SOME PROJECTS

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Contents

1. Chromatic homology of $\Omega^2 S^3$

Recall that $\Omega^2 S^3$ is the space of based continuous maps from S^2 to S^3. This space plays an important role in a number of places in stable homotopy theory, and many things are known about it. In particular, the associated suspension spectrum $\Sigma^\infty_+ \Omega^2 S^3$ splits as a wedge of finite spectra, called Brown-Gitler spectra, with interesting homological properties. Using this, Goerss described $E_*(\Omega^2 S^3)$ (where E is a complex oriented generalised homology theory) in terms of Dieudonné modules of Hopf algebras. However, there are various parts of this story that remain incomplete, which makes it hard to understand the role of $\Omega^2 S^3$ in a really conceptual way. I have various possible projects related to this.

2. The spectra $T(q,n)$ and the telescope conjecture

If we let H denote the mod p Eilenberg-MacLane spectrum (where p is an odd prime), then we have

$$H_*H = P[t_1, t_2, t_3, \dots] \otimes P[e_0, e_1, e_2, \dots]$$

with $|t_i| = 2p^i - 2$ and $|e_j| = 2p^j - 1$. Suppose we have $0 \leq n \leq q \leq \infty$. We will say that a ring spectrum R has type $T(q,n)$ if it satisfies various conditions, of which the most important is that

$$H_*R = P[t_1, \dots, t_q] \otimes E[e_0, \dots, e_{n-1}].$$

There are spectra of type $T(q,n)$ when $n \in \{0,1\}$ or $n = q$ or $q = \infty$. These play a role in the proof of the Hopkins-Devinatz-Smith Nilpotence Theorem, Ravenel's method of calculating homotopy groups of spheres by infinite descent, and the strategy of Mahowald-Ravenel-Shick for attacking the Telescope Conjecture (which is a famous open problem). It would be useful to know more about these objects. If we can choose the technical details of the definitions correctly, it may be possible to prove that there is a unique $T(q,n)$ up to isomorphism for each (q,n), and there is a unique map $T(q,n) \to T(q',n')$ whenever $q \leq q'$ and $n \leq n'$. There are a number of other properties of these spectra that one could hope to analyse, which would illuminate the role of $T(q,0)$ in the nilpotence theorem, and of $T(n,n)$ in the telescope conjecture.

3. Chromatic cohomology of $BSO(n)$

I have some notes on this, following work of Inoue and Yagita. I think that it may be possible to derive more detailed and explicit results using more sophisticated formal group theory. This might be a good initial project for a PhD student wanting to learn about Morava E-theory.

4. Algebraic model structures

It would be nice to have some very concrete and explicit examples of algebraic model structures, as discussed by Emily Riehl. For example, one could try to make algebraic versions of the standard model structures on groupoids, chain complexes of abelian groups, and simplicial sets. I have some ideas for this, which may or may not work. In particular, I have a non-standard account of the model structure on chain complexes which is a promising start for an algebraic version. This would probably be a small project; one would need to add other things to build up to a PhD.

5. Tensor structures on triangulated categories

If we want to consider categories that have both a triangulation and a tensor product, then we should impose axioms about the interaction between these structures. There are some obvious axioms, but Peter May has pointed out that there are some further axioms that are less obvious but are satisfied in all the main examples. I have already understood some extensions and consolidations of May's axioms, but there are many additional things that could be done. As part of this, it has become clear that our understanding of the triangulation of the category of chain complexes may not be optimal. With the standard definition of the triangulation, it is a theorem that the construction

$$(A \to B \to C \to \Sigma A) \mapsto (B \to C \to \Sigma A \to \Sigma B)$$

preserves distinguished triangles. However, it would be better if we had a different definition for which this fact is an immediate observation rather than a theorem. I have an approach for this, but there are some subtle points that need to be resolved.

Moreover, my current understanding involves only the tensor product of two factors. It seems clear that the correct theory will involve tensor products of multiple factors, with some kind of operadic formalism. The theory of derivators is probably relevant here.

6. Self-injective rings

Suppose we silently complete all spectra at a prime p. In the 1960s Freyd conjectured that if $f\colon X \to Y$ is a map of finite spectra and $\pi_*(f) = 0$ then $f = 0$. This is called the Generating Hypothesis, (GH) and it remains completely open. One consequence would be that that the stable homotopy ring $\pi_*(S)$ is injective as a module over itself, and more generally that $\pi_*(X)$ is injective over $\pi_*(S)$. On the other hand, it is known that $\pi_*(S)$ is quite large by most measures. This is surprising, because the main examples of self-injective rings are all very small; many of them are finite-dimensional algebras over a field, for example. However, in his thesis, Leigh Shepperson constructed some interesting large examples, which makes the case of $\pi_*(S)$ look less anomalous.

