HOMOTOPY THEORY AND LINEAR ALGEBRA

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In this note we collect together a number of results about the geometry and homotopy theory of certain spaces related to linear algebra.

1. Basic definitions

We write $\mathcal{L}$ for the category of finite-dimensional inner product spaces over $\mathbb{R}$. The morphisms are the inner product preserving linear maps. For any linear map $\alpha\colon V \to W$ (with $V, W \in \mathcal{L}$) we write $\alpha^\dagger$ for the adjoint map $W \to V$, which is characterised by the identity $\langle \alpha(v), w\rangle = \langle v, \alpha^\dagger(w)\rangle$. If $\alpha \in \mathcal{L}(V, W)$ one checks that $\alpha^\dagger\alpha = 1_V$. Given $V \in \mathcal{L}$ we define

$$\begin{aligned}
S(V) &= \{v \in V \mid \|v\| = 1\} \\
B(V) &= \{v \in V \mid \|v\| \leq 1\} \\
\mathring{B}(V) &= \{v \in V \mid \|v\| < 1\} \\
O(V) &= \mathcal{L}^+(V, V) \\
SO(V) &= \{\alpha \in O(V) \mid \det(\alpha) = 1\} \\
\mathfrak{so}(V) &= \{\alpha \in \mathrm{End}(V) \mid \alpha + \alpha^\dagger = 0\}.
\end{aligned}$$

In particular, we have $B(0) = \mathring{B}(0) = \{0\}$ and $S(0) = \emptyset$ and $B(\mathbb{R}) = [-1, 1]$, $\mathring{B}(\mathbb{R}) = (-1, 1)$ and $S(\mathbb{R}) = \{-1, 1\}$. For the case $V = \mathbb{R}^n$ we put

$$S^{n-1} = S(\mathbb{R}^n) \qquad B^n = B(\mathbb{R}^n) \qquad \mathring{B}^n = \mathring{B}(\mathbb{R}^n) \qquad O(n) = O(\mathbb{R}^n)$$

and so on.

We next consider complex vector spaces and Hermitian products.

Let V be a finite-dimensional complex vector space, and let $V_{\mathbb{R}}$ denote the underlying real vector space. A Hermitian product $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ gives an inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ on $V_{\mathbb{R}}$ by $\langle u, v\rangle_{\mathbb{R}} = \mathrm{Re}(\langle u, v\rangle_{\mathbb{C}})$. Conversely, an inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ on $V_{\mathbb{R}}$ gives a Hermitian product on V by $\langle u, v\rangle_{\mathbb{C}} = \langle u, v\rangle_{\mathbb{R}} + i\langle u, iv\rangle_{\mathbb{R}}$. These constructions are inverse to each other. Now let W be another Hermitian space. If $\alpha\colon V \to W$ is $\mathbb{C}$-linear, we find that we get the same adjoint $\alpha^\dagger$ whether we use $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ or $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ to define it. We also find that α preserves $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ if and only if it preserves $\langle \cdot, \cdot \rangle_{\mathbb{C}}$. Similarly, if $U \leq V$ is a complex subspace, we find that we get the same orthogonal complement $U^\perp$ whether we use $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ or $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ to define it.

We also write

$$\begin{aligned}
U(V) &= \mathcal{H}(V, V) \\
SU(V) &= \{\alpha \in U(V) \mid \det(\alpha) = 1\} \\
\mathfrak{u}(V) &= \{\alpha \in \mathrm{End}_{\mathbb{C}}(V) \mid \alpha + \alpha^\dagger = 0\} \\
\mathfrak{su}(V) &= \{\alpha \in \mathfrak{u}(V) \mid \mathrm{trace}(\alpha) = 0\}.
\end{aligned}$$

We will also use the notation $I = [0, 1]$ and

$$\begin{aligned}
\Delta_n &= \{(x_0, \ldots, x_n) \in I^{n+1} \mid \sum_i x_i = 1\} \\
\Delta'_n &= \{(y_1, \ldots, y_n) \in I^n \mid y_1 \leq y_2 \leq \cdots \leq y_n\}.
\end{aligned}$$

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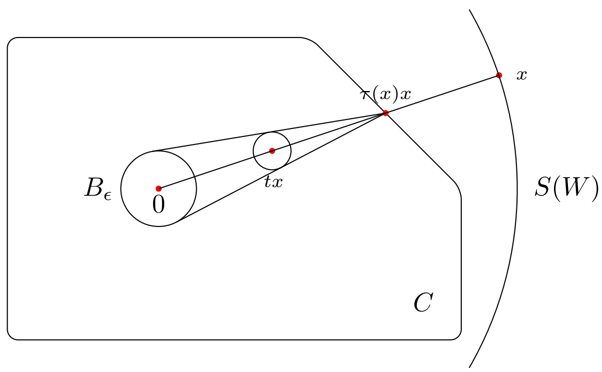
There is a well-known homeomorphism $\Delta_n \to \Delta'_n$ given by

$$(x_0, \dots, x_n) \mapsto (x_0, x_0 + x_1, x_0 + x_1 + x_2, \dots, \sum_{i<n} x_i).$$

2. Balls and spheres

Lemma 2.1. *Let V be a a finite-dimensional real inner product space, and let C be a compact convex subset of V. Let W be the span of the set $\{x - y \mid x, y \in C\}$. Then there is a homeomorphism $r\colon C \to B(W)$, which carries ∂C to $S(W)$.*

Proof. After translating if necessary we may assume that $0 \in C$ and thus that W is just the span of C. We may then replace V by W and thus assume that C spans V. Choose a basis $\{c_1, \dots, c_n\}$ contained in C and put $c_0 = 0$. If $t_1, \dots, t_n > 0$ and $\sum t_i < 1$ then we can put $t_0 = 1 - \sum_{i>0} t_i$ and by convexity we have $\sum_{i>0} t_i c_i = \sum_{i\geq 0} t_i c_i \in C$. It follows that the interior of C is nonempty, and after translating if necessary we can assume that it contains 0. Thus, for some $\epsilon > 0$ the open ball $\mathring{B}_\epsilon$ round zero is contained in C. Suppose $x \in S(W)$, and put $T_x = \{t \geq 0 \mid tx \in C\}$. This is a compact convex subset of $[0, \infty)$ containing $[0, \epsilon)$, so it has the form $[0, \tau(x)]$ for some $\tau(x) \geq \epsilon$. If $t < \tau(x)$ we claim that tx lies in the interior of C. Indeed, we have $t = (1 - s)\tau(x)$ for some $s > 0$ and by convexity $sB_\epsilon + tx = sB_\epsilon + (1 - s)\tau(x)x$ is a neighbourhood of tx contained in C. We conclude that $\{t \mid tx \in \partial C\} = \{\tau(x)\}$. Now define $r\colon \partial C \to S(W)$ by $r(y) = y/\|y\|$; this is easily seen to be a continuous bijection, with inverse $r'(x) = \tau(x)x$. Moreover, a continuous bijection of compact Hausdorff spaces is automatically a homeomorphism. Next, extend r' over $B(W)$ by defining $r'(x) = \|x\| r'(x/\|x\|)$ when $x \neq 0$, and $r'(0) = 0$. It is easy to deduce that this is a homeomorphism $B(W) \to C$, and we define $r\colon C \to B(W)$ to be its inverse.



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Example 2.2. We could take C to be I or Δ_n or $I^k \times \Delta_m \times \Delta_n \times B^p$.

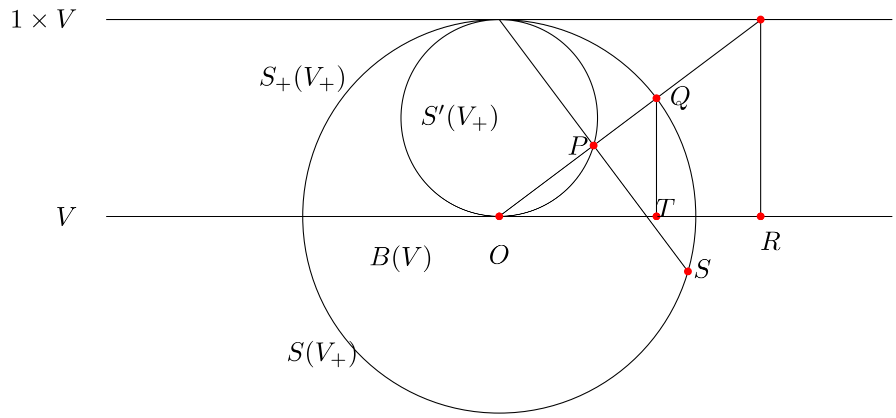
We next consider various kinds of stereographic projection, extending the familiar homeomorphism of the "Riemann sphere" $\mathbb{C} \cup \{\infty\} \simeq \mathbb{R}^2 \cup \{\infty\}$ with S^2. Let V be a real inner product space and define

$$\begin{aligned}
V_+ &= \mathbb{R} \oplus V \\
S^V &= V \cup \{\infty\} = \text{ the one-point compactification of } V \\
S(V_+) &= \{(t, v) \in V_+ \mid t^2 + v^2 = 1\} \\
S_+(V_+) &= \{(t, v) \in S(V_+) \mid t \geq 0\} \\
S'(V_+) &= \{(t, v) \in V_+ \mid (t - 1/2)^2 + v^2 = 1/4\} \\
&= \{(t, v) \in V_+ \mid t(1 - t) = v^2\}.
\end{aligned}$$

We can define homeomorphisms

$$S'(V_+) \simeq S_+(V_+)/S(V) \simeq S^V \simeq S(V_+) \simeq B(V)/S(V)$$

by letting the points P, Q, R, S and T in the following diagram correspond to each other.



Formulae can be read off from the following table:

P	Q	R	S	T
$S'(V_+)$	$S_+(V_+)/S(V)$	S^V	$S(V_+)$	$B(V)/S(V)$
(p,u)	$(p,u)/\sqrt{p}$	u/p	$(2p-1,2u)$	$u/\sqrt{p}$
$q(q,v)$	(q,v)	v/q	$(2q^2-1,2qv)$	v
$(1,w)/(1+w^2)$	$(1,w)/\sqrt{1+w^2}$	w	$(w^2-1,2w)/(1+w^2)$	$w/\sqrt{1+w^2}$
$(1+s,x)/2$	$(1+s,x)/\sqrt{2(1+s)}$	$x/(1+s)$	(s,x)	$x/\sqrt{2(1+s)}$
$(1-y^2,y\sqrt{1-y^2})$	$(\sqrt{1-y^2},y)$	$y/\sqrt{1-y^2}$	$(1-2y^2,2y\sqrt{1-y^2})$	y

By taking $V=\mathbb{R}^n$ and using the evident homeomorphism $\mathbb{R}^{n+1}\simeq\mathbb{R}\oplus\mathbb{R}^n$ we obtain homeomorphisms

$$S^n \simeq S^n_+/S^{n-1} \simeq \mathbb{R}^n \cup \{\infty\} \simeq B^n/S^{n-1}.$$

Remark 2.3. In these homeomorphisms, the first coordinate behaves differently from the remaining n coordinates. We could obtain a different set of homeomorphisms by regarding $\mathbb{R}^{n+1}$ as $\mathbb{R}^n\oplus\mathbb{R}$, so the last coordinate would behave differently from the first n. This makes little difference except to introduce a sign $(-1)^n$ in various cohomological calculations; some care is needed to keep this sort of thing straight.

Proposition 2.4. *Let $\sim$ denote the equivalence relation on $B(\mathbb{R}\oplus V)$ generated by $(t,v)\sim(-t,v)$ for $(t,v)\in S(\mathbb{R}\oplus V)$. Then there is a homeomorphism $f\colon B(\mathbb{R}\oplus V)/\sim\to S(\mathbb{R}^2\oplus V)$ given by*

$$f(x,v)=\left(\frac{2x^2+v^2-1}{\sqrt{1-v^2}},\, 2x\frac{\sqrt{1-v^2-x^2}}{\sqrt{1-v^2}},\, v\right)$$

(or $f(x,v)=(0,0,v)$ if $v^2=1$).

Remark 2.5. In the case $V=\mathbb{R}$, one can check that f can be described geometrically as follows. We first use the map $(y,v)\mapsto(\sqrt{1-y^2-v^2},y,v)$ from B^2 to the right-hand hemisphere in S^2; then we apply the map given in spherical polar coordinates by $(\theta,\phi)\mapsto(2\theta,\phi)$. The general case is obtained by reinterpreting the formulae from this case in a straightforward way.

Proof. It is straightforward algebra to check that when $v^2\neq1$ we have $\|f(x,v)\|^2=1$. Note that the last entry in $f(x,v)$ is just v so if v^2 is close to 1 then the other two entries must be small; this proves that the stated rule for $v^2=1$ gives a continuous extension of the formula. If $(x,v)\in S(\mathbb{R}\oplus V)$ then the second entry in $f(x,v)$ is zero and the first entry is clearly an even function of x; thus f respects our equivalence relation. We next show that f is surjective. Put

$$\begin{aligned}U &= \{(s,t,u)\in S(\mathbb{R}^2\oplus V)\mid s<\sqrt{s^2+t^2}\}\\ &= \{(s,t,u)\in S(\mathbb{R}^2\oplus V)\mid s<0 \text{ or } t\neq0\}\end{aligned}$$

and define $g\colon U \to \mathbb{R}^2 \oplus V$ by

$$g(s,t,v) = \left(\frac{t}{\sqrt{2}} \left(1 - \frac{s}{\sqrt{s^2+t^2}} \right)^{-1/2}, v \right).$$

If we let x be the first entry in $g(s,t,v)$ and put $r = \sqrt{s^2+t^2} = \sqrt{1-v^2}$, we find after some manipulation that $1 - x^2/r^2 = (1-s/r)/2$. From this one can derive the formulae $f(g(s,t,v)) = (s,t,v)$ and $g(f(x,v)) = (x,v)$, and after the range of validity of these calculations we see that f and g give a homeomorphism from the interior of $B(\mathbb{R} \oplus V)$ to U. When $x^2 + v^2 = 1$ the formula for f reduces to $f(x,v) = (|x|, 0, v)$. It follows easily from this that the induced map $B(\mathbb{R} \oplus V)/\sim \to S(\mathbb{R}^2 \oplus V)$ is a continuous bijection from a compact space to a Hausdorff space, so it is a homeomorphism. □

We now briefly consider the problem of embedding a product of spheres in euclidean space.

Definition 2.6. Let V be a vector space with inner product. We define

$$j(V)\colon (-1,1) \times S(V_+) \to V \times (-1,1)$$

by

$$j(V)(s,t,v) = \tfrac{1}{2}(1+s)(v,t).$$

More generally, suppose we have inner product spaces $V_1, \dots, V_n$. We define (recursively) a map

$$j(\underline{V})\colon (-1,1) \times \prod_{k \le n} S(V_{k+}) \to \left(\prod_{k \le n} V_k \right) \times (-1,1)$$

as the composite

$$(-1,1) \times \prod_{k \le n} S(V_{k+}) \xrightarrow{j(V_0,\dots,V_{n-1}) \times 1} \prod_{k<n} V_k \times (-1,1) \times S(V_{n+}) \xrightarrow{1 \times j(V_n)} \prod_{k \le n} V_k \times (-1,1).$$

Proposition 2.7. *The map $j(\underline{V})$ is an open embedding. It therefore restricts to give an embedding of $\prod_k S(V_{k+}) \simeq \prod_k S^{V_k}$ as a submanifold of codimension one in $(\prod_k V_k) \times \mathbb{R}$.*

Proof. It will suffice to show that $j(V)$ is an open embedding. In fact, it gives a homeomorphism from $(-1,1) \times S(V_+)$ to $\mathring{B}(V_+) \setminus \{0\}$, with inverse given by

$$(v,p) \mapsto (2\sqrt{v^2+p^2} - 1, p/\sqrt{v^2+p^2}, v/\sqrt{v^2+p^2}).$$

□

Now consider a complex vector space V. If we have a Hermitian inner product on V we can define $S(V)$ as before and all the above discussion still applies. It is also interesting to consider the situation where we have a perfect symmetric bilinear form $(\cdot,\cdot)\colon V \otimes V \to \mathbb{C}$ and the space $\widetilde{S}(V) = \{v \in V \mid (v,v) = 1\}$. In particular, if W is a real inner product space then we can consider the symmetric bilinear form on $\mathbb{C} \otimes W$ given by

$$(x+iy, u+iv) = (\langle x,u \rangle - \langle y,v \rangle) + (\langle x,v \rangle + \langle y,u \rangle)i,$$

and the space $\widetilde{S}(\mathbb{C} \otimes W)$. We also put

$$TS(W) = \{(u,v) \in S(W) \times W \mid \langle u,v \rangle = 0\}.$$

Proposition 2.8. *There is a homeomorphism $f\colon TS(W) \to \widetilde{S}(\mathbb{C} \otimes W)$ given by*

$$f(u,v) = \sqrt{1+v^2}\, u + iv$$
$$f^{-1}(x+iy) = (x/\|x\|, y).$$

This gives a homotopy equivalence $S(W) \to \widetilde{S}(\mathbb{C} \otimes W)$.

Proof. We have $(x+iy,x+iy) = x^2 - y^2 + 2i\langle x,y\rangle$, so $x+iy \in \widetilde{S}(\mathbb{C}\otimes W)$ if and only if $\langle x,y\rangle = 0$ and $x^2 = 1 + y^2 \geq 1$. Given this, one can check directly that the given formulae have the required effect. We also have an inclusion $i\colon S(W) \to TS(W)$ and a projection $p\colon TS(W) \to S(W)$ given by $i(u) = (u,0)$ and $p(u,v) = u$. Here $pi = 1$ and ip can be connected to the identity by the homotopy $h(t,u,v) = (u,tv)$, so $TS(W)$ is homotopy equivalent to $S(W)$. □

3. Norms of operators

Definition 3.1. Let V and W be Hermitian spaces. We define a Hermitian form on $\mathrm{Hom}(V,W)$ by $\langle \alpha,\beta\rangle = \mathrm{trace}(\beta^\dagger\alpha)$. (If we choose orthonormal bases for V and W and let A and B be the matrices corresponding to α and β, we find that $\langle\alpha,\beta\rangle = \sum_{i,j} A_{ij}\overline{B_{ij}}$, which makes it clear that this is indeed a Hermitian form.) We write $\|\alpha\|_2 = \sqrt{\mathrm{trace}(\alpha^\dagger\alpha)}$ for the corresponding norm.

