r \sin(\phi))^{-1} q_\theta
\]
The rules for div, grad and curl in spherical polar coordinates are as follows.

(a) For any three-dimensional scalar field $f$ (expressed as a function of $r$, $\phi$ and $\theta$) we have

$$\nabla(f) = \text{grad}(f) = f_r e_r + r^{-1} f_\phi e_\phi + (r \sin(\phi))^{-1} f_\theta e_\theta.$$  

(b) For any three-dimensional vector field $u = m e_r + p e_\phi + q e_\theta$ (where $m$, $p$ and $q$ are expressed as functions of $r$, $\phi$ and $\theta$) we have

$$\text{div}(u) = r^{-2} (m r) + (r \sin(\phi))^{-1} (\sin(\phi) p \phi) + (r \sin(\phi))^{-1} q_\theta$$

$$\text{curl}(u) = \frac{1}{r^2 \sin(\phi)} \det \begin{bmatrix} e_r & r e_\phi & r \sin(\phi) e_\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ m & r p & r \sin(\phi) q \end{bmatrix}.$$
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(c) For any three-dimensional scalar field $f$ we have

$$\nabla^2(f) = r^{-2}(r^2 f_r)_r + (r^2 \sin(\phi))^{-1}(\sin(\phi)f_\phi)_\phi + (r^2 \sin^2(\phi))^{-1} f_{\theta\theta}.$$
Potential of a point charge at the origin is $V = A/r$, ($A$ constant, $r = \sqrt{x^2 + y^2 + z^2}$).
Potential of a point charge at the origin is $V = A/r$, ($A$ constant, $r = \sqrt{x^2 + y^2 + z^2}$). The electric field is $\mathbf{E} = \text{grad}(V)$. 
Example of div, grad and curl in spherical polar coordinates

Potential of a point charge at the origin is \( V = A/r \), (\( A \) constant, \( r = \sqrt{x^2 + y^2 + z^2} \)). The electric field is \( \mathbf{E} = \nabla V \). No magnetism or other charges, so Maxwell says \( \text{div}(\mathbf{E}) = 0 \) and \( \text{curl}(\mathbf{E}) = 0 \). We will check this.
Potential of a point charge at the origin is \( V = A/r \), \((A\ \text{constant,} \ r = \sqrt{x^2 + y^2 + z^2})\). The electric field is \( \mathbf{E} = \text{grad}(V) \). No magnetism or other charges, so Maxwell says \( \text{div}(\mathbf{E}) = 0 \) and \( \text{curl}(\mathbf{E}) = 0 \). We will check this. First, we have \( V_r = -A/r^2 \)
Example of div, grad and curl in spherical polar coordinates

Potential of a point charge at the origin is $V = A/r$, ($A$ constant, $r = \sqrt{x^2 + y^2 + z^2}$). The electric field is $\mathbf{E} = \text{grad}(V)$. No magnetism or other charges, so Maxwell says $\text{div}(\mathbf{E}) = 0$ and $\text{curl}(\mathbf{E}) = 0$. We will check this.

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Potential of a point charge at the origin is $V = A/r$, ($A$ constant, $r = \sqrt{x^2 + y^2 + z^2}$). The electric field is $\mathbf{E} = \nabla V$. No magnetism or other charges, so Maxwell says $\text{div}(\mathbf{E}) = 0$ and $\text{curl}(\mathbf{E}) = 0$. We will check this.

First, we have $V_r = -A/r^2$ and $V_\phi = V_\theta = 0$, so the rule

$$
\nabla V = V_r \mathbf{e}_r + r^{-1} V_\phi \mathbf{e}_\phi + (r \sin(\phi))^{-1} V_\theta \mathbf{e}_\theta
$$

just gives $\mathbf{E} = \nabla V = -A r^{-2} \mathbf{e}_r$. 


Potential of a point charge at the origin is $V = A/r$, ($A$ constant, $r = \sqrt{x^2 + y^2 + z^2}$). The electric field is $\mathbf{E} = \text{grad}(V)$. No magnetism or other charges, so Maxwell says $\text{div} (\mathbf{E}) = 0$ and $\text{curl} (\mathbf{E}) = 0$. We will check this. First, we have $V_r = -A/r^2$ and $V_\phi = V_\theta = 0$, so the rule

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just gives $\mathbf{E} = \text{grad}(V) = -Ar^{-2} \mathbf{e}_r$. In other words, we have $\mathbf{E} = me_r + pe_\phi + qe_\theta$ with $m = -Ar^{-2}$ and $p = q = 0$. 
Potential of a point charge at the origin is \( V = A/r \), (\( A \) constant, \( r = \sqrt{x^2 + y^2 + z^2} \)). The electric field is \( \mathbf{E} = \nabla V \). No magnetism or other charges, so Maxwell says \( \text{div}(\mathbf{E}) = 0 \) and \( \text{curl}(\mathbf{E}) = 0 \). We will check this. First, we have \( V_r = -A/r^2 \) and \( V_\phi = V_\theta = 0 \), so the rule