One could attempt to continue in the same vein: look for surprising algebraic consequences of GH, and find purely algebraic situations where similar phenomena occur, in the hope of shedding light on GH. As a key example, GH implies that a suitable full subcategory of injective $\pi_*(S)$-modules can be made into a triangulated category. This is unusual, because in all other constructions of triangulated categories, one has to start with a concrete category and impose a nontrivial equivalence relation on the morphisms. It would be a good project to look for a purely algebraic construction of a self-injective ring with a triangulation of a full subcategory of injective modules.

7. The ring structure of MSO_*

The structure of MSO_* was determined a long time ago by Atiyah and Wall. Some features have a clear explanation in terms of formal group theory, but others do not. It would be interesting to tidy this up. It might also be possible to obtain formal group theoretic descriptions of rings such as $MSpin_*$ and MSU_*, after inverting 2 or 6.

8. Secondary operations and representations of categories

Consider the abelian category $\mathcal{A}$ of additive functors from the Spanier-Whitehead category $\mathcal{F}$ to the category of Abelian groups. The category $\mathcal{H}$ of homology theories (or spectra mod phantoms) embeds in $\mathcal{A}$. The smash product and extended power functors can be extended to $\mathcal{A}$.

Any functor in $\mathcal{A}$ can be expressed as the image (or kernel, or cokernel) of a morphism of homology theories. A higher-order homology operation is the same thing as a morphism in $\mathcal{A}$. The project would be to reexamine a lot of classical results about such operations, and their relationship with differentials in the Adams spectral sequence, in terms of this point of view.

Given a module over the Steenrod algebra (or equivalent for other cohomology theories) there is a spectral sequence converging to the homotopy groups of the space of realizations, which seems closely related to the Adams spectral sequence. It would also be good to make contact with this picture.

Partly in the same vein, it would be good to have sharper algebraic models for various categories of low-dimensional spectra. I have published a paper that does this for Moore spectra, which involves a surprisingly large amount of work to achieve an actual equivalence of categories. It should be possible to do something with connective j-theory to handle p-local spectra of dimension at most $2p-2$ or so. One can also consider the special case of low-dimensional spectra with torsion-free homology. Baues and Drozd have a complete classification of indecomposables up to dimension 5, but their methods are not well-adapted to give a more functorial formulation.

9. Unstable homotopy

One can attempt to calculate unstable 2-local homotopy groups of spheres using the EHP sequence. This leads to a kind of lexicographically ordered filtration of the relevant groups, in which the quotient of successive terms in the filtration always has order one or two (this is a reformulation of the notion of geneology of homotopy elements). Thus, one of the key problems is to determine which quotients are nontrivial. The literature on this kind of question is somewhat tricky. Even at the end of the calculation one has various kinds of indeterminacy in the answer, and at intermediate stages the indeterminacy is worse. It would be useful to have a more clearly specified language for talking about calculations of this type, and to apply the tools of mathematical logic, functional programming and proof verification to that language. Also, one can describe the 2-local homotopy groups of spheres in terms of simplicial homotopy groups of certain pro-nilpotent simplicial groups. The operators in the EHP sequence arise from maps between such groups that are not homomorphisms, but which are in some sense polynomial maps. There are a number of interesting questions arising from this.

10. Combinatorial models for operads

One of the first operads that was ever considered is the little n-cubes operad. This is originally defined as an operad in the category of topological spaces, but it then gives rise to an operad in the homotopy category. There are many other constructions of topological operads that become isomorphic to the little n-cubes operad in the homotopy category; we will call any of these an E_n-operad. One example, which has unusually rich and beautiful geometric structure, is called the Fulton-Macpherson operad.

The thesis of Dean Barber contains, among other things, a combinatorially defined operad E_n operad, which was designed to have other properties parallel to those of the Fulton-Macpherson operad. Unfortunately, this did not work as well as was originally hoped. Nonetheless, there are intriguing hints that a subtle modification of the definitions (whose details remain to be determined) might make everything work much better. To address this, one would need to use an expanded toolkit of techniques from combinatorial topology, such as discrete Morse theory and the theory of lexicographic shellings. Even if we cannot do what we hope with the Fulton-Macpherson operad, there are potentially some useful and interesting results to be found about the topology of various related simplicial complexes.

References