Definition 3.2. We define another norm on $\mathrm{Hom}(V,W)$ by

$$\|\alpha\|_\infty = \max\{\|\alpha(v)\| \mid v \in S(V)\}.$$

We call this the *operator norm* of α.

Remark 3.3. The maximum here is well-defined, because the rule $v \mapsto \|\alpha(v)\|$ defines a continuous function on the compact space $S(V)$, which therefore attains its supremum. We leave it to the reader to check that this does indeed give a norm.

Example 3.4. Let $\alpha \in \mathrm{End}(\mathbb{C}^2)$ be given by the matrix $\begin{bmatrix} 1 & 2a \\ 0 & 1\end{bmatrix}$ with $a \geq 0$. Then $\|\alpha\|_2 = 2 + 4a^2$, and a nice exercise in calculus shows that

$$\|\alpha\|_\infty = \max\{(\cos(t) + 2a\sin(t))^2 + \sin(t)^2 \mid t \in \mathbb{R}\} = \frac{\sqrt{a^2+1}+a}{\sqrt{a^2+1}-a}.$$

In fact, we have

$$(\cos(t) + 2a\sin(t))^2 + \sin(t)^2 = \tfrac{1}{2}(r + r^{-1}) - \tfrac{1}{2}(r - r^{-1})\cos(2t + \arctan(1/a)),$$

where $r = (\sqrt{a^2+1}+a)/(\sqrt{a^2+1}-a)$.

One can prove by an abstract argument that any two norms on a finite-dimensional space are bounded by constant multiples of each other. The next proposition gives a concrete estimate.

Proposition 3.5. *For any $\alpha\colon V \to W$ we have $\|\alpha\|_\infty \leq \|\alpha\|_2 \leq \sqrt{\dim(V)}\|\alpha\|_\infty$.*

Proof. Choose a unit vector $v_1 \in V$ with $\|\alpha(v_1)\| = \|\alpha\|_\infty$, and extend it to give an orthonormal basis $v_1,\dots,v_n$ for V. Choose an orthonormal basis $w_1,\dots,w_m$ for W, and put $A_{ij} = \langle\alpha(v_i),w_j\rangle$, so $\alpha(v_i) = \sum_j A_{ij}w_j$. It follows that $\alpha^\dagger(w_j) = \sum_i \overline{A_{ij}}v_i$ and thus that $\|\alpha\|_2^2 = \sum_{i,j}|A_{ij}|^2$. However, we also find that $\|\alpha(v_i)\|^2 = \sum_j |A_{ij}|^2$ so $\|\alpha\|_2^2 = \sum_{i=1}^n \|\alpha(v_i)\|^2$. Here $\|\alpha(v_i)\|^2 \leq \|\alpha\|_\infty^2$ for all i, with equality when $i = 1$. It follows that $\|\alpha\|_\infty^2 \leq \|\alpha\|_2^2 \leq n\|\alpha\|_\infty^2$ as claimed. □

Proposition 3.6. *The space $\mathcal{H}(V,W)$ is compact Hausdorff.*

Proof. The description $\mathcal{H}(V,W) = \{\alpha \in \mathrm{Hom}(V,W) \mid \alpha^\dagger\alpha = 1\}$ makes it clear that $\mathcal{H}(V,W)$ is a closed subspace of $\mathrm{Hom}(V,W)$, so it is Hausdorff. For $\alpha \in \mathcal{H}(V,W)$ we also have $\|\alpha\|_2 = \sqrt{\mathrm{trace}(\alpha^\dagger\alpha)} = \sqrt{\dim(V)}$, so $\mathcal{H}(V,W)$ is bounded and thus compact. □

We next specialise to the case $V = W$.

Definition 3.7. Let V be a Hermitian space, and let α be an endomorphism of V. We put

$$\mathrm{spec}(\alpha) = \{\text{ eigenvalues of } \alpha\} = \{\lambda \in \mathbb{C} \mid \alpha - \lambda \text{ is not invertible }\}$$

$$\rho(\alpha) = \text{ the spectral radius of } \alpha = \max\{|\lambda| \mid \lambda \in \mathrm{spec}(\alpha)\}.$$

Lemma 3.8. *For any α we have $\rho(\alpha) \leq \|\alpha\|_\infty$.*

Proof. Just choose an eigenvalue λ with $|\lambda| = \rho(\alpha)$, and then a unit vector v with $\alpha(v) = \lambda v$; then $\rho(\alpha) = |\lambda| = \|\alpha(v)\| \leq \|\alpha\|_\infty$. □

Remark 3.9. Let α be a nonzero map represented by an upper triangular matrix with zeros on the diagonal. Then $\text{spec}(\alpha) = \{0\}$ so $\rho(\alpha) = 0$ but $\|\alpha\|_2,\ \|\alpha\|_\infty > 0$. Thus, neither norm can be bounded above by a constant multiple of the spectral radius.

Definition 3.10. We say that an endomorphism $\alpha\colon V \to V$ is *normal* if $\alpha^\dagger\alpha = \alpha\alpha^\dagger$.

Remark 3.11. Recall that

- α is unitary if it is invertible, with $\alpha^\dagger = \alpha^{-1}$
- α is hermitian (or self-adjoint) if $\alpha^\dagger = \alpha$
- α is antihermitian (or anti self-adjoint) if $\alpha^\dagger = -\alpha$.

It is clear that α is normal in all these cases.

Proposition 3.12. *α is normal if and only if there exists an orthonormal basis for V consisting of eigenvectors for α.*

Proof. Suppose that $v_1, \dots, v_n$ is an orthonormal basis for V with $\alpha(v_i) = \lambda_i v_i$. We then define $\beta\colon V \to V$ by $\beta(v_i) = \overline{\lambda_i} v_i$, and check that $\langle \alpha(v_i), v_j\rangle = \delta_{ij}\lambda_i = \langle v_i, \beta(v_j)\rangle$, so $\beta = \alpha^\dagger$. We also see that $\alpha\beta = \beta\alpha$, so α is normal.

Conversely, suppose that α is normal. Let λ be an eigenvalue, and put $W = \ker(\alpha - \lambda) \neq 0$. For $w \in W$ we have $\alpha\alpha^\dagger(w) = \alpha^\dagger\alpha(w) = \lambda\alpha^\dagger(w)$, so $\alpha^\dagger(w) \in W$. It follows that for $x \in W^\perp$ we have $\langle \alpha(x), W\rangle = \langle x, \alpha^\dagger(W)\rangle \leq \langle x, W\rangle = 0$, so $\alpha^\dagger(x) \in W^\perp$. This means that α preserves the splitting $V = W \oplus W^\perp$, and so restricts to give a normal operator on $W^\perp$. By induction on dimension we can choose an orthonormal basis of $W^\perp$ consisting of eigenvectors for α, and we can combine this with an arbitrary orthonormal basis of W to get the required basis of V. □

Corollary 3.13. *Let α be normal, with eigenvalues $\lambda_1, \dots, \lambda_n$ (repeated according to multiplicity). Then*

$$\|\alpha\|_2 = \left(\sum_i |\lambda_i|^2\right)^{1/2}$$

$$\|\alpha\|_\infty = \rho(\alpha) = \max\{|\lambda_1|, \dots, |\lambda_n|\}.$$

Proof. We can choose an orthonormal basis $v_1, \dots, v_n$ with $\alpha(v_i) = \lambda_i v_i$. The formula for $\|\alpha\|_2$ follows immediately. Now consider a vector $v = \sum_i x_i v_i \in V$. We have $\|v\|^2 = \sum_i |x_i|^2$ and

$$\|\alpha(v)\|^2 = \sum_i |\lambda_i|^2 |x_i|^2 \leq \sum_i \rho(\alpha)^2 |x_i|^2 = \rho(\alpha)^2\|v\|^2,$$

so $\|\alpha\|_\infty^2 \leq \rho(\alpha)^2$. The reverse inequality is given by Lemma 3.8. □

Proposition 3.14. *Let $\alpha\colon V \to V$ be self-adjoint, and consider the set*

$$R(\alpha) = \{\langle v, \alpha(v)\rangle \mid v \in S(V)\}$$

(known as the numerical range *of α). Then all eigenvalues of α are real, and $R(\alpha) = [\rho_-(\alpha), \rho_+(\alpha)]$, where $\rho_-(\alpha)$ and $\rho_+(\alpha)$ are respectively the smallest and largest eigenvalues.*

Proof. Choose an orthonormal basis $v_1, \dots, v_n$ with $\alpha(v_i) = \lambda_i v_i$ say. We then have

$$\overline{\lambda_i} = \langle v_i, \lambda_i v_i\rangle = \langle v_i, \alpha(v_i)\rangle = \langle \alpha^\dagger(v_i), v_i\rangle = \langle \alpha(v_i), v_i\rangle = \lambda_i,$$

so λ_i is real as claimed. We can order the basis so that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, so that $\rho_-(\alpha) = \lambda_1$ and $\rho_+(\alpha) = \lambda_n$. Now consider an element $v = \sum_i x_i v_i \in V$. We have

$$\langle v, \alpha(v)\rangle = \langle \sum_i x_i v_i, \sum_j \lambda_j x_j v_j\rangle = \sum_i \lambda_i |x_i|^2 \leq \sum_i \lambda_n |x_i|^2 = \lambda_n \|v\|^2.$$

It follows that $R(\alpha) \subseteq (-\infty, \lambda_n]$. By a similar argument we have $R(\alpha) \subseteq [\lambda_1, \infty)$, so $R(\alpha) \subseteq [\lambda_1, \lambda_n]$. By taking $v = \cos(\theta)v_1 + \sin(\theta)v_n$ for $0 \leq \theta \leq \pi/2$ we see that $[\lambda_1, \lambda_n] \subseteq R(\alpha)$, which completes the proof. □

Definition 3.15. We write $w(V)$ for the space of self-adjoint endomorphisms of V. We say that $\alpha \in w(V)$ is *positive* if $\langle v, \alpha(v)\rangle > 0$ for all $v \neq 0$, or equivalently $\rho_-(\alpha) > 0$, or equivalently $\text{spec}(\alpha) \subset (0, \infty)$. We write $w^+(V)$ for the space of positive operators. We also define *nonnegative* operators in the analogous way.

Remark 3.16. For a self-adjoint operator α we have $\|\alpha\|_\infty = \rho(\alpha) = \max(|\rho_-(\alpha)|, |\rho_+(\alpha)|)$. For a positive operator this reduces to $\|\alpha\|_\infty = \rho(\alpha) = \rho_+(\alpha)$.

Corollary 3.17. *For any linear map $\alpha\colon V \to W$, we have $\|\alpha\|_\infty = \|\alpha^\dagger\alpha\|_\infty^{1/2} = \sqrt{\rho(\alpha^\dagger\alpha)}$.*

Proof. Apply Proposition 3.14 to the nonnegative operator $\alpha^\dagger\alpha$, noting that $\langle v, \alpha^\dagger\alpha(v)\rangle = \langle \alpha(v), \alpha(v)\rangle = \|\alpha(v)\|^2$. ∎

4. Continuity of eigenvalues

We next give two standard results about the sense in which the eigenvalues of α depend continuously on α.

Proposition 4.1. *Let $\gamma\colon [0,1] \to \mathbb{C}$ be a simple closed curve, and m a natural number. Let $U(\gamma, m)$ be the set of endomorphisms $\alpha \in \mathrm{End}(V)$ such that α has precisely m eigenvalues insider γ (counted with multiplicity), and no eigenvalues on γ. Then $U(\gamma, m)$ is open in $\mathrm{End}(V)$.*

Proof. First let $U(\gamma)$ be the set of endomorphisms that have no eigenvalues on γ, so $U(\gamma) = \bigcup_m U(\gamma, m)$. We also put $\phi_\alpha(z) = \det(z - \alpha)$, so the eigenvalues of α are just the roots of the polynomial ϕ_α. If $\alpha \in U(\gamma)$ then we have a continuous function $\phi_\alpha \circ \gamma\colon [0,1] \to \mathbb{C}$, which is nowhere zero. As $[0,1]$ is compact we see that there exists $\epsilon > 0$ such that $|\phi_\alpha(\gamma(t))| \geq \epsilon$ for all t. If β is sufficiently close to α then $\phi_\beta \circ \gamma$ will be within ϵ of $\phi_\alpha \circ \gamma$ and so will be nowhere zero. It follows that $U(\gamma)$ is open.

Next, for $\alpha \in U(\gamma)$ put

$$n_\gamma(\alpha) = \frac{1}{2\pi i}\int_\gamma \frac{\phi'_\alpha(z)}{\phi_\alpha(z)}\, dz.$$

The integral formula shows us that $n_\gamma\colon U(\gamma) \to \mathbb{C}$ is continuous. However, the argument principle shows that $n_\gamma(\alpha)$ is the number of eigenvalues of α inside the contour γ, so in particular $n_\gamma(\alpha) \in \mathbb{N}$. As $n_\gamma\colon U(\gamma) \to \mathbb{N}$ is continuous, it must be locally constant. Thus, the sets $U(\gamma, m) = n_\gamma^{-1}\{m\}$ are all open. ∎

Definition 4.2. If α is self-adjoint then the eigenvalues of α are real, so we can list them in increasing order, repeated according to multiplicity. We write $e_k(\alpha)$ for the k'th element in this list, so $e_1(\alpha) \leq \ldots \leq e_n(\alpha)$ and $\det(t - \alpha) = \prod_k (t - e_k(\alpha))$.

Proposition 4.3. *The functions $e_k\colon w(V) \to \mathbb{R}$ are continuous.*

Proof. Fix a real number r; it will suffice to show that the sets $A_+(r) = \{\alpha \in w(V) \mid e_k(\alpha) > r\}$ and $A_-(r) = \{\alpha \in w(V) \mid e_k(\alpha) < r\}$ are open. For each $n > 0$ consider the rectangle $R_n \subseteq \mathbb{C}$ with vertices $r - 1/n + i$, $r - 1/n - i$, $r - n + i$ and $r - n - i$, so $R_n \cap \mathbb{R} = (r - n, r - 1/n)$. Let γ_n be the boundary curve of R_n. We have $\alpha \in A_-(r)$ if and only if α has at least k eigenvalues in the set $(-\infty, r) = \mathbb{R} \cap \bigcup_n R_n$, and using this we see that $A_-(r) = \bigcup_n \bigcup_{m \geq k} U(\gamma_n, m)$, which is open. A similar argument works for $A_+(r)$. ∎

Corollary 4.4. *Let X be the space of nonnegative operators on V of norm at most one, and let $U(V)$ act by conjugation on X. Put $n = \dim(V)$ and recall that*

$$\Delta'_n = \{y \in I^n \mid 0 \leq y_1 \leq \cdots \leq y_n \leq 1\}.$$

Then there is a canonical homeomorphism $e\colon X/U(V) \to \Delta'_n$.

Proof. We can define $e\colon X \to \mathbb{R}^n$ by $e(\alpha) = (e_1(\alpha), \ldots, e_n(\alpha))$, and the proposition tells us that this is continuous. As eigenvalues are invariant under conjugation this factors through a map $e\colon X/U(V) \to \mathbb{R}^n$. Using Proposition 3.14 and standard theory of eigenvalues we see that this gives a bijection $X/U(V) \to \Delta'_n$. Here $X/U(V)$ is compact and Δ'_n is Hausdorff, so our map is actually a homeomorphism. ∎

5. Functional calculus

We now briefly recall some basic facts about functional calculus for normal operators. There is a large literature on functional calculus, focussing on analytic aspects that are important for operators on infinite-dimensional spaces. In our context the analysis is easy but the algebra and geometry remain interesting and useful.

Let α be a normal operator on a finite-dimensional hermitian space V. For each eigenvalue $\lambda \in \operatorname{spec}(\alpha)$ we put $V_\lambda = \ker(\alpha - \lambda)$. Proposition 3.12 tells us that V splits as the orthogonal direct sum of these spaces.

Let X be a subset of $\mathbb{C}$ containing $\operatorname{spec}(\alpha)$, and let $f\colon X \to \mathbb{C}$ be a continuous function. We define $f(\alpha)$ to be the endomorphism of V that has eigenvalue $f(\lambda)$ on the space V_λ. From this definition it is clear that the following equations are valid whenever they make sense:

$$\begin{aligned}
c(\alpha) &= c.1_V \text{ if } c \text{ is constant} \\
\operatorname{id}(\alpha) &= \alpha \\
\operatorname{Re}(\alpha) &= (\alpha + \alpha^\dagger)/2 \\
\operatorname{Im}(\alpha) &= (\alpha - \alpha^\dagger)/(2i) \\
(f+g)(\alpha) &= f(\alpha) + g(\alpha) \\
(fg)(\alpha) &= f(\alpha)g(\alpha) = g(\alpha)f(\alpha) \\
\overline{f}(\alpha) &= f(\alpha)^\dagger \\
(f \circ g)(\alpha) &= f(g(\alpha)) \\
\|f(\alpha)\|_\infty &= \rho(f(\alpha)) \leq \sup\{|f(x)| \mid x \in X\}.
\end{aligned}$$

The continuity properties of $f(\alpha)$ are less clear from our definition. However, they are provided by the following result.

Proposition 5.1. *Let X be a closed subset of $\mathbb{C}$, and V a vector space. Let $N(X,V)$ be the set of normal operators on V whose eigenvalues lie in X, and let $C(X,\mathbb{C})$ be the set of continuous functions from X to $\mathbb{C}$ (with the topology of uniform convergence on compact sets). Define a function $E\colon C(X,\mathbb{C}) \times N(X,V) \to \operatorname{End}(V)$ by $E(f,\alpha) = f(\alpha)$. Then E is continuous.*

Proof. As the topology on $C(X,\mathbb{C})$ is defined in terms of compact subsets of X, we can reduce to the case where X itself is compact.

