Let $\mathcal{C}$ be a category with symmetric monoidal product which we write as $a.b$

we are given isomorphisms

$$\alpha_{a,b,c} : a.(b.c) \xrightarrow{\sim} (a.b).c$$
$$\tau_{a,b} : a.b \xrightarrow{\sim} b.a$$
$$\lambda_a : 1.a \xrightarrow{\sim} a$$
$$\rho_a : a.1 \xrightarrow{\sim} a$$

so that the following commute:

C0 :
$$\begin{tikzcd}
a.(b.(c.d)) \arrow[r, "\alpha"] \arrow[d, "1.\alpha"'] & (a.b).(c.d) \arrow[r, "\alpha"] & ((a.b).c).d \\
a.((b.c).d) \arrow[r, "\alpha"'] & (a.(b.c)).d \arrow[ur, "\alpha.1"'] &
\end{tikzcd}$$

C1.0:
$$\begin{tikzcd}
1.(b.c) \arrow[rr, "\alpha"] \arrow[dr, "\lambda"'] & & (1.b).c \arrow[dl, "\lambda.1"] \\
& b.c &
\end{tikzcd}$$

C1.1:
$$\begin{tikzcd}
a.(1.c) \arrow[rr, "\alpha"] \arrow[dr, "1.\lambda"'] & & (a.1).c \arrow[dl, "\rho.1"] \\
& a.c &
\end{tikzcd}$$

C1.2
$$\begin{tikzcd}
a.(b.1) \arrow[rr, "\alpha"] \arrow[dr, "1.\rho"'] & & (a.b).1 \arrow[dl, "\rho"] \\
& a.b &
\end{tikzcd}$$

C2 :
$$\begin{tikzcd}
a.b \arrow[rr, "\tau"] \arrow[dr, "1"'] & & b.a \arrow[dl, "\tau"] \\
& a.b &
\end{tikzcd}$$

C3
$$\begin{tikzcd}
1.1 \arrow[r, shift left, "\tau"] \arrow[r, shift right, "1"'] & 1
\end{tikzcd}$$

C4 :
$$\begin{tikzcd}
& (b.a).c & b.(a.c) \arrow[l, "\alpha"'] \arrow[dr, "1.\tau"] & \\
(a.b).c \arrow[ur, "\tau.1"] & & & b.(c.a) \arrow[dl, "\alpha"] \\
& a.(b.c) \arrow[ul, "\alpha"] & (b.c).a \arrow[l, "\tau"] &
\end{tikzcd}$$

C5 :
$$\begin{tikzcd}
a.1 \arrow[rr, "\tau"] \arrow[dr, "\rho"'] & & 1.a \arrow[dl, "\lambda"] \\
& a &
\end{tikzcd}$$

By C3 & C5, $\lambda_1 = \rho_1 : 1.1 \to 1$.

$E = End(1)$ is an Abelian monoid: given $f, g : 1 \to 1$, the maps fg, gf, & $1 \xrightarrow{\lambda^{-1}} 1.1 \xrightarrow{f.g} 1.1 \xrightarrow{\lambda} 1$ coincide.

E acts naturally on $\mathcal{C}(a,b)$; the actions defined on the left or the right coincide, & are compatible with composition & product.

Let $\mathcal{P}$ be the subcategory of objects a such that $\exists b \ a.b \simeq 1$.

Given $\varphi : a.b \simeq 1$, set $\varphi' = (ba \xrightarrow[\varphi^{-1}.1]{\sim} abba \xrightarrow[\tau.1]{\sim} baba \xrightarrow[1.\varphi.1]{\sim} ba \xrightarrow[\tau]{\sim} ab \xrightarrow[\varphi]{\sim} 1)$

Then $\varphi' : b.a \simeq 1$ & $\varphi'.1 = 1.\varphi : bab \to b$ & $\varphi.1 = 1.\varphi' : aba \to a$ & $\varphi'' = \varphi$.