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\[
\text{div}(\mathbf{E}) = r^{-2}(r^2 m)_r + (r \sin(\phi))^{-1}(\sin(\phi)p)_\phi + (r \sin(\phi))^{-1} q_\theta.
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Example of div, grad and curl in spherical polar coordinates

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As \( p = q = 0 \), the second and third terms are zero. In the first term, we have \( r^2 m = -A \), which is constant, so \( (r^2 m)_r = 0 \) as well.
Potential of a point charge at the origin is $V = A/r$, ($A$ constant, $r = \sqrt{x^2 + y^2 + z^2}$). The electric field is $\mathbf{E} = \text{grad}(V)$. No magnetism or other charges, so Maxwell says $\text{div}(\mathbf{E}) = 0$ and $\text{curl}(\mathbf{E}) = 0$. We will check this.

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Potential of a point charge at the origin is $V = A/r$, ($A$ constant, $r = \sqrt{x^2 + y^2 + z^2}$). The electric field is $\mathbf{E} = \nabla V$. No magnetism or other charges, so Maxwell says $\text{div}(\mathbf{E}) = 0$ and $\text{curl}(\mathbf{E}) = 0$. We will check this.

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just gives $\mathbf{E} = \nabla V = -Ar^{-2} \mathbf{e}_r$. In other words, we have $\mathbf{E} = me_r + p\mathbf{e}_\phi + q\mathbf{e}_\theta$ with $m = -Ar^{-2}$ and $p = q = 0$. The general rule for the divergence is

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$$\frac{1}{r^2 \sin(\phi)} \det \begin{bmatrix} \mathbf{e}_r & r \mathbf{e}_\phi & r \sin(\phi) \mathbf{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ m & rp & r \sin(\phi) q \end{bmatrix}$$
Potential of a point charge at the origin is $V = A/r$, ($A$ constant, $r = \sqrt{x^2 + y^2 + z^2}$). The electric field is $\mathbf{E} = \text{grad}(V)$. No magnetism or other charges, so Maxwell says $\text{div}(\mathbf{E}) = 0$ and $\text{curl}(\mathbf{E}) = 0$. We will check this.

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Potential of a point charge at the origin is $V = A/r$, ($A$ constant, $r = \sqrt{x^2 + y^2 + z^2}$). The electric field is $E = \nabla V$. No magnetism or other charges, so Maxwell says $\text{div}(E) = 0$ and $\text{curl}(E) = 0$. We will check this. First, we have $V_r = -A/r^2$ and $V_\phi = V_\theta = 0$, so the rule

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just gives $E = \nabla V = -Ar^{-2} \mathbf{e}_r$. In other words, we have $E = m \mathbf{e}_r + p \mathbf{e}_\phi + q \mathbf{e}_\theta$ with $m = -Ar^{-2}$ and $p = q = 0$. The general rule for the divergence is

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As $\frac{\partial}{\partial \phi}(-Ar^{-2}) = \frac{\partial}{\partial \theta}(-Ar^{-2}) = 0$, all terms vanish.
Potential of a point charge at the origin is $V = A/r$, ($A$ constant, $r = \sqrt{x^2 + y^2 + z^2}$). The electric field is $\mathbf{E} = \text{grad}(V)$. No magnetism or other charges, so Maxwell says $\text{div}(\mathbf{E}) = 0$ and $\text{curl}(\mathbf{E}) = 0$. We will check this. First, we have $V_r = -A/r^2$ and $V_\phi = V_\theta = 0$, so the rule

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just gives $\mathbf{E} = \text{grad}(V) = -Ar^{-2} \mathbf{e}_r$. In other words, we have $\mathbf{E} = m \mathbf{e}_r + p \mathbf{e}_\phi + q \mathbf{e}_\theta$ with $m = -Ar^{-2}$ and $p = q = 0$. The general rule for the divergence is

$$\text{div}(\mathbf{E}) = r^{-2}(r^2 m)_r + (r \sin(\phi))^{-1}(\sin(\phi)p)_\phi + (r \sin(\phi))^{-1} q_\theta.$$

As $p = q = 0$, the second and third terms are zero. In the first term, we have $r^2 m = -A$, which is constant, so $(r^2 m)_r = 0$ as well. This means that $\text{div}(\mathbf{E}) = 0$ as expected. Finally, $\text{curl}(\mathbf{E})$ is

$$\frac{1}{r^2 \sin(\phi)} \det \begin{bmatrix} \mathbf{e}_r & r \mathbf{e}_\phi & r \sin(\phi) \mathbf{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ m & rp & r \sin(\phi)q \end{bmatrix} = \frac{1}{r^2 \sin(\phi)} \det \begin{bmatrix} \mathbf{e}_r & r \mathbf{e}_\phi & r \sin(\phi) \mathbf{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ -Ar^{-2} & 0 & 0 \end{bmatrix}.$$

As $\frac{\partial}{\partial \phi} (-Ar^{-2}) = \frac{\partial}{\partial \theta} (-Ar^{-2}) = 0$, all terms vanish so $\text{curl}(\mathbf{E}) = 0$ as well.