Let A be the set of functions $f \in C(X,\mathbb{C})$ for which the function $\alpha \mapsto f(\alpha)$ is continuous. Using the above algebraic properties, we see that A is a subalgebra of $C(X,\mathbb{C})$ containing the functions $z \mapsto \operatorname{Re}(z)$ and $z \mapsto \operatorname{Im}(z)$. By the Stone-Weierstrass theorem, it is dense in $C(X,\mathbb{C})$. Now suppose we have $f \in C(X,\mathbb{C})$, $\alpha \in N(X,V)$ and $\epsilon > 0$. Put $Y = \{x \in X \mid |x| \leq \|\alpha\| + 1\}$, which is compact. As A is dense we can choose $p \in A$ with $|f - p| < \epsilon/4$ on Y. As $p \in A$ can choose δ such that $\|p(\beta) - p(\alpha)\| < \epsilon/4$ whenever $\|\beta - \alpha\| < \delta$. We may also assume that $\delta < 1$, which means that when $\|\beta - \alpha\| < \delta$ we have $\beta \in Y$. Now if $|f - g| < \epsilon/4$ on Y and $\|\alpha - \beta\| < \delta$ then

$$\begin{aligned}
\|f(\alpha) - g(\beta)\| \leq& \|f(\alpha) - p(\alpha)\| + \|p(\alpha) - p(\beta)\| + \\
& \|p(\beta) - f(\beta)\| + \|f(\beta) - g(\beta)\| \\
<& \epsilon/4 + \epsilon/4 + \epsilon/4 + \epsilon/4 = \epsilon,
\end{aligned}$$

as required. □

Example 5.2. Recall that $w^+(V)$ denotes the space of positive operators on V. Functional calculus tells us that the square root function $r\colon [0,\infty) \to [0,\infty)$ induces a map $r\colon w^+(V) \to w^+(V)$. It is interesting to see how this works out in the case $V = \mathbb{C}^2$ (with the standard Hermitian product). Any element $A \in w(V)$ has the form $A = \left(\begin{smallmatrix} a & b \\ \overline{b} & c \end{smallmatrix}\right)$, and such an element is positive iff $a, c > 0$ and $ac > |b|^2$. If so, we have

$$\begin{aligned}
r(A) &= \frac{1}{\sqrt{a + c + 2\sqrt{ac - |b|^2}}} \begin{bmatrix} a + \sqrt{ac - |b|^2} & b \\ \overline{b} & c + \sqrt{ac - |b|^2} \end{bmatrix} \\
&= (\operatorname{trace}(A) + 2\sqrt{\det(A)})^{-1/2}(A + \sqrt{\det(A)}).
\end{aligned}$$

Example 5.3. Functional calculus also gives a homeomorphism $\exp\colon w(V) \to w^+(V)$, with inverse $\log\colon w^+(V) \to w(V)$.

The following proposition is an elementary exercise in linear algebra.

Proposition 5.4. *Let $\alpha\colon V \to W$ be a linear map. Then $\alpha^\dagger\alpha$ and $\alpha\alpha^\dagger$ are self-adjoint endomorphisms of V and W with nonnegative eigenvalues. For each $t > 0$ the map α gives an isomorphism of $\ker(\alpha^\dagger\alpha - t)$ with $\ker(\alpha\alpha^\dagger - t)$, so the nonzero eigenvalues of $\alpha^\dagger\alpha$ and their multiplicities are the same as those of $\alpha\alpha^\dagger$. If $f\colon [0,\infty) \to \mathbb{R}$ then $\alpha \circ f(\alpha^\dagger\alpha) = f(\alpha\alpha^\dagger) \circ \alpha$.* □

Proposition 5.5. *Suppose that $\gamma\colon V \to W$ is injective. Then $\gamma^\dagger\gamma\colon V \to V$ is positive (and thus an isomorphism), and the map*

$$\epsilon = \gamma(\gamma^\dagger\gamma)^{-1}\gamma^\dagger\colon W \to W$$

is the orthogonal projection onto $\gamma(V)$. In particular, we have $\epsilon^2 = \epsilon^\dagger = \epsilon$ and $\operatorname{trace}(\epsilon) = \dim(V)$.

Proof. Certainly $(\gamma^\dagger\gamma)^\dagger = \gamma^\dagger\gamma^{\dagger\dagger} = \gamma^\dagger\gamma$, so $\gamma^\dagger\gamma$ is self-adjoint. If $v \neq 0$ we also have $\gamma(v) \neq 0$ and so $\langle v, \gamma^\dagger\gamma(v)\rangle = \langle \gamma(v), \gamma(v)\rangle > 0$, so $\gamma^\dagger\gamma$ is positive. Now put $\epsilon = \gamma(\gamma^\dagger\gamma)^{-1}\gamma^\dagger$. We then have $\epsilon\gamma = \gamma(\gamma^\dagger\gamma)^{-1}\gamma^\dagger\gamma = \gamma$. This shows that ϵ acts as the identity on $\gamma(V)$. Suppose instead that $w \in \gamma(V)^\perp$. This gives

$$\|\gamma^\dagger(w)\|^2 = \langle \gamma^\dagger(w), \gamma^\dagger(w)\rangle = \langle \gamma\gamma^\dagger(w), w\rangle \in \langle \gamma(V), w\rangle = 0,$$

so $\gamma^\dagger(w) = 0$. It follows that $\epsilon(w) = \gamma(\gamma^\dagger\gamma)^{-1}\gamma^\dagger(w) = 0$. Thus, with respect to the decomposition $W = \gamma(V) \oplus \gamma(V)^\perp$ we have $\epsilon = 1 \oplus 0$, as claimed. From this description it is clear that $\epsilon^2 = \epsilon^\dagger = \epsilon$ and $\operatorname{trace}(\epsilon) = \dim(\gamma(V)) = \dim(V)$. The first two of these properties can also be deduced directly from the formula $\epsilon = \gamma(\gamma^\dagger\gamma)^{-1}\gamma^\dagger$. □

Proposition 5.6. *Let $\mathcal{J}(V, W)$ be the space of complex-linear injective maps from $V \to W$, and define $\mu\colon \mathcal{H}(V, W) \times w^+(V) \to \mathcal{J}(V, W)$ by $\mu(\alpha, \beta) = \alpha\beta$. Then μ is a homeomorphism.*

Proof. Suppose we have $\gamma \in \mathcal{J}(V, W)$, so $\gamma^\dagger\gamma \in w^+(V)$ by the previous proposition. We can use functional calculus to make sense of the expression $\beta = (\gamma^\dagger\gamma)^{1/2} \in w^+(V)$, and we put $\alpha = \gamma\beta^{-1}\colon V \to W$. We find that $\alpha^\dagger = \beta^{-1}\gamma^\dagger$ and thus that $\alpha^\dagger\alpha = 1$. This means that $\alpha \in \mathcal{H}^+(V, W)$, and $\mu(\alpha, \beta) = \gamma$. The pair (α, β) depends continuously on γ, so we have defined a map $\nu\colon \mathcal{J}(V, W) \to \mathcal{H}^+(V, W) \times w^+(V)$ with $\mu\nu = 1$. From the definitions we also see that $\nu\mu = 1$. □

Lemma 5.7. *The space $\operatorname{Aut}(W)$ is path-connected.*

Proof. Put $Z = \{z.1_W \mid z \in \mathbb{C}^\times\}$; this is clearly a path-connected subgroup of $\operatorname{Aut}(W)$. Consider any $\gamma \in \operatorname{Aut}(W)$, and let $\lambda_1, \dots, \lambda_r$ be the eigenvalues of $-\gamma$. Choose any $\mu \in \mathbb{C}^\times$ such that μ is not a positive real multiple of any λ_i, so the line segment joining μ to $-\lambda_i$ does not pass through the origin. Then put $u(t) = t\mu.1_W + (1-t)\alpha$, and note that this is invertible for $0 \leq t \leq 1$. We can thus connect $u(0) = \alpha$ to a point $u(1) = \mu.1_W$ in the connected group Z, and it follows that $\operatorname{Aut}(W)$ is itself connected. □

Proposition 5.8. *If $\dim(V) \leq \dim(W)$ then $\mathcal{J}(V, W)$ and $\mathcal{H}(V, W)$ are path-connected; but if $\dim(V) > \dim(W)$ then they are empty.*

Proof. If $\dim(V) \leq \dim(W)$ we choose an injective linear map $i\colon V \to W$ and consider the map $i^*\colon \operatorname{Aut}(W) \to \mathcal{J}(V, W)$ given by $i^*(\gamma) = \gamma \circ i$. Linear algebra tells us that this is surjective (and continuous). As $\operatorname{Aut}(W)$ is path-connected, the same is therefore true of $\mathcal{J}(V, W)$. Next, Proposition 5.6 (together wit the observation that $w^+(V)$ is contractible) shows that $\mathcal{H}(V, W)$ is homotopy equivalent to $\mathcal{J}(V, W)$ and so is also path-connected. It is clear that $\mathcal{J}(V, W) = \mathcal{H}(V, W) = \emptyset$ when $\dim(V) > \dim(W)$. □

Corollary 5.9. *Let $\mathcal{C}$ be a topological category, and let $F\colon \mathcal{L} \to \mathcal{C}$ be a continuous functor. Then for any pair of vector spaces $V, W \in \mathcal{H}$ with $\dim(V) \leq \dim(W)$, there is a well-defined homotopy class $j_{VW} \in [F(V), F(W)]$, obtained by applying F to any linear embedding $\alpha \in \mathcal{L}(V, W)$; and we have $j_{VV} = 1$ and $j_{VW}j_{UV} = j_{UW}$.* □

Proposition 5.10. *There is a homeomorphism (called the* Cayley transform*)*

$$\phi\colon \mathfrak{u}(V) \to \{\gamma \in U(V) \mid \gamma + 1 \text{ is invertible }\}$$

given by $\phi(\epsilon) = (1 + \epsilon/2)(1 - \epsilon/2)^{-1}$. The inverse is $\phi^{-1}(\gamma) = 2(\gamma - 1)(\gamma + 1)^{-1}$.

Proof. Put $X = \{\gamma \in U(V) \mid \gamma + 1 \in \mathrm{Aut}(V)\}$. If $\epsilon \in \mathfrak{u}(V)$ then the eigenvalues of ϵ are purely imaginary, so $1 + \epsilon/2$ and $1 - \epsilon/2$ is invertible. Thus our formula gives a well-defined map $\phi\colon \mathfrak{u}(V) \to \mathrm{Aut}(V)$. Using $\epsilon^\dagger = -\epsilon$ we see that $\phi(\epsilon)^\dagger = \phi(\epsilon)^{-1}$, so $\phi(\epsilon) \in U(V)$. Also $\phi(\epsilon) + 1 = 2(1 - \epsilon/2)^{-1}$ which is again invertible, so $\phi(\epsilon) \in X$. Now define $\xi\colon X \to \mathrm{End}(V)$ by $\xi(\gamma) = 2(\gamma - 1)(\gamma + 1)^{-1}$. Using $\gamma^\dagger = \gamma^{-1}$ and a little algebra, we see that $\xi(\gamma)^\dagger = -\xi(\gamma)$, so $\xi\colon X \to \mathfrak{u}(V)$. It is now straightforward to check that $\phi\xi$ and $\xi\phi$ are identity maps. □

This gives a nice picture of a neighbourhood of the identity in $U(V) = \mathcal{H}(V, V)$. We would like to have an analogous construction giving a neighbourhood of α in $\mathcal{H}(V, W)$, for any $\alpha \in \mathcal{H}(V, W)$. If $\dim(W) = \dim(V)$ then α is an isomorphism and the problem is easy. We put

$$\begin{aligned} T_\alpha &= \{\epsilon \in \mathrm{Hom}(V, W) \mid \alpha^\dagger\epsilon \in \mathfrak{u}(V)\} \\ U_\alpha &= \{\beta \in \mathcal{H}(V, W) \mid \alpha + \beta \text{ is injective }\}, \end{aligned}$$

and we define $\phi_\alpha\colon T_\alpha \to U_\alpha$ by $\phi_\alpha(\epsilon) = (1+\alpha^{-1}\epsilon/2)(1-\alpha^{-1}\epsilon/2)^{-1}\alpha$. One can deduce from Proposition 5.10 that this is a homeomorphism.

We now study the more general case, where $\dim(W)$ may be larger than $\dim(V)$. We define T_α and U_α just as before, and we put

$$\begin{aligned} T &= \{(\alpha, \epsilon) \mid \alpha \in \mathcal{H}(V, W),\ \epsilon \in T_\alpha\} \\ &= \{(\alpha, \epsilon) \in \mathcal{H}(V, W) \times \mathrm{Hom}(V, W) \mid \alpha^\dagger\epsilon + \epsilon^\dagger\alpha = 0\} \\ U &= \{(\alpha, \beta) \mid \alpha \in \mathcal{H}(V, W),\ \beta \in T_\alpha\} \\ &= \{(\alpha, \beta) \in \mathcal{H}(V, W)^2 \mid \alpha + \beta \text{ is injective }\}. \end{aligned}$$

Theorem 5.11. *There is a homeomorphism $\psi\colon T \to U$ given by*

$$\psi(\alpha, \epsilon) = (\alpha,\ (1 + \epsilon\alpha^\dagger/4 - \alpha\epsilon^\dagger/4)(1 - \epsilon\alpha^\dagger/4 + \alpha\epsilon^\dagger/4)^{-1}\alpha).$$

Moreover, we have $\psi(\alpha, \epsilon) \simeq \alpha + \epsilon$ for small ϵ.

The proof will be given after some preliminary results.

Lemma 5.12. *The above formula gives a continuous map $\psi\colon T \to U$, with $\psi(\alpha, \epsilon) \simeq \alpha + \epsilon$ for small ϵ.*

Proof. It is clear that the map $\delta = (\epsilon\alpha^\dagger - \alpha\epsilon^\dagger)/2$ satisfies $\delta + \delta^\dagger$, so the Cayley transform $\gamma = (1+\delta/2)(1-\delta/2)$ is defined and is unitary, and $\gamma + 1$ is invertible. It follows that the map $\beta = \gamma\alpha\colon V \to W$ is an isometric embedding, and that $\alpha + \beta = (1 + \gamma)\alpha\colon V \to W$ is injective, so $(\alpha, \beta) \in U$ as required. To first order we have $\gamma \simeq 1 + \delta$ so $\beta \simeq \alpha + \delta\alpha$, but we can use the relation $\alpha^\dagger\epsilon + \epsilon^\dagger\alpha = 0$ to see that $\delta\alpha = \epsilon$, so $\beta \simeq \alpha + \epsilon$. □

Notation . The *standard case* means the case where $W = V \oplus X$ for some hermitian space X, and $\alpha\colon V \to W$ is just the inclusion. In this case we can write any endomorphism ζ of W as a matrix $\zeta = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, with

$$\begin{aligned} &p \in \mathrm{Hom}(V, V) && q \in \mathrm{Hom}(W, V) \\ &r \in \mathrm{Hom}(V, W) && s \in \mathrm{Hom}(W, W). \end{aligned}$$

We can write maps $V \to W$ or $W \to V$ as matrices in a similar way, and then composition is given by an evident kind of matrix multiplication. In particular, we have

$$\alpha = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad\qquad \alpha^\dagger = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

We will generally make constructions and formulate identities using notation that is visibly natural and functorial, but then it will suffice to check the identities in the standard case.

Lemma 5.13. *Suppose that $(\alpha, \beta) \in U$ (so the map $\sigma = \alpha + \beta\colon V \to W$ is injective). Then in* $\mathrm{End}(V)$ *we have*

$$\beta^\dagger\sigma = \sigma^\dagger\alpha = 1 + \beta^\dagger\alpha = (1 + \alpha^\dagger\beta)^\dagger = (\alpha^\dagger\sigma)^\dagger = (\sigma^\dagger\beta)^\dagger,$$

and this map is an isomorphism.

Proof. We have $\alpha^\dagger\alpha = 1_V = \beta^\dagger\beta$, and all the stated identities follow easily from this. For $v \in S(V)$ we have

$$\begin{aligned} 0 < &\|\alpha(v) + \beta(v)\|^2 = \langle \alpha(v) + \beta(v), \alpha(v) + \beta(v) \rangle \\ &= \|\alpha(v)\|^2 + 2\operatorname{Re}(\langle \alpha(v), \beta(v) \rangle) + \|\beta(v)\|^2 \\ &= 2 + 2\operatorname{Re}(\langle v, \alpha^\dagger\beta(v) \rangle) = 2\operatorname{Re}(\langle v, (1 + \alpha^\dagger\beta)(v) \rangle), \end{aligned}$$

so $(1 + \alpha^\dagger\beta)(v) \neq 0$. It follows that $1 + \alpha^\dagger\beta$ is injective and thus an isomorphism, as required. □

Definition 5.14. Given $(\alpha, \beta) \in U$ we define $\gamma(\alpha, \beta) \in \operatorname{End}(W)$ by

$$\gamma(\alpha, \beta) = ((\alpha + \beta)(\beta^\dagger\alpha + 1)^{-1}(\alpha + \beta)^\dagger - 1)(2\alpha\alpha^\dagger - 1).$$

Proposition 5.15. *We have $\gamma(\beta, \alpha) = \gamma(\alpha, \beta)^\dagger = \gamma(\alpha, \beta)^{-1}$, so $\gamma(\alpha, \beta)$ is unitary. Moreover, $\gamma(\alpha, \beta)\alpha = \beta$.*

Proof. Put $\sigma = \alpha + \beta$ and

$$\begin{aligned} \theta &= \sigma(\beta^\dagger\sigma)^{-1}\sigma^\dagger = (\alpha + \beta)(\beta^\dagger\alpha + 1)^{-1}(\alpha^\dagger + \beta^\dagger) \\ \rho &= 2\alpha\alpha^\dagger - 1, \end{aligned}$$

so $\gamma(\alpha, \beta) = (\theta - 1)\rho$. Using the fact that $\alpha^\dagger\alpha = 1$ we see that $\rho^2 = 1$ and $\rho = \rho^\dagger$ so ρ is unitary. We also see that $\rho\alpha = \alpha$. Next, using Lemma 5.13 we see that $\theta^\dagger\beta = \sigma = \theta\alpha$ and $\beta^\dagger\theta = \sigma^\dagger = \alpha^\dagger\theta^\dagger$. This gives

$$\gamma(\alpha, \beta)\alpha = (\theta - 1)\rho\alpha = (\theta - 1)\alpha = \sigma - \alpha = \beta,$$

which is one of the claims in the proposition. Next, using $\beta^\dagger\beta = 1 = \alpha^\dagger\alpha$ we find that

$$\sigma^\dagger\beta + \beta^\dagger\sigma - \sigma^\dagger\sigma.$$

We now multiply on the left by $\sigma(\beta^\dagger\sigma)^{-1}$ and on the right by $(\sigma^\dagger\beta)^{-1}\sigma^\dagger$ and do some multiplicative cancellation to obtain the relation

$$\sigma(\beta^\dagger\sigma)^{-1}\sigma^\dagger + \sigma(\sigma^\dagger\beta)^{-1}\sigma^\dagger - \sigma(\beta^\dagger\sigma)^{-1}\sigma^\dagger\sigma(\sigma^\dagger\beta)^{-1}\sigma^\dagger = 0.$$

The left hand side here can also be written as $1 - (\theta - 1)(\theta^\dagger - 1)$, so we have $(\theta - 1)(\theta^\dagger - 1) = 1$, proving that $\theta - 1$ is unitary and so $\gamma(\alpha, \beta)$ is unitary.