Define $t : End(a) \xrightarrow{\sim} E = End(1)$ $\quad (a \xrightarrow{u} a) \mapsto (1 \xleftarrow[\varphi]{} a.b \xrightarrow{u.1} a.b \xrightarrow[\varphi]{} 1)$

t^{-1} is the action map $\quad (1 \xrightarrow{v} 1) \mapsto (a \xleftarrow[\lambda]{} 1.a \xrightarrow{v.1} 1.a \xrightarrow[\lambda]{} a)$

$$\begin{tikzcd}
End(a_0) \times End(a_1) \arrow[r, "\bullet"] \arrow[d, "t \times t"'] & End(a_0.a_1) \arrow[d, "t"] \\
E \times E \arrow[r, "\mu"'] & E
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{C}(a_0, a_1) \times \mathcal{C}(a_1, a_0) \arrow[r] \arrow[d] & End(a_0) \arrow[d] \\
End(a_1) \arrow[r] & E
\end{tikzcd}$$

commute.

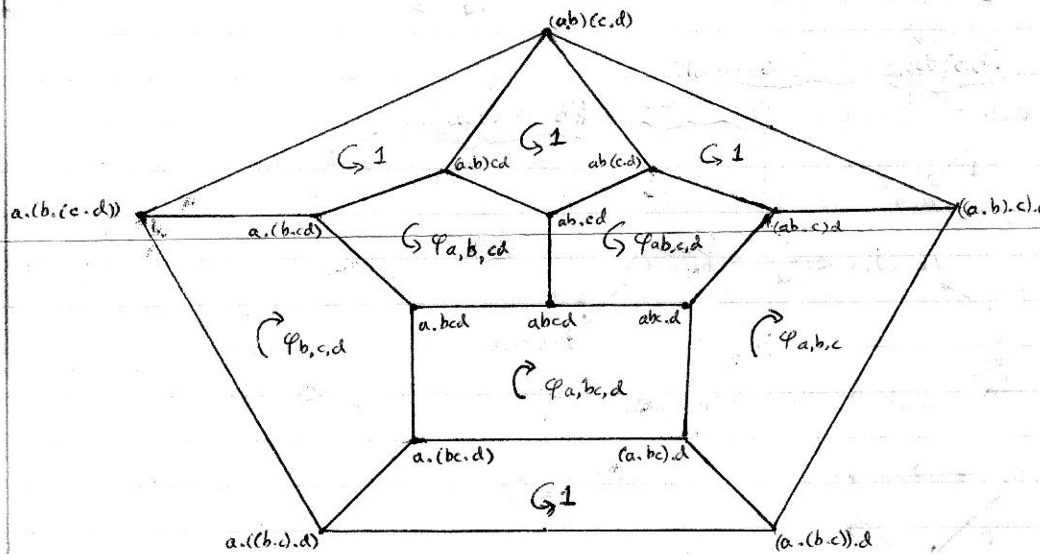
It follows that given a loop  of objects of $\mathcal{P}$ & isomorphisms, there is a unique element $u \in \mathrm{Aut}(1)$ such that the automorphism of any node given by the loop maps under t to u. We write . Then  $\Rightarrow$ 

Define $\mathrm{Pic} = \{$isomorphism classes in $\mathcal{P}\}$. We choose a representative in $\mathcal{P}$ for each class in Pic & so identify Pic with a subset of obj $\mathcal{P}$. The $\cdot$ product induces a product in Pic, so that $a.b \simeq ab$. We choose a system of such isomorphisms $\vartheta_{a,b} : a.b \simeq ab$.

Define $\varphi : \mathrm{Pic}^3 \longrightarrow \mathrm{Aut}(1)$ by

$$\begin{tikzcd}
a.(b.c) \arrow[rr, "\alpha"] \arrow[d, "1.\vartheta"'] & & (a.b).c \arrow[d, "\vartheta.1"] \\
a.bc \arrow[dr, "\vartheta"'] & \circlearrowleft \varphi_{a,b,c} & ab.c \arrow[dl, "\vartheta"] \\
& abc &
\end{tikzcd}$$

We interpret φ as a cochain in the bar construction for $H^3(\mathrm{Pic}\,;\mathrm{Aut}(1))$



In the diagram above, the top centre kite commutes by functorality of $\cdot$.
The other three regions which commute do so by naturality of α.
The outer pentagon commutes by CO. This shows that
$\varphi_{b,c,d}\,\varphi_{ab,c,d}^{-1}\,\varphi_{a,bc,d}\,\varphi_{a,b,cd}^{-1}\,\varphi_{a,b,c} = 1$ ie that φ is a cocycle.