Next, the relation $(\theta - 1)\alpha = \beta$ gives

$$(\theta - 1)(2\alpha\alpha^\dagger - 1)(\theta^\dagger - 1) = 2\beta\beta^\dagger - 1.$$

We can multiply on the left by $\theta^\dagger - 1 = (\theta - 1)^{-1}$ to get

$$(2\alpha\alpha^\dagger - 1)(\theta^\dagger - 1) = (\theta^\dagger - 1)(2\beta\beta^\dagger - 1).$$

The left hand side is $\gamma(\alpha, \beta)^\dagger$. After noting that $\theta^\dagger = \sigma(\alpha^\dagger\beta + 1)\sigma^\dagger$ we also see that the right hand side is $\gamma(\beta, \alpha)$, so $\gamma(\alpha, \beta)^\dagger = \gamma(\beta, \alpha)$, which completes the proof. □

Remark 5.16. In the standard case, if $\beta = \begin{bmatrix} p \\ r \end{bmatrix}$ then we have

$$\begin{aligned} \sigma &= \begin{bmatrix} p + 1 \\ r \end{bmatrix} \\ \theta &= \begin{bmatrix} p + 1 \\ r \end{bmatrix} (p^\dagger + 1)^{-1} \begin{bmatrix} p^\dagger + 1 & r^\dagger \end{bmatrix} = \begin{bmatrix} p + 1 & (p + 1)(p^\dagger + 1)^{-1}r^\dagger \\ r & r(p^\dagger + 1)^{-1}r^\dagger \end{bmatrix} \\ \rho &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ \gamma &= \begin{bmatrix} p & -(p + 1)(p^\dagger + 1)^{-1}r^\dagger \\ r & 1 - r(p + 1)(p^\dagger + 1)^{-1}r^\dagger \end{bmatrix}. \end{aligned}$$

Proposition 5.17. *$\gamma(\alpha, \beta)$ is the Cayley transform of the antihermitian endomorphism*

$$\delta(\alpha, \beta) = 2(\beta(\alpha^\dagger\beta + 1)^{-1}\alpha^\dagger - \alpha(\beta^\dagger\alpha + 1)^{-1}\beta^\dagger).$$

(In particular, $\gamma(\alpha, \beta) + 1$ is invertible.) Moreover, if we put $\pi = 1 - \alpha\alpha^\dagger$ we have $\pi\delta\pi = 0$.

Proof. We will just write γ for $\gamma(\alpha,\beta)$, and δ for $\delta(\alpha,\beta)$. One can see directly from the definition that $\delta+\delta^\dagger=0$, so δ is antihermitian. We also have $\alpha^\dagger\pi=0$ and $\pi\alpha=0$, which together give $\pi\delta\pi=0$. We next claim that $(\gamma+1)(1-\delta/2)=2$. It will suffice to check this in the standard case. We then have

$$\begin{aligned}
\beta(\alpha^\dagger\beta+1)^{-1}\alpha^\dagger &= \begin{bmatrix} p \\ r \end{bmatrix} (p+1)^{-1} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1-(p+1)^{-1} & 0 \\ r(p+1)^{-1} & 0 \end{bmatrix} \\
\delta &= 2\begin{bmatrix} 1-(p+1)^{-1} & 0 \\ r(p+1)^{-1} & 0 \end{bmatrix} - 2\begin{bmatrix} 1-(p^\dagger+1)^{-1} & (p^\dagger+1)^{-1}r^\dagger \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 2((p^\dagger+1)^{-1}-(p+1)^{-1}) & -2(p^\dagger+1)^{-1}r^\dagger \\ 2r(p+1)^{-1} & 0 \end{bmatrix} \\
(\gamma+1)(1-\delta/2) &= \begin{bmatrix} p+1 & -(p+1)(p^\dagger+1)r^\dagger \\ r & 2-r(p^\dagger+1)^{-1}r^\dagger \end{bmatrix} \begin{bmatrix} 1-(p^\dagger+1)^{-1}+(p+1)^{-1} & (p^\dagger+1)^{-1}r^\dagger \\ -r(p+1)^{-1} & 1 \end{bmatrix}.
\end{aligned}$$

When we expand this out, we encounter the expression $(p^\dagger+1)^{-1}r^\dagger r(p+1)^{-1}$. By assumption we have $p^\dagger p+r^\dagger r=\beta^\dagger\beta=1$, so $r^\dagger r=1-p^\dagger p$. We also have $p(p+1)^{-1}=1-(p+1)^{-1}$ and similarly $(p^\dagger+1)^{-1}p^\dagger=1-(p^\dagger+1)^{-1}$. Putting these observations together, we get

$$(p^\dagger+1)^{-1}r^\dagger r(p+1)^{-1}=(p^\dagger+1)^{-1}+(p+1)^{-1}-1.$$

Apart from this, we need only straightforward algebra to reduce the above expression for $(\gamma+1)(1-\delta/2)$ to twice the identity, as claimed. This shows that $\gamma+1$ is invertible, and it can be rearranged to give $\gamma=(1+\delta/2)(1-\delta/2)^{-1}$ as claimed. □

Definition 5.18. We define $\chi\colon U\to T$ by

$$\chi(\alpha,\beta)=(\alpha,(2-\alpha\alpha^\dagger)\,\delta(\alpha,\beta)\,\alpha)$$

We note here that

$$\alpha^\dagger(2-\alpha\alpha^\dagger)\,\delta(\alpha,\beta)\,\alpha=\alpha^\dagger\,\delta(\alpha,\beta)\,\alpha,$$

which is antihermitian because $\delta(\alpha,\beta)$ is. This means that $\chi(U)\subseteq T$ as indicated.

For the next few results, we fix α and consider the map $\psi_\alpha\colon T_\alpha\to U_\alpha$ induced by ψ. In this context we can assume without loss of generality that α is just the inclusion $V\to V\oplus X=W$ for some Hermitian space X. We can then write an endomorphism γ of W as a matrix

Lemma 5.19. *If* $\beta=\begin{bmatrix} p \\ r \end{bmatrix}\in U_\alpha$ *then the map* $\alpha^\dagger\beta+1=p+1\colon V\to V$ *is invertible.*

Proof. As $\beta\in U_\alpha$ we know that the map $\alpha+\beta=\begin{bmatrix} p+1 \\ r \end{bmatrix}$ is injective. Suppose that $(p+1)(v)=0$, so $p(v)=-v$. As $\beta\in\mathcal{H}(V,W)$ we know that $\begin{bmatrix} p(v) \\ r(v) \end{bmatrix}$ must have the same norm as v, but $p(v)=-v$ already has the same norm as v, so $r(v)=0$. This means that $(\alpha+\beta)(v)=0$ but $\alpha+\beta$ is assumed to be injective so $v=0$. This proves that $p+1\colon V\to V$ is injective and thus an isomorphism. □

Note that a map $\epsilon=\begin{bmatrix} a \\ b \end{bmatrix}\colon V\to W$ lies in T_α iff $a+a^\dagger=0\colon V\to V$. If so, the map $\delta=(\epsilon\alpha^\dagger-\alpha\epsilon^\dagger)/2$ is given by

$$\delta=\frac{1}{2}\left(\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}-\begin{bmatrix} a^\dagger & b^\dagger \\ 0 & 0 \end{bmatrix}\right)=\begin{bmatrix} a & -b^\dagger/2 \\ b/2 & 0 \end{bmatrix}.$$

Definition 5.20. We define $\chi_\alpha\colon U_\alpha\to T_\alpha$ by

$$\chi_\alpha\begin{bmatrix} p \\ r \end{bmatrix}=\begin{bmatrix} 2((p^\dagger+1)^{-1}-(p+1)^{-1}) \\ 4r(p+1)^{-1} \end{bmatrix}.$$

Lemma 5.21. *If* $\lambda,\mu\colon V\to V$, *and* λ *is a strictly positive operator and* $\mu^\dagger=-\mu$ *then* $\lambda+\mu$ *is invertible.*

Proof. Let v be a unit vector in V. Then $\langle\lambda(v),v\rangle>0$ and $\langle\mu(v),v\rangle\in i\mathbb{R}$ so $\langle(\lambda+\mu)(v),v\rangle\neq 0$, so $(\lambda+\mu)(v)\neq 0$. □

Lemma 5.22. *If* $\delta = \begin{bmatrix} a & -b^\dagger/2 \\ b/2 & 0 \end{bmatrix}$ *and* $c = 1 - a/2 + b^\dagger b/16 \colon V \to V$ *then* c *is invertible and*

$$(1+\delta/2)(1-\delta/2)^{-1}\alpha = \begin{bmatrix} 2c^{-1}-1 \\ bc^{-1}/2 \end{bmatrix}.$$

Proof. First note that $1 + b^\dagger b/16$ is strictly positive and a is anti self adjoint, so Lemma 5.21 tells us that the map $c = 1 - a/2 + b^\dagger b/16 \colon V \to V$ is invertible. One can check directly that

$$(1-\delta/2)\begin{bmatrix} c^{-1} \\ bc^{-1}/4 \end{bmatrix} = \begin{bmatrix} 1-a/2 & b^\dagger/4 \\ -b/4 & 1 \end{bmatrix}\begin{bmatrix} c^{-1} \\ bc^{-1}/4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \alpha,$$

so

$$\begin{bmatrix} c^{-1} \\ bc^{-1}/4 \end{bmatrix} = (1-\delta/2)^{-1}\alpha.$$

We can also write $(1+\delta/2)(1-\delta/2)^{-1}$ as $2(1-\delta/2)^{-1} - 1$, and the claim then follows easily. □

Proposition 5.23. *For all* $\epsilon \in T_\alpha$ *we have* $\chi_\alpha\psi_\alpha(\epsilon) = \epsilon$.

Proof. Write ϵ as $\begin{bmatrix} a \\ b \end{bmatrix}$, so $a + a^\dagger = 0$. Put $\delta = \begin{bmatrix} a & -b^\dagger/2 \\ b/2 & 0 \end{bmatrix}$ as before. Put $c = 1 - a/2 + b^\dagger b/16$ and $p = 2c^{-1} - 1$ and $r = bc^{-1}/2$ so that $\psi_\alpha(\epsilon) = \begin{bmatrix} p \\ r \end{bmatrix}$. Now $(p+1)^{-1} = c/2$ and $(p^\dagger+1)^{-1} = c^\dagger/2$, so

$$2((p^\dagger+1)^{-1} - (p+1)^{-1}) = c^\dagger - c = a.$$

We also have

$$4r(p+1)^{-1} = 4bc^{-1}/2 \,.\, c/2 = b.$$

Thus $\chi_\alpha\psi_\alpha(\epsilon) = \epsilon$ as claimed. □

Proposition 5.24. *For all* $\beta \in U_\alpha$ *we have* $\psi_\alpha\chi_\alpha(\beta) = \beta$.

Proof. Write β as $\begin{bmatrix} p \\ r \end{bmatrix}$ as before. Put

$$\begin{aligned} z &= r(p+1)^{-1} \\ \epsilon &= \chi\begin{bmatrix} p \\ r \end{bmatrix} = \begin{bmatrix} 2((p^\dagger+1)^{-1} - (p+1)^{-1}) \\ 4z \end{bmatrix} \\ \delta &= (\epsilon\alpha^\dagger - \alpha\epsilon^\dagger)/2 = \begin{bmatrix} 2((p^\dagger+1)^{-1} - (p+1)^{-1}) & -2z^\dagger \\ 2z & 0 \end{bmatrix} \end{aligned}$$

We claim that

$$z^\dagger z = (p^\dagger+1)^{-1} + (p+1)^{-1} - 1.$$

To see this, note that $\beta^\dagger\beta = 1$ so $r^\dagger r = 1 - p^\dagger p$. Also $p(p+1)^{-1} = 1 - (p+1)^{-1}$ and similarly $(p^\dagger+1)^{-1}p^\dagger = 1 - (p^\dagger+1)^{-1}$. The claim now follows by straightforward expansion of the definitions. It follows in turn that

$$(1-\delta/2)\begin{bmatrix} 1 \\ z \end{bmatrix} = \begin{bmatrix} 1 - (p^\dagger+1)^{-1} + (p+1)^{-1} & 2z^\dagger \\ -2z & 1 \end{bmatrix}\begin{bmatrix} 1 \\ z \end{bmatrix} = \begin{bmatrix} 2(p+1)^{-1} \\ 0 \end{bmatrix} = 2\alpha(p+1)^{-1}.$$

We can then apply the map $\gamma = (1+\delta/2)(1-\delta/2)^{-1}$ and rearrange to get

$$\psi_\alpha(\epsilon) = \gamma\alpha = (1+\delta/2)\begin{bmatrix} 1 \\ z \end{bmatrix}(p+1)/2.$$

To evaluate the right hand side, it is convenient to write $(1+\delta/2)$ as $2 - (1-\delta/2)$ and use our previous calculation to get

$$\psi_\alpha(\epsilon) = \left(2\begin{bmatrix} 1 \\ z \end{bmatrix} - \begin{bmatrix} 2(p+1)^{-1} \\ 0 \end{bmatrix}\right)(p+1)/2 = \begin{bmatrix} p \\ r \end{bmatrix} = \beta,$$

as required. □

Lemma 5.25. *If $\beta \in U_\alpha$ then*

$$\chi_\alpha(\beta) = (4\beta + 2\alpha)(\alpha^\dagger\beta + 1)^{-1} + 2\alpha(\beta^\dagger\alpha + 1)^{-1} - 4\alpha.$$

(The inverses mentioned here exist by Lemma 5.19 and its adjoint.)

Proof. If we write $\beta = \begin{bmatrix} p \\ r \end{bmatrix}$ as before then the right hand side is

$$\begin{bmatrix} (4p+2)(p+1)^{-1} \\ 4r(p+1)^{-1} \end{bmatrix} + \begin{bmatrix} 2(p^\dagger+1)^{-1} \\ 0 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \end{bmatrix},$$

which reduces easily to the definition of $\chi_\alpha(\beta)$. □

Proof of Theorem 5.11. The above lemmas show that ψ is a bijection, with inverse

$$\psi^{-1}(\alpha, \beta) = (\alpha, (4\beta + 2\alpha)(\alpha^\dagger\beta + 1)^{-1} + 2\alpha(\beta^\dagger\alpha + 1)^{-1} - 4\alpha),$$

which is continuous. □

Corollary 5.26. *The maps ψ_α are charts for an atlas on $\mathcal{H}(V, W)$, making it a compact smooth manifold.*

Proof. We need only check that $\psi_\beta^{-1}\psi_\alpha$ is smooth on the subset of T_α where it is defined. This is clear from the formulae in Theorem 5.11 and Lemma 5.25. □

6. Grassmannians

Definition 6.1. Let V be a complex vector space of dimension n. For $0 \le k \le n$ we put

$$G_k(V) = \mathrm{Grass}_k(V) = \{A \le V \mid \dim(A) = k\}.$$

There is a surjective map $\pi\colon \mathcal{J}(\mathbb{C}^k, V) \to G_k(V)$ given by $\pi(\alpha) = \alpha(\mathbb{C}^k)$, and we topologise $G_k(V)$ as a quotient of $\mathcal{J}(\mathbb{C}^k, V)$. Given a splitting $V = A \oplus B$ with $\dim(A) = k$, we define a map $\Gamma_{AB}\colon \mathrm{Hom}(A, B) \to G_k(V)$ by

$$\Gamma_{AB}(\phi) = (1 + \phi)(A) \le A \oplus B = V.$$

Remark 6.2. We will be particularly interested in the projective space $PV = G_1(V)$.

Proposition 6.3. *The space $G_k(V)$ is a compact complex manifold with charts Γ_{AB}. Moreover, if we choose a Hermitian product on V we get a homeomorphism between $G_k(V)$ and the space*

$$\mathrm{Proj}_k(V) = \{\alpha \in \mathrm{End}(V) \mid \alpha = \alpha^\dagger = \alpha^2,\ \ \mathrm{trace}(\alpha) = k\}.$$

Proof. We first prove that Γ_{AB} is continuous. Choose a linear isomorphism $\lambda\colon \mathbb{C}^k \to A$, and define $\sigma\colon \mathrm{Hom}(A, B) \to \mathcal{J}(\mathbb{C}^k, V)$ by $\sigma(\phi) = (1 + \phi) \circ \lambda$. Then $\Gamma_{AB} = \pi\sigma$, and continuity follows from this.

Next, because $V = A \oplus B$ we have projections $A \xleftarrow{\alpha} V \xrightarrow{\beta} B$. We put

$$N_B = \{U \in G_k(V) \mid U \cap B = 0\}.$$

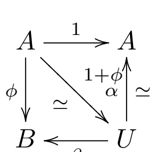
The preimage of this in $\mathcal{J}(\mathbb{C}^k, V) = \mathcal{J}(\mathbb{C}^k, A \oplus B)$ is just $\mathcal{J}(\mathbb{C}^k, A) \times \mathrm{Hom}(\mathbb{C}^k, B)$, which is open in $\mathcal{J}(\mathbb{C}^k, V)$. As we are using the quotient topology, this means that N_B is open in $G_k(V)$.