It is easy to check that any other choice $\vartheta'_{ab} : a.b \xrightarrow{\simeq} ab$ has the form $a.b \xrightarrow{\vartheta} ab \xrightarrow{\chi_{ab}} ab$ for a unique function $\chi : \mathrm{Pic}^2 \longrightarrow \mathrm{Aut}(1)$ & then $\varphi' = \varphi.\delta\chi$. Thus $[\varphi]$ is a well defined element of $H^3(\mathrm{Pic}\,;\mathrm{Aut}(1))$ independent of the choice of ϑ; & ϑ can be chosen so that φ vanishes iff this class is zero.

The vanishing of φ is equivalent to the associativity of the pairing
$\pi_a x \times \pi_b y \longrightarrow \pi_{ab}\, x.y$ $\qquad (a \xrightarrow{f} x,\ b \xrightarrow{g} y) \mapsto (ab \xrightarrow{\vartheta^{-1}} a.b \xrightarrow{f.g} x.y)$

If $[\varphi] = 0$ then $Z^2(\mathrm{Pic}\,;\mathrm{Aut}(1))$ acts freely & transitively on $\{\vartheta : \varphi = 0\}$. Modifying ϑ by $\delta\omega$ where $\omega : \mathrm{Pic} \to \mathrm{Aut}(1)$ changes the resulting Pic-graded monoid only up to a natural isomorphism. The possible isomorphism classes of graded objects thus have a free transitive action of $H^2(\mathrm{Pic}\,;\mathrm{Aut}(1))$.
By suitable choice of ω we may ensure $\vartheta_{1,a} = \lambda_a$, $\vartheta_{b,1} = \rho_b$ (viz $\omega_a = \vartheta_{1,a}\lambda_a^{-1}$)

Consider the case $\mathcal{C} = K(1)$ - local spectra. Choose a generator k of $\mathbb{Z}_p^\times$.
Then for $\vartheta \in \mathrm{End}(\mathbb{Z}_p^\times)$ we have a fibration $S_\vartheta \xrightarrow{j_\vartheta} K_p^\wedge \xrightarrow{\psi^k - \vartheta(k)} K_p^\wedge$.

& $\vartheta \mapsto S_\vartheta$ gives a bijection $\mathrm{End}(\mathbb{Z}_p^\times) \simeq \mathrm{Pic}_1^0$

$j_\vartheta \times j_\varphi \in (K_p^\wedge)^0(S_\vartheta \wedge S_\varphi)$ satisfies $\psi^k(j_\vartheta \times j_\varphi) = \psi^k(j_\vartheta) \times \psi^k(j_\varphi) = (\vartheta . \varphi)(k)\, j_\vartheta \times j_\varphi$

so we get

$$\begin{tikzcd}
 & S_\vartheta \wedge S_\varphi \arrow[r] \arrow[d, dotted, "\exists\sigma"'] & K_p^\wedge \wedge K_p^\wedge \arrow[d, "\mu"] & \\
\Sigma^{-1} K_p^\wedge \arrow[r] & S_{\vartheta\varphi} \arrow[r] & K_p^\wedge \arrow[r, "\psi^k - \vartheta\varphi(k)"'] & K_p^\wedge
\end{tikzcd}$$

Given two choices of σ, the difference factors through $\Sigma^{-1}K_p^\wedge$ but $(K_p^\wedge)^*(S_\vartheta \wedge S_\varphi)$ is in even dimension so σ is unique. Using this it is easy to see that the σ's give a coherent system of isomorphisms & $\therefore$ an associative ~~gra~~ graded ring.

In the case $\mathcal{C} = K(n)$ - local spectra

(1) it is conjectured (& known for $p \gg 0$) that $\mathrm{Aut}(L_{K(n)} S) = A \times B$
 A a finite p'-group & B a finitely generated $\mathbb{Z}_p$-module.

(2) I believe Hopkins has constructed a subgroup $\mathbb{Z}_p^2 \rightarrowtail \mathrm{Pic}$.

Here $H^*(\mathbb{Z}_p^2; A \times B) \simeq H^*(\mathbb{Z}^2; B) \simeq E(x, y) \otimes B$
so $H^3 = 0$ & $H^2 = B$.