For $U \in N_B$ we see that the map $\alpha|_U\colon U \to A$ is injective, and thus an isomorphism by dimension count. We can thus define

$$\Delta_{AB}(U) = (\beta|_U) \circ (\alpha|_U)^{-1}\colon A \to B.$$

This gives a map $\Delta_{AB}\colon N_B \to \mathrm{Hom}(A, B)$. If $u = (a, b) \in U$ then $a = (\alpha|_U)(u)$ and $b = (\beta|_U)(u)$ so $b = \phi(a)$. This shows that $\Gamma_{AB}\Delta_{AB}(U) = U$.

Now suppose we have $\phi\colon A \to B$, and we take $U = (1 + \phi)(A)$. By contemplating the diagram



we see that $U \in N_B$ and $\Delta_{AB}\Gamma_{AB}(\phi) = \Delta_{AB}(U) = \phi$. Putting this together, we see that Γ_{AB} and Δ_{AB} give homeomorphisms $\mathrm{Hom}(A,B) \simeq N_B$.

Now suppose we have another splitting $V = C \oplus D$, and a linear map $\psi\colon C \to D$. We find that $\Delta_{AB}\Gamma_{CD}(\psi)$ is defined if and only if $\alpha \circ (1+\psi)\colon C \to A$ is invertible, and if so, we have

$$\Delta_{AB}\Gamma_{CD}(\psi) = (\beta \circ (1+\psi)) \circ (\alpha \circ (1+\psi))^{-1}\colon A \to B.$$

If we choose bases, and identify linear maps with matrices, and use the standard determinant formula for inverses, we see that the entries in $\Delta_{AB}\Gamma_{CD}(\psi)$ are rational functions of the entries of ψ. It follows that $\Delta_{AB}\Gamma_{CD}$ is holomorphic where defined, so the maps Γ_{AB} are charts for a holomorphic atlas on $G_k(V)$.

We now choose a Hermitian product on V. Given $A \in G_k(V)$ we let $\epsilon(A)$ be the orthogonal projection onto A, so $\epsilon(A) = 1 \oplus 0$ with respect to the decomposition $V = A \oplus A^\perp$. This lies in $\mathrm{Proj}_k(V)$, so we have defined a map $\epsilon\colon G_k(V) \to \mathrm{Proj}_k(V)$. This is easily seen to be bijective, with inverse $\theta \mapsto \mathrm{image}(\theta) = \ker(1-\theta)$. Now consider the diagram

$$\begin{array}{ccccc}
\mathcal{H}(\mathbb{C}^k,V) \times w^+(\mathbb{C}^k) & & \xrightarrow{\quad\mathrm{proj}\quad} & & \mathcal{H}(\mathbb{C}^k,V) \\
\big\downarrow{\scriptstyle \mu} & & & & \big\downarrow{\scriptstyle \psi} \\
\mathcal{J}(\mathbb{C}^k,V) & \xrightarrow[\pi]{} & G_k(V) & \xrightarrow[\epsilon]{} & \mathrm{Proj}_k(V),
\end{array}$$

where $\psi(\alpha) = \alpha\alpha^\dagger$. Using Proposition 5.5 we see that our formula does indeed define a map $\mathcal{H}(\mathbb{C}^k,V) \to \mathrm{Proj}_k(V)$, and that it makes the diagram commute. It is clear that $\psi \circ \mathrm{proj}$ is continuous, and Proposition 5.6 tells us that μ is a homeomorphism, so $\epsilon\pi\colon \mathcal{J}(\mathbb{C}^k,V) \to \mathrm{Proj}_k(V)$ is continuous. As $G_k(V)$ is topologised as a quotient of $\mathcal{J}(\mathbb{C}^k,V)$, we deduce that ϵ is continuous. As ϵ is injective and $\mathrm{Proj}_k(V) \subseteq \mathrm{End}(V)$ is Hausdorff, it follows that $G_k(V)$ is Hausdorff. We also see from the diagram that $\psi\colon \mathcal{H}(\mathbb{C}^k,V) \to G_k(V)$ is surjective, and $\mathcal{H}(\mathbb{C}^k,V)$ is compact by Proposition 3.6, so $G_k(V)$ is compact. Now $\epsilon\colon G_k(V) \to \mathrm{Proj}_k(V)$ is a continuous bijection of compact Hausdorff spaces, so it is a homeomorphism. □

7. Rotating subspaces

Let A and B be finite-dimensional subspaces of a universe $\mathcal{U}$, with $\dim(A) = \dim(B)$. If A and B are sufficiently close then we will have $A \cap B^\perp = 0$ so the orthogonal projection $\pi\colon B \to A$ is an isomorphism; this gives a fairly canonical way to identify B with A, which has been used for many purposes. Unfortunately it does not give an *isometry* from A to B, which is inconvenient for us, so we will give a slightly different construction.

Definition 7.1. For any universe $\mathcal{U}$, we define

$$G(\mathcal{U}) = \{A \mid A \ll \mathcal{U}\} = \text{ the Grassmannian of } \mathcal{U}$$
$$N(\mathcal{U}) = \{(A,B) \in G(\mathcal{U})^2 \mid \dim(A) = \dim(B) \text{ and } A \cap B^\perp = 0\}.$$

If U is finite then $G(U)$ is a compact manifold, and in general we topologise $G(\mathcal{U})$ as $\varinjlim_{U \ll \mathcal{U}} G(U)$. It is easy to see that $N(\mathcal{U})$ is a neighbourhood of the diagonal in $G(\mathcal{U})^2$.

Proposition 7.2. *There is a natural continuous map $\rho\colon N(\mathcal{U}) \to O(\mathcal{U})$ such that $\rho(A,B)A = B$ and $\rho(B,A) = \rho(A,B)^{-1}$. (Naturality is to be interpreted using Definition 16.5.)*

The proof will follow after a lemma.

Lemma 7.3. *Let A and B be subspaces of a finite universe U, and suppose that $\dim(A) = \dim(B)$. Let $\alpha, \beta \in End(U)$ be the orthogonal projections onto A and B. Then $1 - (\alpha - \beta)^2$ is a positive self-adjoint operator on U, and the following are equivalent:*

(a) $A \cap B^\perp = 0$
(b) $A^\perp \cap B = 0$
(c) $\|(\alpha - \beta)^2\| < 1$
(d) $1 - (\alpha - \beta)^2$ *is invertible.*

Proof. First note that $\alpha - \beta$ is self-adjoint (because α and β are) so we can choose an orthonormal basis of eigenvectors e_i with eigenvalues t_i say. As α is a projector we have $0 \leq \langle \alpha e_i, e_i \rangle \leq 1$ and similarly for β, so

$$t_i = \langle \alpha e_i, e_i \rangle - \langle \beta e_i, e_i \rangle \in [-1, 1],$$

so $1 - t_i^2 \in [0, 1]$. It follows that $\xi := 1 - (\alpha - \beta)^2$ is a positive self-adjoint operator. Given all this it is clear that (c)$\Leftrightarrow$(d).

Next let $j \colon A \to U$ and $k \colon B \to U$ be the inclusions, and write j^* and k^* for their adjoints. Then $A^\perp \cap B$ is the kernel of $j^* k$, so it vanishes iff $j^* k$ is iso, iff the adjoint map $k^* j$ is iso, iff $A \cap B^\perp = 0$. Thus (a)$\Leftrightarrow$(b).

Next note that $\xi = (1 - \alpha + \beta)(1 + \alpha - \beta)$ is invertible iff the two factors are invertible. On $A \cap B^\perp$ we have $\alpha = 1$ and $\beta = 0$ so $1 - \alpha + \beta = 0$. Thus if ξ is invertible we have $A \cap B^\perp = 0$ and similarly $A^\perp \cap B = 0$. Conversely, suppose that $1 - \alpha + \beta$ is not invertible, so $(1 - \alpha)u = -\beta u = v$ say for some $u \in U \setminus \{0\}$. Clearly $v \in \operatorname{image}(1 - \alpha) \cap \operatorname{image}(\beta) = A^\perp \cap B$. If $v \neq 0$ then $A^\perp \cap B \neq 0$. On the other hand, if $v = 0$ then $u \in \ker(1 - \alpha) \cap \ker(\beta) = A \cap B^\perp$ so $A \cap B^\perp \neq 0$ by the first paragraph. A similar argument works if $1 + \alpha - \beta$ is not invertible. This completes the proof. □

Proof of proposition 7.2. Suppose that $(A, B) \in N(\mathcal{U})$. Choose $U \in \mathcal{U}$ with $A + B \leq U$. Let $\alpha, \beta \colon U \to U$ be the orthogonal projections onto A and B, and put

$$\delta = (\alpha - \beta)^2 = \alpha + \beta - \alpha\beta - \beta\alpha.$$

As $\|\delta\| < 1$ we can define $\chi = \sqrt{1 - \delta}$ by the usual power series expansion. This is clearly self-adjoint and invertible. We also write $\gamma = \alpha\beta - \beta\alpha$, which clearly has $\gamma^* = -\gamma$. We see by direct calculation that α and β commute with δ and thus with χ and χ^{-1}. It follows that any expression in α and β commutes with δ, χ and χ^{-1}; in particular this is true of γ. We also check directly that $\gamma^2 = -\delta(1 - \delta) = -\delta\chi^2$. Now put $\rho = \chi - \gamma\chi^{-1}$. We have

$$\rho\rho^* = (\chi - \gamma\chi^{-1})(\chi + \gamma\chi^{-1}) = \chi^2 - \gamma^2\chi^{-2} = (1 - \delta) + \delta = 1,$$

so $\rho \in O(U)$. We also calculate that

$$\begin{aligned} \rho\alpha\chi = \rho\chi\alpha &= (1 - \delta - \gamma)\alpha \\ &= (1 - \alpha - \beta + \alpha\beta + \beta\alpha - \alpha\beta + \beta\alpha)\alpha \\ &= \beta\alpha, \end{aligned}$$

so $(1 - \beta)\rho\alpha = (1 - \beta)\beta\alpha\chi^{-1} = 0$. As $A = \operatorname{image}(\alpha)$ and $B = \ker(1 - \beta)$ we see that $\rho(A) \leq B$, and thus $\rho(A) = B$ by dimension count.

Now define $\rho(A, B) \colon \mathcal{U} \to \mathcal{U}$ to be ρ on U and 1 on $U^\perp$. This is easily seen to be natural and independent of the choice of U, and to depend continuously on A and B.

If we exchange A and B then δ and χ are unchanged and γ becomes $-\gamma$, so ρ becomes $\chi + \gamma\chi^{-1}$ which we have seen is inverse to the original ρ. Thus $\rho(B, A) = \rho(A, B)^{-1}$ as claimed. □

Remark 7.4. A more abstract point of view is as follows. Let M be a manifold with a transitive action of a compact Lie group G, and suppose we have chosen an invariant inner product on G. Given sufficiently close points $x, y \in M$, there will be a unique element v close to zero in the Lie algebra of G such that $\exp(v).x = y$ and v is orthogonal to the Lie algebra of the stabiliser of x. We can then define $\rho(x, y) = \exp(v)$ to get a map from a neighbourhood of the diagonal in $M \times M$ to G, with the property that $\rho(x, y).x = y$. Note also that the map $t \mapsto \exp(tv).x$ gives a geodesic from x to y.

In particular, this applies when $G = O(W)$ and M is the Grassmannian of n-planes in W and the inner product on $LG \leq \operatorname{End}(W)$ is given by $\langle \alpha, \beta \rangle = \operatorname{trace}(\alpha\beta)$. One can check that the resulting map ρ in this case is the same as that defined more explicitly above. We will record some of the formulae necessary to check this, using the notation of the above proof. Let $\sqrt{\delta}$ be the unique positive self-adjoint square root of δ and let $\phi = \frac{1}{2}\sin^{-1}(\sqrt{\delta})$ be defined by the obvious power series. Then $\delta = \sin^2(2\phi)$ and $\chi = \cos(2\phi)$. If ϕ is invertible we can define $i = 2\gamma/\sin(4\phi)$ and we find that $i^2 = -1$ and $-i\alpha i = 1 - \alpha$, so i gives a complex structure on W. We then find that $2i\phi$ is orthogonal to the Lie algebra of $O(V) \times O(V^\perp)$ (because it carries V to $V^\perp$ and *vice versa*) and that $\exp(2i\phi) = \rho(U, V)$. We also find that the orthogonal projection onto the space $\exp(2i\phi t).U$ is given by

$$\pi_t = \frac{\sin(4\phi(1-t))}{\sin(4\phi)}\alpha + \frac{\sin(4\phi t)}{\sin(4\phi)}\beta - \frac{\sin(2\phi t)\sin(2\phi(1-t))}{\cos(2\phi)}.$$

In particular, one checks that

$$\pi_{1/2} = (2\chi)^{-1}(\alpha+\beta+\chi-1).$$

If ϕ is not invertible we merely remark that $f(z) = 4z/\sin(4z)$ is a power series in z with radius of convergence π so we can still interpret $f(\phi)\gamma$ as an endomorphism of W, which is just $2i\phi$ in the invertible case. We again have $\rho(U,V) = \exp(f(\phi)\gamma)$ and everything goes through much as before.

It is an illuminating exercise to work through all the formulae when $W = \mathbb{R}^2$ and U and V are the lines at angles $+\theta$ and $-\theta$ to the x-axis.

We now digress slightly to prove another useful result using the same ideas.

Proposition 7.5. *Fix integers a,b,n with $0 \leq a,b \leq n$, and put $d = \min(a,b,n-a,n-b)$. Let $\mathcal{C}$ be the category of triples (A,B,V), where V is an n-dimensional complex universe, and A and B are subspaces of dimensions a and b. Then $\pi_0\mathcal{C}$ is naturally identified with the simplex Δ_d.*

Proof. Firstly, we identify Δ_d with

$$\{x \in I^d \mid 0 \leq x_1 \leq \cdots \leq x_d \leq 1\} \subset I^d,$$

and we note that the resulting map

$$\Delta_d \to I^d \to I^d/\Sigma_d = \mathrm{SP}^d(I)$$

is a homeomorphism. We can regard $\mathrm{SP}^d(I)$ as a subset of $\mathbb{Z}[I]$, and thus of $\mathbb{Q}[I]$ (the free rational vector space on the set I). We will define an invariant $\phi\colon \pi_0\mathcal{C} \to \mathbb{Z}[I]$ and show that it actually gives a bijection $\pi_0\mathcal{C} \to \mathrm{SP}^d(I)$.

As before, we let α and β be the orthogonal projectors on to A and B, and we let $\chi = \chi(\alpha,\beta)$ be the unique positive self-adjoint square root of $(\alpha-\beta)^2$. We note that χ commutes with α and β, and has all eigenvalues in I. It is also clear that

$$\chi(\alpha,\beta) = \chi(\beta,\alpha) = \chi(1-\alpha,1-\beta) = \chi(1-\beta,1-\alpha).$$

We put

$$\xi(\alpha,\beta) = \chi(\alpha,1-\beta) = \chi(1-\alpha,\beta) = \chi(\beta,1-\alpha) = \chi(1-\beta,\alpha).$$

Observe that

$$\begin{aligned}\chi^2+\xi^2 &= (\alpha-\beta)^2 + (\alpha+\beta-1)^2\\ &= \alpha^2 - \alpha\beta - \beta\alpha + \beta^2 + \alpha^2 + \beta^2 + 1 + \alpha\beta + \beta\alpha - 2\alpha - 2\beta\\ &= \alpha - \alpha\beta - \beta\alpha + \beta + \alpha + \beta + 1 + \alpha\beta + \beta\alpha - 2\alpha - 2\beta\\ &= 1.\end{aligned}$$

The n-tuple of eigenvalues of χ (repeated with appropriate multiplicities) defines a poin $\mathrm{spec}(\chi) \in \mathrm{SP}^n(I) \subset \mathbb{Q}[I]$. We define

$$\phi[A,B,V] = (\mathrm{spec}(\chi) - |a-b|.[1] - |a+b-n|.[0])/2 \in \mathbb{Q}[I].$$

To analyse this in more detail, we let $\nu(s)$ denote the multiplicity of s as an eigenvalue of ξ (which is the same as the multiplicity of $t = \sqrt{1-s^2}$ as an eigenvalue of χ). Put $V_s = \ker(\xi - s)$, so $V = \bigoplus_s V_s$ and $\dim(V_s) = \nu(s)$. As α and β commute with ξ we see that they preserve this splitting, so $A = \bigoplus_s A_s$ and $B = \bigoplus_s B_s$, where $A_s = A \cap V_s$ and $B_s = B \cap V_s$. Let α_s and β_s be the restrictions of α and β to V_s.

We next claim that $\nu(1) \geq |a-b|$. It will suffice to prove this when $a \geq b$. We then have $\dim(A \cap B^\perp) \geq a-b = |a-b|$, and on $A \cap B^\perp$ we have $\alpha = 1$ and $\beta = 0$ so $\chi = \sqrt{(1-0)^2} = 1$ as required. By applying the same logic to the triple $(A, B^\perp, V)$ we see that $\nu(0) \geq |a+b-n|$. It follows that $2\phi[A,B,V] \in \mathbb{N}[I] \subset \mathbb{Q}[I]$.

Now suppose that $0 < s < 1$ and put $V_s^+ = \ker(\alpha-\beta-s)$ and $V_s^- = \ker(\alpha-\beta+s)$. As $(\alpha-\beta)^2 = \chi^2$ acts as s^2 on V_s we see that $V_s = V_s^+ \oplus V_s^-$. Put $t = \sqrt{1-s^2}$. We claim that relative to the above splitting we have

$$\alpha_s = \frac{1}{2}\begin{pmatrix} 1+s & t\mu \\ t\mu^{-1} & 1-s \end{pmatrix} \qquad \beta_s = \frac{1}{2}\begin{pmatrix} 1-s & t\mu \\ t\mu^{-1} & 1+s \end{pmatrix}$$

for some isometric isomorphism $\mu\colon V_s^- \to V_s^+$. Indeed, we certainly have $\alpha_s - \beta_s = \left(\begin{smallmatrix} s & 0 \\ 0 & -s \end{smallmatrix}\right)$, and it follows easily from this and self-adjointness that

$$\alpha_s = \frac{1}{2}\left(\begin{smallmatrix} \sigma+s & t\mu \\ t\mu^* & \rho-s \end{smallmatrix}\right) \qquad \beta_s = \frac{1}{2}\left(\begin{smallmatrix} \sigma-s & t\mu \\ t\mu^* & \rho+s \end{smallmatrix}\right)$$

for some $\sigma\colon V_s^+ \to V_s^+$ and $\rho\colon V_s^- \to V_s^-$ and $\mu\colon V_s^- \to V_s^+$, with σ and ρ self-adjoint. We must show that σ and ρ are identity maps and that μ is an isometric isomorphism. **I have this claim on a piece of paper but without a proof. It probably follows from the idempotence of α and β but I don't see a good way to organise the calculation at the moment. Given this, we see that $\nu(s) = 2\dim(V_s^+)$ is even for $0 < s < 1$. We still need a separate argument to show that $\nu(0) - |a+b-n|$ and $\nu(1) - |a-b|$ are even.** □

Remark 7.6. One can identify $\pi_0\mathcal{C}$ with $(G_a(\mathbb{C}^n) \times G_b(\mathbb{C}^n))/U(n)$, which gives a topology. Presumably $\pi_0\mathcal{C}$ is actually homeomorphic to Δ_d, but I have not attempted to check any details.

8. More rotations and reflections

Given vectors $u, v \in S(V)$ with $u+v \neq 0$, we observe that $\langle u, v\rangle > -1$ so we can define $\mathrm{rot}(u,v) \in \mathrm{End}(V)$ by

$$\mathrm{rot}(u,v)(x) = x - \frac{\langle u+v, x\rangle(u+v)}{1+\langle u,v\rangle} + 2\langle u,x\rangle v.$$

One checks that this actually lies in $SO(V)$ and that it sends u to v and acts as the identity on $\mathbb{R}\{u,v\}^\perp$. This defines a continuous map

$$\mathrm{rot}\colon \{(u,v) \in S(V)^2 \mid u+v \neq 0\} \to SO(V).$$

We also define $\mathrm{ref}(u)(x) = x - 2\langle u,x\rangle u$, so $\mathrm{ref}\colon S(V) \to O(V)$ and $\mathrm{ref}(u)(u) = -u$ and $\mathrm{ref}(u)$ acts as the identity on $\mathbb{R}\{u\}^\perp$.

Lemma 8.1. *If $V = \mathbb{H}$ or $V = \mathbb{O}$ and $u \in S(V)$ then $\mathrm{ref}(u)(x) = -u\overline{x}u$. (In the octonionic case, we recall that $\mathbb{O}$ is alternative, which means that $(u\overline{x})u = u(\overline{x}u)$, so brackets can be omitted.)*

Proof. Put $g(x) = -u\overline{x}u$; it will suffice to show that $g(u) = -u$ and $g(x) = -x$ for $x \in u^\perp$. The first of these is clear (becaues $u\overline{u} = 1$ for $u \in S(V)$). For the second recall that $\langle x, u\rangle = \mathrm{Re}(\overline{u}x)$, so $x \in u^\perp$ iff $x = uz$ for some z with $\overline{z} = -z$. The alternative property implies that the subalgebra of V generated by z and $\mathrm{Im}(u)$ is associative, so $\overline{x}u = (\overline{z}\overline{u})u = \overline{z}(\overline{u}u) = \overline{z} = -z$. This gives $g(x) = -u((\overline{z}\overline{u})u) = uz = x$, as required. □

We define a stereographic projection map $\phi_V\colon S(V \oplus \mathbb{R}) \to S^V$ by $\phi(v,t) = v/(1-t)$, which has inverse

$$\phi^{-1}(w) = (2w, \|w\|^2 - 1)/(\|w\|^2 + 1).$$

Note that for $v \in S(V)$ we have $\phi(v,0) = v$ and so $\phi^{-1}(v) = (v,0)$.

Now suppose that $V = \mathbb{K}_0 = \{a \in \mathbb{K} \mid a + \overline{a} = 0\}$ for $\mathbb{K} = \mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$. We identify $V \oplus \mathbb{R}$ with $\mathbb{K}$ in the obvious way. Then for $w \in \mathbb{K}_0$ we have $\|w\|^2 = -w^2$ and so

$$\phi^{-1}(w) = \frac{2w - w^2 - 1}{1 - w^2} = \frac{(w-1)^2}{w^2-1} = \frac{w-1}{w+1}.$$

It follows that for $a \in S(\mathbb{K})$ we have

$$\phi(a) = (1+a)/(1-a) = (1+a)^2/(1+\|a\|^2).$$

(Note that all the above takes place in a subalgebra of $\mathbb{K}$ generated by a single element, and all such subalgebras are commutative and associative even when $\mathbb{K} = \mathbb{O}$.)

9. Complex structures

Definition 9.1. Let U be a real universe of even dimension. Put

$$\begin{aligned} \mathcal{C}(U) &= \{ \text{ complex structures on } U \} = \{ j \in \operatorname{End}(U) \mid j^2 = -1 \} \\ \mathcal{C}_1(U) &= \mathcal{C}(U) \cap O(U) = \{ j \in \mathcal{C}(U) \mid j^\dagger j = 1 \}. \end{aligned}$$

Proposition 9.2. *The space $\mathcal{C}_1(U)$ is naturally a deformation retract of $\mathcal{C}(U)$. More precisely, we can define $h_t \colon \mathcal{C}(U) \to \mathcal{C}(U)$ by $h_t(j) = j(j^\dagger j)^{-t/2}$, and then h_0 is the identity and $h_1(j) \in \mathcal{C}_1(U)$.*

Proof. Put $m = j^\dagger j$. Using $j^2 = -1$ we find that $j^{-1} m j = j j^\dagger = m^{-1}$, so m is invertible and conjugate to its inverse. Let E be the set of eigenvalues of m. For $\lambda \in E$ we put $U_\lambda = \ker(m - \lambda)$. As m is self-adjoint and positive and invertible, we see that the elements λ are strictly positive real numbers, and that U is the orthogonal direct sum of the spaces U_λ. As $j^{-1} m j = m^{-1}$ we see that $\lambda^{-1} \in E$ and $j(U_\lambda) = U_{\lambda^{-1}}$. The power m^s (for $s \in \mathbb{R}$) can be defined by functional calculus; it acts on U_λ as multiplication by λ^s. Using $j(U_\lambda) = U_{\lambda^{-1}}$ we see that $h_t(j)^2$ acts as -1 on U_λ, but $U = \bigoplus_\lambda U_\lambda$, so $h_t(j)^2 = -1$, or in other words $h_t(j) \in \mathcal{C}(U)$ as claimed.

It is also straightforward to check that

$$h_1(j)^\dagger h_1(j) = m^{-1/2} j^\dagger j m^{-1/2} = m^{-1/2} m m^{-1/2} = 1,$$

so $h_1(j) \in \mathcal{C}_1(U)$. ☐

Question: is there a corresponding result for quaternionic structures, or more general Clifford module structures?

Remark 9.3. There is a slogan that giving a complex structure on V is equivalent to giving one on $V^\perp$. I think that the correct formulation is like this. Fix a basepoint in $\mathcal{C}(U \oplus V)$, and let F be the fibre of the inclusion $\mathcal{C}(U) \times \mathcal{C}(V) \to \mathcal{C}(U \oplus V)$. Then under appropriate assumptions on the dimensions, the projections $\mathcal{C}(U) \leftarrow F \to \mathcal{C}(V)$ will be highly connected.

10. Quotients of projective spaces

Proposition 10.1. *There is a natural homeomorphism $P(V \oplus W)/PW = PV^{\operatorname{Hom}(T,W)}$.*

Proof. **Write this out** ☐

Proposition 10.2. *Let V, W and X be unitary representations of A, where V and W have finite dimension and X is a colimit of finite-dimensional subrepresentations. Put $U = V \oplus W \oplus X$. Then there is a homotopy-commutative diagram as follows, in which the maps marked q are the obvious collapses, the maps marked j are the obvious inclusions, and δ is the diagonal map.*

$$\begin{array}{ccccc} PU & & \xrightarrow{\quad\delta\quad} & & PU \times PU \\ {\scriptstyle q_{V\oplus W}}\downarrow & & & & \downarrow{\scriptstyle q_V \wedge q_W} \\ PU/P(V \oplus W) & \xrightarrow{\overline{\delta}} & P(V \oplus X)/PV \wedge P(W \oplus X)/PW & \xrightarrow{j \wedge j} & PU/PV \wedge PU/PW \end{array}$$

Moreover, if $\dim(X) = 1$ *then $\overline{\delta}$ is just the standard homeomorphism*

$$S^{\operatorname{Hom}(X, V \oplus W)} = S^{\operatorname{Hom}(X,V)} \wedge S^{\operatorname{Hom}(X,W)}.$$

All maps and homotopies are natural for isometric embeddings of V, W and X.

Remark 10.3. The above diagram gives a map

$$\overline{\delta}^* \colon E^*(P(V \oplus X), PV) \otimes E^*(P(W \oplus X), PW) \to E^*(PU, P(V \oplus W)).$$

In his unpublished thesis [1], Cole writes $a * b$ for $\overline{\delta}^*(a \otimes b)$. The idea of using this construction seems to be original to that thesis; our approach differs only in being somewhat more geometric.

Proof. Assume for the moment that X is finite-dimensional. We start by defining a map

$$\overline{\gamma}\colon PU/P(V\oplus W)\to PU/PV\wedge PU/PW,$$

which will be homotopic to $(j\wedge j)\circ\overline{\delta}$. For $u=(v,w,x)\in U^\times:=U\setminus\{0\}$ we put

$$s=s(u)=(\|w\|-\|v\|)/(\|v\|+\|w\|+\|x\|).$$

Note that $s(u)\in[-1,1]$, and $s(\lambda u)=s(u)$ for all $\lambda\in\mathbb{C}^\times$, and $s(u)\geq 0$ iff $\|w\|\geq\|v\|$. We next define $\alpha,\beta\colon U^\times\to U$ by

$$\alpha(v,w,x)=\begin{cases}((1-s)v,sw,x) & \text{if } s\geq 0\\ (v,0,x) & \text{if } s\leq 0\end{cases}$$

$$\beta(v,w,x)=\begin{cases}(0,w,x) & \text{if } s\geq 0\\ (-sv,(1+s)w,x) & \text{if } s\leq 0.\end{cases}$$

Note that $\alpha(\lambda u)=\lambda\alpha(u)$ and similarly for β.

We claim that when $u\neq 0$, the line joining u to $\alpha(u)$ never passes through 0 (so in particular $\alpha(u)\neq 0$). Indeed, if $s\leq 0$, then the points on the line have the form (v,tw,x) for $0\leq t\leq 1$. Thus, the line can only pass through zero if $v=x=0$. The relation $s\leq 0$ means that $\|w\|\leq\|v\|=0$, so $w=0$ as well, contradicting the assumption that $u\neq 0$. In the case $s>0$, the points on the line have the form $((1-ts)v,(1-t+ts)w,x)$. As $s>0$ and $0\leq t\leq 1$ we have $1-t+ts>0$. For the line to pass through zero we must thus have $x=w=0$, and the relation $s\geq 0$ means that $\|v\|\leq\|w\|=0$, again giving a contradiction. Similarly, the line from u to $\beta(u)$ never passes through 0.

It follows that α and β induce self-maps of PU that are homotopic to the identity, so the map $\gamma=(\alpha,\beta)\colon PU\to PU\times PU$ is homotopic to the diagonal map δ.

Next, note that if $u\in V\oplus W$, then for $s\geq 0$ we have $\gamma(u)\in U\times W$, and for $s\leq 0$ we have $\gamma(u)\in V\times U$. It follows that the induced map on projective spaces has

$$\gamma(P(V\oplus W))\subseteq(PU\times PW)\cup(PV\times PU),$$

so there is an induced map

$$\overline{\gamma}\colon PU/P(V\oplus W)\to PU/PV\wedge PU/PW.$$

As γ is homotopic to δ, we see that $\overline{\gamma}\circ q_{V\oplus W}\simeq(q_V\wedge q_W)\circ\delta$.

To construct the map $\overline{\delta}$, we need a slightly different model. Clearly

$$PU\setminus P(V\oplus W)=(V\times W\times X^\times)/\mathbb{C}^\times=(V\times W\times S(X))/S^1,$$

and $PU/P(V\oplus W)$ is the one-point compactification of this. Similarly, $P(V\oplus X)/PV\wedge P(W\oplus X)/PW$ is the one-point compactification of the space $(V\times S(X))/S^1\times(W\times S(X))/S^1$. We can thus define $\overline{\delta}$ by giving a proper map

$$V\times W\times S(X)\to V\times S(X)\times W\times S(X)$$

with appropriate equivariance. The map in question just sends (v,w,x) to (v,x,w,x).

If X is one-dimensional and $(v,x)\in V\times S(X)$ then we have a linear map $\alpha\colon X\to V$ given by $\alpha(x)=v$, which does not change if we multiply (v,x) by an element of S^1. This gives a homeomorphism $(V\times S(X))/S^1=\mathrm{Hom}(X,V)$, and thus $P(V\oplus X)/PV=S^{\mathrm{Hom}(X,V)}$. It is easy to see that with this identification, $\overline{\delta}$ is just the standard homeomorphism

$$S^{\mathrm{Hom}(X,V\oplus W)}=S^{\mathrm{Hom}(X,V)}\wedge S^{\mathrm{Hom}(X,W)}.$$

We now show that $(j\wedge j)\circ\overline{\delta}\simeq\overline{\gamma}$. Put

$$T=\{((v_0,w_0,x_0),(v_1,w_1,x_1))\in U^2\mid\|(w_0,x_0)\|=\|(v_1,x_1)\|=1\},$$

so that $PU/PV\wedge PU/PW$ is the one-point compactification of $T/(S^1\times S^1)$. Define maps

$$\theta_t\colon V\times W\times S(X)\to T$$

for $0\leq t\leq 1$ by

$$\theta_t(v,w,x)=\begin{cases}\left(\frac{((1-st)v,stw,x)}{\|(stw,x)\|},(0,w,x)\right) & \text{if } s\geq 0\\ \left((v,0,x),\frac{(-stv,(1+st)w,x)}{\|(-stv,x)\|}\right) & \text{if } s\leq 0,\end{cases}$$

where $s = (\|w\| - \|v\|)/(\|v\| + \|w\| + \|x\|)$ as before. (Note that both clauses give $\theta_t(v, w, x) = ((v, 0, x), (0, w, x))$ if $s = 0$, so the two clauses are consistent.)

We claim that the maps θ_t are proper. To see this, put

$$\nu((v_0, w_0, x_0), (v_1, w_1, x_1)) = \max(\|v_0\|, \|w_1\|),$$

and $T_k = \{t \in T \mid \nu(t) \leq k\}$. It is easy to see that every compact subset of T is contained in some T_k, so it will be enough to show that $\theta_t^{-1} T_k$ is compact. In the case $s \geq 0$ we have $0 \leq 1 - st \leq 1$ and $\|(stw, x)\| \geq \|x\| = 1$ so $\|((1 - st)v/\|(stw, x)\|)\| \leq \|v\| \leq \|w\|$, so $\nu(\theta_t(v, w, x)) = \|w\|$. Similarly, when $s \leq 0$ we have $\nu(\theta_t(v, w, x)) = \|v\|$, so in general $\nu(\theta_t(v, w, x)) = \max(\|v\|, \|w\|)$. It follows immediately that θ_t is proper, and we get an induced family of maps

$$\theta_t \colon PU/P(V \oplus W) \to PU/PV \wedge PU/PW.$$

It is easy to see that $\theta_0 = (j \wedge j) \circ \overline{\delta}$ and $\theta_1 = \overline{\gamma}$. The proposition follows easily (for the case where X has finite dimension).

If X has infinite dimension, we apply the above to all finite dimensional subrepresentations of X. We see by inspection that all constructions pass to the colimit, so the conclusion is valid for X itself. $\square$

By an evident inductive extension, we obtain the following:

Corollary 10.4. *Let $L_1, \dots, L_d$ be one-dimensional representations of A, and let X be as above. Put $Y = \bigoplus_i L_i$ and $U = Y \oplus X$. Then there is a homotopy-commutative diagram as follows:*

$$\begin{array}{ccccc}
PU & & \xrightarrow{\quad\delta\quad} & & PU^r \\
\downarrow q & & & & \downarrow q \\
PU/PY & \xrightarrow[\overline{\delta}]{} & \bigwedge_i P(L_i \oplus X)/PL_i & \xrightarrow[j]{} & \bigwedge_i PU/PL_i
\end{array}$$

Moreover, if $\dim(X) = 1$ *then* $\overline{\delta}$ *is just the standard homeomorphism*

$$S^{\mathrm{Hom}(X,Y)} = \bigwedge_i S^{\mathrm{Hom}(X,L_i)}. \quad \square$$

11. Unitary groups

Fibrations $U(V) \to U(V \oplus \mathbb{C}) \to S(V \oplus \mathbb{C})$. The complex reflection map. The Miller filtration. Filtration-preserving approximation to the diagonal.

12. The clutching map

Here I just record some formulae for the map $\Sigma U(V) \to G_d(V^2)$. We regard V^2 as $V \otimes_{\mathbb{R}} \mathbb{C}$. We define $\kappa \colon \mathbb{C} \to \mathbb{C}$ by $\kappa(z) = \overline{z}$, and $\gamma \colon U(1) \to \mathbb{R} \cup \{\infty\}$ by $\gamma(z) = i(1 + z)/(1 - z)$, so $\gamma^{-1}(t) = (it + 1)/(it - 1)$. We define $\beta(t)$ to be the square root of $\gamma^{-1}(t)$, using the branch such that $\beta(t) \to \mp 1$ as $t \to \pm\infty$. We then define

$$\phi(t, g) = ((1 + g) \otimes \kappa + (1 - g) \otimes \beta(t))/2 \in U(V \otimes \mathbb{C}),$$

and we let $\psi(t, g) = W_{(t,g)}$ be the image of $V \otimes 1$ under $\phi(t, g)$. We find that

$$\begin{aligned}
\phi(t, g) &\to g \oplus (-1) && \text{as } t \to \infty \\
\phi(t, g) &\to 1 \oplus (-g) && \text{as } t \to -\infty.
\end{aligned}$$

It follows that ψ extends to a based map $\Sigma U(V) \to G_d(V^2)$, if we use $V \otimes 1$ as a basepoint in $G_d(V^2)$. Now define I_+ to be the closure of the image of $[0, \infty)$ in S^1, and let I_- be the closure of the image of $(-\infty, 0]$, so $I_+ \cap I_- = \{0, \infty\} = S^0$. Put $C_\pm = I_\pm \wedge U(V)$, and define trivialisations $\theta_\pm \colon C_\pm \times V \to W|_{C_\pm}$ by

$$\begin{aligned}
\theta_+(t, g)(v) &= \phi(t, g)(g^{-1} v) \\
\theta_-(t, g)(v) &= \phi(t, g)(v).
\end{aligned}$$

Note that $\theta_\pm(\infty, g) = v$, and that $\theta_+(0, g)^{-1} \theta_-(0, g) = g \colon V \to V$. This means that the identity map of $U(V)$ is the clutching function for W.

13. The J-homomorphism

We define an unbased map $J\colon O(V) \to F(S^V, S^V)$ as follows. Given $v \in S(V)$ and $r \in [0,\infty]$ we put

$$J(g)(rv) = \begin{cases} v\sqrt{r^{-2}-1} & \text{if } r \leq 1 \\ g(v)\sqrt{1-r^{-2}} & \text{if } r \geq 1. \end{cases}$$

We are using some obvious conventions about infinity, so $J(g)(0) = J(g)(\infty) = \infty$ and $J(g)(v) = 0$ for $v \in S(V)$. Note that the formula is independent of v when $r = 0$ or $r = \infty$, which implies that we have a well-defined, continuous map.

We next show that $J(1)$ is joined to the basepoint in $F(S^V, S^V)$ by a canonical path. Indeed, we can just define

$$H_t(rv) = (t + |r^{-2} - 1|^{1/2})v$$

for $0 \leq t, r \leq \infty$ and $v \in S(V)$. We then have $H_t(0) = H_t(\infty) = \infty$ and $H_0(v) = J(1)(v)$ and $H_\infty(v) = \infty$, as required.

Note that the function $r \mapsto \sqrt{1-r^{-2}}$ gives an increasing homeomorphism $(1,\infty) \to (0,\infty)$, whereas the function $r \mapsto \sqrt{r^{-2}-1}$ gives a decreasing homeomorphism $(0,1) \to (0,\infty)$. This means that $J = \alpha - \beta$, where $\alpha, \beta\colon O(V) \to F(S^V, S^V)$ are given by $\alpha(g) = S^g$ and $\beta(g) = 1_{S^V}$, and subtraction uses the co-H-structure of the first S^V. There is a wrinkle here because the co-H-structure depends on a choice of embedding $\mathbb{R} \to V$, whereas our definition uses the collapse $S^V \to S^V/S(V)$ which is not a comultiplication because the basepoint does not lie in $S(V)$. However, I think that everything is OK up to a non-canonical homotopy.

Using our homotopy H we can extend J to a based map defined on the whiskered space $O(V)' = (O(V) \amalg [0,\infty])/(1_V \sim 0)$ (where ∞ is taken as the basepoint). We can also define a canonical homotopy equivalence $O(V) \to O(V)'$ by functional calculus. Define $u\colon O(V) \to [-1,1]$ by $u(g) = \min(\mathrm{Re}(\mathrm{spec}(g)))$. Define $p\colon S^1 \to S^1$ by

$$p(-e^{i\theta}) = \begin{cases} -e^{2i\theta} & \text{if } 0 \leq |\theta| \leq \pi/2 \\ 1 & \text{if } \pi/2 \leq |\theta| \leq \pi, \end{cases}$$

so p collapses the right-hand semicircle to a point and stretches out the left-hand semicircle to fill the whole circle. By functional calculus we get a map $p\colon U(\mathbb{C} \otimes V) \to U(\mathbb{C} \otimes V)$ satisfying $\mathrm{spec}(p(g)) = p(\mathrm{spec}(g))$. This is homotopic to the identity, and I think it preserves $O(V)$. Now choose an increasing homeomorphism $f\colon [0,1] \to [0,\infty]$ and define $p'\colon O(V) \to O(V)'$ by

$$p'(g) = \begin{cases} p(g) \in O(V) & \text{if } u(g) \leq 0 \\ f(u(g)) \in [0,\infty] & \text{if } u(g) \geq 0. \end{cases}$$

This is the required equivalence.

14. Loop groups

Polynomial loops etc.

15. Schubert cells

16. Universes

Definition 16.1. We write $\mathbb{R}^\infty$ for the direct sum of countably many copies of $\mathbb{R}$, with basis $\{e_0, e_1, \ldots\}$. We give this the obvious inner product, and regard $\mathbb{R}^n$ as a subspace in the obvious way. We write $\mathbb{R}^{\infty-n}$ for the orthogonal complement of $\mathbb{R}^n$, which is spanned by $\{e_k \mid k \geq n\}$.

A *universe* is a real vector space $\mathcal{U}$ of finite or countable dimension, equipped with an inner product. Any universe is isometrically isomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N} \cup \{\infty\}$, by a Gram-Schmidt argument. It will be technically convenient to require that the underlying set of $\mathcal{U}$ lies in the stage V_{ω_1} of the von Neumann hierarchy, where ω_1 is the first uncountable ordinal; this ensures that there is only a set of universes. With any of the usual definitions of $\mathbb{R}$, $\oplus$ and $\otimes$, we find that $\mathbb{R}$ is a universe and that the set of universes is closed under countable direct sums and under tensor products.

We say that a universe $\mathcal{U}$ is *infinite* if it has infinite dimension. We write $U \ll \mathcal{U}$ to indicate that U is a finite-dimensional subspace of $\mathcal{U}$. We topologise $\mathcal{U}$ as the colimit of its finite-dimensional subspaces; this is different from the metric topology if $\mathcal{U}$ is infinite. We write $\mathcal{L}^+$ for the category of universes; the morphisms $\mathcal{L}^+(\mathcal{U},\mathcal{V})$ are the linear maps that preserve the inner product. We also write $\mathcal{L}$ for the full subcategory of infinite universes.

If $\mathcal{U}$ and $\mathcal{V}$ are finite then $\mathcal{L}^+(\mathcal{U},\mathcal{V})$ has an evident topology making it a compact manifold. If $\mathcal{U}$ or $\mathcal{V}$ is infinite then we topologise $\mathcal{L}^+(\mathcal{U},\mathcal{V})$ as

$$\mathcal{L}^+(\mathcal{U},\mathcal{V}) = \varprojlim_{U\ll\mathcal{U}} \varinjlim_{V\ll\mathcal{V}} \mathcal{L}^+(U,V).$$

We also write

$$\begin{aligned}
\mathcal{L}^+(\mathcal{U}) &= \mathcal{L}^+(\mathcal{U},\mathcal{U}) \\
V_n(\mathcal{U}) &= \mathcal{L}^+(\mathbb{R}^n,\mathcal{U}) = \{(u_0,\dots,u_{n-1}) \in \mathcal{U}^n \mid \langle u_i,u_j\rangle = \delta_{ij}\} \\
\mathcal{L}(n) &= \mathcal{L}((\mathbb{R}^\infty)^n,\mathbb{R}^\infty) \\
S^\infty &= V_1\mathbb{R}^\infty = \{u \in \mathbb{R}^\infty \mid \langle u,u\rangle = 1\}.
\end{aligned}$$

We next prove the well-known fact that many of these spaces are contractible.

Lemma 16.2. *The space $\mathcal{L}^+(\mathcal{U},\mathcal{V})$ is a retract of the space $\mathcal{J}(\mathcal{U},\mathcal{V})$ of injective linear maps from $\mathcal{U}$ to $\mathcal{V}$.*

Proof. After choosing bases, we may assume that $\mathcal{U} = \mathbb{R}^m$ for some $m \in \mathbb{N}\cup\{\infty\}$. This identifies $\mathcal{J}(\mathcal{U},\mathcal{V})$ with the space X of linearly independent sequences $(v_i)_{i<m}$ in $\mathcal{V}$, and $\mathcal{L}^+(\mathcal{U},\mathcal{V})$ of orthonormal sequences. Given a sequence $\underline{v} \in X$, we define an orthonormal sequence $\gamma(\underline{u}) \in Y$ by the usual Gram-Schmidt procedure:

$$\begin{aligned}
v_0 &= u_0/\|u_0\| \\
\tilde{v}_i &= u_i - \sum_{j<i} \langle u_i,v_j\rangle v_j \\
v_i &= \tilde{v}_i/\|\tilde{v}_i\| \\
\gamma(u_0,u_1,\dots) &= (v_0,v_1,\dots).
\end{aligned}$$

More abstractly, $\underline{v}$ is the unique orthonormal sequence such that $\operatorname{span}(v_j \mid j < i) = \operatorname{span}(u_j \mid j < i)$ and $\langle u_i,v_i\rangle > 0$ for all i. It is easy to check that γ gives a continuous retraction $X \to Y$. □

Proposition 16.3. *If $\mathcal{V}$ is infinite then $\mathcal{L}^+(\mathcal{U},\mathcal{V})$ is contractible for all $\mathcal{U}$. In particular, the spaces $\mathcal{L}^+(\mathcal{V})$, $V_n(\mathcal{V})$, $\mathcal{L}(n)$ and S^∞ are contractible.*

Proof. We may assume that $\mathcal{V} = \mathbb{R}^\infty$ and that $\mathcal{U} = \mathbb{R}^n$ for some $n \le \infty$. We will show that the space $F := \mathcal{J}(\mathbb{R}^n,\mathbb{R}^\infty)$ is contractible; it will follow from Lemma 16.2 that $\mathcal{L}^+(\mathcal{U},\mathcal{V})$ is contractible.

First define $d(t)\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ by $d(t)(e_i) = (1-t)e_i + te_{2i}$. If $v \in \mathbb{R}^\infty \setminus \{0\}$ then, by considering $\max\{i \mid v_i \ne 0\}$, we see that $d(t)(v) \ne 0$ for all t. Thus, we have $d(t) \in \mathcal{J}(\mathbb{R}^\infty,\mathbb{R}^\infty)$, and we can define a homotopy $h\colon I \times F \to F$ by $h(t,f) = d(t)\circ f$.

Next, we define $j\colon \mathbb{R}^n \to \mathbb{R}^\infty$ by $j(e_i) = e_{2i+1}$, and we define another homotopy $k\colon I \times F \to F$ by

$$k(t,f) = (1-t)d(1)\circ f + tj\colon \mathbb{R}^n \to \mathbb{R}^\infty.$$

(This is an injective linear map, as required, because the images of $d(1)$ and j have trivial intersection.)

By composing h and k, we see that the identity map of F is homotopic to the constant map with value j. □

It is not hard to see that the composition map

$$\mathcal{L}^+(\mathcal{U},\mathcal{V}) \times \mathcal{L}^+(\mathcal{V},\mathcal{W}) \to \mathcal{L}^+(\mathcal{U},\mathcal{W})$$

is continuous, and thus that $\mathcal{L}^+(\mathcal{U})$ is a topological monoid. If $\mathcal{U}$ is infinite then it is rather far from being a group, and a number of our results seem very strange if one forgets this. However, it does contain a useful subgroup:

Definition 16.4. For any universe $\mathcal{U}$, we define the orthogonal group of $\mathcal{U}$ as
$$O(\mathcal{U}) = \{f \in \mathcal{L}^+(\mathcal{U}) \mid f|_{U^\perp} = 1 \text{ for some } U \ll \mathcal{U}\} = \varinjlim_{U \ll \mathcal{U}} O(U).$$
We also write $O(n) = O(\mathbb{R}^n)$ for $n \in \mathbb{N} \cup \{\infty\}$. If $\mathcal{U}$ is finite then it is clear that $O(\mathcal{U})$ is a topological group, and the same conclusion holds in general by passage to colimits.

Definition 16.5. Let $\mathcal{V}$ be another universe, and let $f\colon \mathcal{U} \to \mathcal{V}$ be a linear isometry. If $g \in O(\mathcal{U})$ then there is some $W \ll \mathcal{U}$ such that $g = 1$ on $W^\perp$. We can define $f_*(g)\colon fW \to fW$ by $fw \mapsto fgw$, and extend this to an automorphism of $\mathcal{V}$ by defining it to be 1 on $(fW)^\perp$. This is easily seen to be independent of the choice of W, and to give a continuous map $\mathcal{L}(\mathcal{U}, \mathcal{V}) \times O(\mathcal{U}) \to O(\mathcal{V})$ which behaves as a homomorphism in the second variable. In other words, this makes the construction $\mathcal{U} \mapsto O(\mathcal{U})$ into a continuous functor from $\mathcal{L}^+$ to the category of topological groups. If f is an isomorphism we just have $f_*(g) = fgf^{-1}$ but this does not make sense for general f.

Lemma 16.6. *For any universes $\mathcal{U}$ and $\mathcal{V}$, the evident map $\mathcal{L}^+(\mathcal{U}, \mathcal{V}) \to F(\mathcal{U}, \mathcal{V})$ is a closed inclusion.*

Proof. The left hand side is $\varprojlim_{U \ll \mathcal{U}} \mathcal{L}^+(U, \mathcal{V})$ and the right hand side is $\varprojlim_{U \ll \mathcal{U}} F(U, \mathcal{V})$, and inverse limits of regular monomorphisms are regular mono, so it is enough to check that $\mathcal{L}^+(U, \mathcal{V}) \to F(U, \mathcal{V})$ is a closed inclusion. In fact, by Lemma **??**, it is enough to check that the composite $\mathcal{L}^+(U, \mathcal{V}) \to F(U, \mathcal{V}) \to F(S(U)_+, \mathcal{V})$ is a closed inclusion. To do this, choose a sequence $V_0 < V_1 < \ldots \ll \mathcal{V}$ with $\mathcal{V} = \varinjlim_n V_n$. Observe that $\mathcal{L}^+(U, V_n)$ is compact, so the injections $\mathcal{L}^+(U, V_n) \to \mathcal{L}^+(U, V_{n+1})$ and $\mathcal{L}^+(U, V_n) \to F(S(U)_+, V_n)$ are closed inclusions. Moreover, $F(S(U)_+, -)$ preserves closed inclusions, so the following square is a pullback of closed inclusions:
$$\begin{array}{ccc} \mathcal{L}^+(U, V_n) & \rightarrowtail & \mathcal{L}^+(U, V_{n+1}) \\ \downarrow & & \downarrow \\ F(S(U)_+, V_n) & \rightarrowtail & F(S(U)_+, V_{n+1}) \end{array}$$
By applying Lemma **??** and Lemma **??**, we conclude that the map $\mathcal{L}^+(U, \mathcal{V}) \to F(S(U)_+, \mathcal{V})$ of colimits is a closed inclusion as required. □

Lemma 16.7. *For any universes $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$, the evident map $\mathcal{L}^+(\mathcal{U}, \mathcal{V}) \to \mathcal{L}^+(\mathcal{U}, \mathcal{V} \oplus \mathcal{W})$ is a closed inclusion.*

Proof. Consider the following square.
$$\begin{array}{ccc} \mathcal{L}^+(\mathcal{U}, \mathcal{V}) & \longrightarrow & \mathcal{L}^+(\mathcal{U}, \mathcal{V} \oplus \mathcal{W}) \\ \downarrow & & \downarrow \\ F(\mathcal{U}, \mathcal{V}) & \rightarrowtail & F(\mathcal{U}, \mathcal{V} \oplus \mathcal{W}) \end{array}$$
The vertical maps are closed inclusions by Lemma 16.6. The bottom horizontal map is a split monomorphism and thus a closed inclusion. The claim follows by Lemma **??**. □

Lemma 16.8. *The map $\epsilon\colon O(\infty) \to S^\infty$ sending g to $g(e_0)$ is a trivial bundle with fibre isomorphic to $O(\infty)$ (so in particular, it admits a section).*

Proof. For any $u \in \mathbb{R}^\infty \setminus \{0\}$ we let $\tau(u) \in O(\infty)$ be the reflection in the hyperplane orthogonal to u, so $\tau(u)(v) = v - 2\langle u, v\rangle u/\langle u, u\rangle$. If $u, v \in S^\infty$ and $u \neq v$ then it is easy to see that $\tau(u - v)$ exchanges u and v. Next, we let $\rho\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ be the shift map, given by $\rho(e_k) = e_{k+1}$ for all k. If $u \in S^\infty$ it is easy to see that $\rho(u) \in S^\infty$ and that $u \neq \rho(u) \neq e_0$. We can thus define $\sigma(u) = \tau(u - \rho(u))\tau(e_0 - \rho(u)) \in O$ and note that $\sigma(u)(e_0) = u$, so σ is a section of ϵ. We also observe that the map $\rho_*\colon O(\infty) \to O(\infty)$ (as in Definition 16.5) gives a homeomorphism
$$O(\infty) \to O(e_0^\perp) = \{g \in O(\infty) \mid g(e_0) = e_0\}.$$
Now define $\phi\colon S^\infty \times O(\infty) \to O(\infty)$ by $\phi(u, g) = \sigma(u)\rho_*(g)$, so ϕ is a homeomorphism and $\epsilon\phi(u, g) = u$. This gives the required trivialisation. □

16.1. **The topology of S^∞.** We next prove the following slightly surprising result.

Proposition 16.9. *S^∞ is homeomorphic to $\mathbb{R}^\infty$.*

To see this, let B_k be the closed ball of radius k in $\mathbb{R}^k$. Then B_k is compact and is contained in the interior of B_{k+1}, and it is easy to check that $\mathbb{R}^\infty = \varinjlim_k B_k$.

Similarly, put $S_k = \{(x_0, \dots, x_k) \in S^k \mid x_k \leq 1/2\}$. One checks that S_k is homeomorphic to B_k and is contained in the interior of S_{k+1}, and that $S^\infty = \varinjlim_k S_k$. We will also need to know that the pair (S_{k+1}, S_k) is homeomorphic to (B_{k+1}, B_k), or in other words that there is a homeomorphism $f\colon S_{k+1} \to B_{k+1}$ such that $f(S_k) = B_k$. One can either construct such an f directly, or appeal to a general result to be mentioned later. Given this, we need only prove the following result.

Proposition 16.10. *Let X be a space, and suppose that X is the colimit of a sequence of subspaces X_k such that each pair (X_{k+1}, X_k) is homeomorphic to (B_{k+1}, B_k) for all k. Then $X \simeq \mathbb{R}^\infty$.*

The proof depends on the following lemma.

Lemma 16.11. *Any homeomorphism $f\colon B_k \to B_k$ can be extended to give a homeomorphism $E(f)\colon B_{k+1} \to B_{k+1}$.*

Proof. Note that f automatically preserves the boundary ∂B_k, so it is an automorphism of the pair $(B_k, \partial B_k)$. Given any pair of spaces (Y, Z), let $E(Y, Z)$ be the unreduced suspension of $(Y \times \{0\}) \cup (Z \times [0, 1])$, which is clearly a functor of (Y, Z). It is easy to identify $E(B_k, \partial B_k)$ with B_{k+1}, and this gives the result.

More explicitly, a point $u \in B_{k+1}$ can be written as $u = r\cos(\theta)x + (k+1)\sin(\theta)e_k$ for some $r \in [0, k+1]$, $\theta \in [-\pi, \pi]$ and $x \in S^{k-1}$. We then have

$$E(f)(u) = \begin{cases} \cos(\theta)f(rx) + (k+1)\sin(\theta)e_k & \text{if } 0 \leq r \leq k \\ r\cos(\theta)f(kx)/k + (k+1)\sin(\theta)e_k & \text{if } k \leq r \leq k+1. \end{cases}$$

□

Proof of Proposition 16.10. By assumption, we can choose homeomorphisms $f_k\colon B_k \to X_k$ such that the restriction of f_k gives a homeomorphism $B_{k-1} \to X_{k-1}$, which we denote by f'_k. Now define homeomorphisms $g_k\colon B_k \to X_k$ by $g_1 = f_1$ and $g_k = f_k \circ E((f'_k)^{-1} \circ g_{k-1})$ for $k > 1$. One checks that $g_k|_{B_{k-1}} = g_{k-1}$ for all k, so we can pass to colimits to get a homeomorphism $\mathbb{R}^\infty \to X$. □

We can use the same technique to show that many other spaces are homeomorphic to $\mathbb{R}^\infty$.

Proposition 16.12. *Let S be any finite subset of $\mathbb{R}^\infty$. Then $\mathbb{R}^\infty \setminus S$ is homeomorphic to $\mathbb{R}^\infty$.*

Proof. First, the homeomorphism type of $\mathbb{R}^\infty \setminus S$ depends only on the integer $n = |S|$; this is well-known for subsets of $\mathbb{R}^k$ when $k < \infty$, and the case $k = \infty$ follows immediately. We may thus assume that $S = \{me_0 \mid m \in \{0, \dots, n-1\}\}$. Now put

$$\begin{aligned} F_k &= [-n-k, n+k]^k \subset \mathbb{R}^k \\ G_k &= S + (-1/2^k, 1/2^k)^{k-1} \times (-1/2^k, \infty) \subset \mathbb{R}^k \\ X_k &= F_k \setminus G_k. \end{aligned}$$

Then X_k is a k-ball with n separate holes drilled part way into it, so it is again a k-ball. One can also check directly that X_k is contained in the interior of X_{k+1}. If K is a compact subspace of $\mathbb{R}^\infty \setminus S$ then there exists k such that $\operatorname{dist}(K, S) > 1/2^k$ and $\operatorname{dist}(K, 0) < k$ and $K \subset \mathbb{R}^k$, and it follows that $K \subset X_{k+1}$. Using this, we see that $\mathbb{R}^\infty \setminus S = \varinjlim_k X_k$.

We still need to check that $(X_{k+1}, X_k) \simeq (B_{k+1}, B_k)$, but I think this should be formal. □

Proposition 16.13. *Let S be a finite subset of $\mathbb{R}^\infty$, and let $F_n(\mathbb{R}^\infty \setminus S)$ denote the configuration space of n-tuples of distinct points in $\mathbb{R}^\infty \setminus S$. Then $F_n(\mathbb{R}^\infty \setminus S)$ is homeomorphic to $\mathbb{R}^\infty$.*

Proof. Define $\pi\colon F_n(\mathbb{R}^\infty \setminus S) \to \mathbb{R}^\infty \setminus S$ by $\pi(x_1, \ldots, x_n) = x_n$. This is well-known to be a fibre bundle projection, with fibres of the form $F_{n-1}(\mathbb{R}^\infty \setminus (S \cup \{x_n\}))$. By induction, we may assume that these fibres are homeomorphic to $\mathbb{R}^\infty$. We have seen that the base is homeomorphic to $\mathbb{R}^\infty$, which is a contractible CW complex. It follows that the bundle is trivialisable, so $F_n(\mathbb{R}^\infty \setminus S) \simeq \mathbb{R}^\infty \times \mathbb{R}^\infty \simeq \mathbb{R}^\infty$. □

Proposition 16.14. *The Stiefel manifold $V_n\mathbb{R}^\infty$ is homeomorphic to $\mathbb{R}^\infty$.*

Proof. Apply the same argument to the well-known fibre bundle $V_{n-1}\mathbb{R}^\infty \to V_n\mathbb{R}^\infty \to S^\infty$. (In the next section we will be more precise about how to trivialise this bundle, but for the moment we need only note that it is trivialisable.) □

Remark 16.15. The space $F_n\mathbb{R}^\infty$ is one of the standard models for $E\Sigma_n$, and $V_n\mathbb{R}^\infty$ is one of the standard models for $EO(n)$.

16.2. **The topology of $\mathcal{L}(1)$.** We now prove that $\mathcal{L}(1)$ is homeomorphic to a product of countably many copies of S^∞. This is needed to clear up a minor technical point later, and in any case it seems like an illuminating fact. We also prove that all orbits of the evident left action of the orthogonal group O on $\mathcal{L}(1)$ are dense, which will have a number of applications.

Let $F\colon \mathbb{R} \oplus \mathbb{R}^\infty \to \mathbb{R}^\infty$ be the obvious isomorphism, and define $T\colon \mathcal{L}(1) \to \mathcal{L}(1)$ by

$$T(g) = F \circ (1 \oplus g) \circ F^{-1}.$$

This clearly gives a homeomorphism of $\mathcal{L}(1)$ with $\{h \in \mathcal{L}(1) \mid h(e_0) = e_0\}$, and more generally T^n identifies $\mathcal{L}(1)$ with $\{h \in \mathcal{L}(1) \mid h|_{\mathbb{R}^n} = 1\}$.

We also choose a section σ of the evaluation map $\epsilon\colon O(\infty) \to S^\infty$ as in Lemma 16.8 and define

$$\begin{aligned} \sigma_n\colon \prod_{k<n} S^\infty &\to O(\infty) & \qquad \sigma_n(v_0, \ldots, v_{n-1}) &= \sigma(v_0) \circ \ldots \circ T^{n-1}\sigma(v_{n-1}) \\ \alpha_n\colon \prod_{k<n} S^\infty &\to V_n\mathbb{R}^\infty & \qquad \alpha_n(\underline{v}) &= \sigma_n(\underline{v})|_{\mathbb{R}^n}. \end{aligned}$$

Lemma 16.16. *The map α_n is a homeomorphism.*

Proof. The inverse β_n is defined recursively as follows. Suppose we have defined β_n. We identify $V_n\mathbb{R}^\infty = \mathcal{L}^+(\mathbb{R}^n, \mathbb{R}^\infty)$ with the set of n-frames in $\mathbb{R}^\infty$ in the obvious way. Let $\underline{u} = (u_0, \ldots, u_{n-1})$ be an n-frame, so that $g = \sigma_n\beta_n(\underline{u}) \in O$ and $g(e_i) = u_i$ for $i < n$, so g induces an isomorphism $\{e_0, \ldots, e_{n-1}\}^\perp \simeq \{u_0, \ldots, u_{n-1}\}^\perp$. Now suppose that we have a vector u_n giving us an $(n+1)$-frame. It follows that there is a unique element $v \in \{e_0, \ldots, e_{n-1}\}^\perp$ such that $gv = u_n$. We define

$$\beta_{n+1}(u_0, \ldots, u_n)) = (\beta_n(u_0, \ldots, u_{n-1}), L^{n-1}v) \in \left(\prod_{k<n} S^\infty\right) \times S^\infty = \prod_{k<n+1} S^\infty.$$

It is not hard to check that this is continuous and inverse to α_n. □

Corollary 16.17. *There are commutative diagrams as follows, in which the vertical maps are homeomorphisms (and therefore the top horizontal maps are trivialisable fibre bundles).*

$$\begin{array}{ccccc} \mathcal{L}(1) & \twoheadrightarrow & \mathcal{L}^+(\mathbb{R}^n, \mathbb{R}^\infty) & \twoheadrightarrow & \mathcal{L}^+(\mathbb{R}^{n-1}, \mathbb{R}^\infty) \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta_n} & & \downarrow{\scriptstyle \beta_{n-1}} \\ \prod_k S^\infty & \underset{\pi}{\twoheadrightarrow} & \prod_{k<n} S^\infty & \underset{\pi}{\twoheadrightarrow} & \prod_{k<n-1} S^\infty \end{array}$$

Proof. This follows immediately from the lemma and the fact that $\mathcal{L}(1) = \varprojlim_n \mathcal{L}^+(\mathbb{R}^n, \mathbb{R}^\infty)$. □

Corollary 16.18. *Given universes $\mathcal{U}$ and $\mathcal{V}$ (where $\mathcal{V}$ is infinite) and a subspace $U \ll \mathcal{U}$, the restriction map*

$$\mathcal{L}^+(\mathcal{U}, \mathcal{V}) \to \mathcal{L}^+(U, \mathcal{V})$$

is a trivialisable fibre bundle.

Proof. As all universes of the same dimension are isomorphic, we may assume that $\mathcal{V} = \mathbb{R}^\infty$ and $U = \mathbb{R}^n$ for some $n \in \mathbb{N}$ and $\mathcal{U} \ominus U = \mathbb{R}^m$ for some $m \in \mathbb{N} \cup \{\infty\}$. In this case the claim is clear from the previous corollary. $\square$

We next define a map $\lambda\colon \mathcal{L}(1) \to \mathcal{L}(1)$ as follows. Observe that $g = \sigma(fe_0)^{-1} \circ f \in \mathcal{L}(1)$ and $g(e_0) = e_0$, so that $g = Th$ for a unique $h \in \mathcal{L}(1)$; we set $\lambda(f) = h$. We also define $\mu\colon \prod_k S^\infty \to \prod_k S^\infty$ by $\mu(v_0, v_1, \ldots) = (v_1, v_2, \ldots)$. The following lemma follows by inspection of definitions:

Lemma 16.19. *There is a commutative diagram*

$$\begin{array}{ccc} \prod_k S^\infty & \xrightarrow[\sim]{\alpha} & \mathcal{L}(1) \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\lambda} \\ \prod_k S^\infty & \xrightarrow[\alpha]{\sim} & \mathcal{L}(1) \end{array}$$

Moreover, if $f = \alpha(\underline{v})$ then $f = \sigma_n(\alpha) \circ T^n\lambda^n f$, so f lies in the same orbit as $T^n\lambda^n f$ under the left action of O. $\square$

The next result would be easy if we were using the ordinary product topology, but requires a little more work in the compactly generated category.

Lemma 16.20. *Let Y be a nonempty subset of $\prod_k S^\infty$, with the property that for any $n \geq 0$, $\mu^n\underline{x} \in \mu^n Y$ implies $\underline{x} \in Y$. Then Y is dense in $\prod_k S^\infty$.*

Proof. Choose $\underline{y} \in Y$. Consider an arbitrary element $\underline{x} \in \prod_k S^\infty$, and set $K = \prod_k\{x_k, y_k\}$. This is compact in the ordinary product topology, so it is compact in the CG topology and its topology as a subspace of the CG product is the ordinary one. Define $\underline{x}_N \in K$ by $x_{N,k} = x_k$ for $k < n$ and $x_{N,k} = y_k$ for $k \geq n$. Clearly $\mu^n\underline{x}_N = \mu^n\underline{y} \in \mu^n Y$, so $\underline{x}_N \in Y \cap K \subseteq \overline{Y} \cap K$. Clearly $\underline{x}_N$ converges to $\underline{x}$, and $\overline{Y} \cap K$ is closed, so $\underline{x} \in \overline{Y}$ as required. $\square$

Proposition 16.21. *Every O-orbit Of in $\mathcal{L}(1)$ is dense. It follows that $\mathcal{L}(1)f$ is always dense, and that*

$$\mathcal{L}^0(1) = \{g \in \mathcal{L}(1) \mid \dim g(\mathbb{R}^\infty)^\perp = \infty\}$$

is dense in $\mathcal{L}(1)$.

Proof. Write $Y = \alpha^{-1}(Of) \subseteq \prod_k S^\infty$. We claim that Lemma 16.20 applies to this Y. Clearly $Y \neq \emptyset$, so suppose that $\mu^n\underline{x} = \mu^n\underline{y}$ with $\underline{y} \in Y$. It follows easily from Lemma 16.19 that $\lambda^n\alpha(\underline{x}) = \lambda^n\alpha(\underline{y}) = \lambda^n(gf)$ for some $g \in O$, and thus that $\alpha(\underline{x})$ lies in the same orbit under O as gf or f, so that $\underline{x} \in Y$ as required. Thus Lemma 16.20 shows that Y is dense in $\prod_k S^\infty$; as α is a homeomorphism we conclude that Of is dense in $\mathcal{L}(1)$. It follows immediately that $\mathcal{L}(1)f$ is dense. Now consider the special case where $f(e_i) = e_{2i}$ for all i. It is easy to see that $f \in \mathcal{L}^0(1)$ and thus $\mathcal{L}(1)f \subseteq \mathcal{L}^0(1)$, so $\mathcal{L}^0(1)$ is dense. $\square$

16.3. **The monoid structure of $\mathcal{L}(1)$.** We now prove various facts about the structure of $\mathcal{L}(1)$ as a monoid.

Lemma 16.22. *Let $\mathcal{L}'$ be obtained from the category $\mathcal{L}$ of infinite universes by inverting all morphisms. Then $\mathcal{L}'(\mathcal{U}, \mathcal{V})$ has precisely one element for all $\mathcal{U}$ and $\mathcal{V}$.*

Proof. We know that $\mathcal{L}(\mathcal{U}, \mathcal{V}) \neq \emptyset$ so it suffices to show that any two maps $f, g\colon \mathcal{U} \to \mathcal{V}$ become equal in $\mathcal{L}'$. We have a map $i_0\colon \mathcal{U} \to \mathcal{U} \times \mathcal{U}$ with $(1 \times f)i_0 = (1 \times g)i_0$ and i_0 is invertible in $\mathcal{L}'$ so $1 \times f = 1 \times g$ there. We next have a commutative diagram in $\mathcal{L}'$ as follows.

$$\begin{array}{ccc} \mathcal{U} & \overset{f}{\underset{g}{\rightrightarrows}} & \mathcal{V} \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle i_1} \\ \mathcal{U} \times \mathcal{U} & \overset{1\times f}{\underset{1\times g}{\rightrightarrows}} & \mathcal{U} \times \mathcal{V} \end{array}$$

As $1 \times f = 1 \times g$ and the maps i_1 are invertible we have $f = g$. $\square$

Corollary 16.23. *The group completion of the monoid $\mathcal{L}(1)$ is trivial.*

Proof. This is immediate from the lemma and the fact that $\mathcal{L}$ is equivalent to the category with one object $\mathbb{R}^\infty$ whose endomorphisms are $\mathcal{L}(1)$. □

Definition 16.24. Let M be a topological monoid and let X and Y be spaces with a right and a left action of M, respectively. We then write $X \times_M Y$ for the coequaliser of the maps $d_0, d_1 \colon X \times M \times Y \to X \times Y$, where

$$d_0(x,m,y) = (xm, y)$$
$$d_1(x,m,y) = (x, my).$$

We now give a proof of a result of Hopkins, which is the key point in the proof that the smash product of $\mathbb{L}$-spectra is associative. While it is not significantly different from that given in [2], it is a nice illustration of the point of view we will take in a number of proofs later in this paper.

Proposition 16.25. *Let $\mathcal{U}, \mathcal{V}, \mathcal{W}, \mathcal{X}$ and $\mathcal{Y}$ be infinite universes, so that $\mathcal{L}(\mathcal{V}) \times \mathcal{L}(\mathcal{X})$ acts on the right on $\mathcal{L}(\mathcal{V} \oplus \mathcal{X}, \mathcal{Y})$ and on the left on $\mathcal{L}(\mathcal{U}, \mathcal{V}) \times \mathcal{L}(\mathcal{W}, \mathcal{X})$. Then*

$$\mathcal{L}(\mathcal{V} \oplus \mathcal{X}, \mathcal{Y}) \times_{\mathcal{L}(\mathcal{V})\times\mathcal{L}(\mathcal{X})} (\mathcal{L}(\mathcal{U}, \mathcal{V}) \times \mathcal{L}(\mathcal{W}, \mathcal{X})) = \mathcal{L}(\mathcal{U} \oplus \mathcal{W}, \mathcal{Y}).$$

Proof. As all infinite universes are isomorphic, we may assume that $\mathcal{U} = \mathcal{V}$ and $\mathcal{W} = \mathcal{X}$. The proposition then becomes an instance of the easy fact that $X \times_M M = X$. □

Corollary 16.26 (Hopkins)**.** *If $k, l > 0$ we have $\mathcal{L}(2) \times_{\mathcal{L}(1)^2} (\mathcal{L}(k) \times \mathcal{L}(l)) = \mathcal{L}(k + l)$.*

Proof. Take $\mathcal{U} = (\mathbb{R}^\infty)^k$, $\mathcal{W} = (\mathbb{R}^\infty)^l$, $\mathcal{V} = \mathcal{X} = \mathcal{Y} = \mathbb{R}^\infty$. □

Definition 16.27. If X is a space with an action of a monoid M, we write X/M for the coequaliser of the action map and the projection map $M \times X \to X$. This is the quotient of X by the smallest closed equivalence relation such that $x \sim mx$ for all $x \in X$ and $m \in M$. We say that a space X is *quasi-transitive* if X/M is a one-element set.

Observe that if G is a group then

$$G \simeq (G \times G)/G \not\simeq (G/G) \times (G/G) = 1.$$

Thus, the functor $X \mapsto X/G$ fails badly to preserve products. This makes the following result rather curious.

Proposition 16.28. *If X and Y are spaces with a right action of $\mathcal{L}(\mathcal{U})$ (where $\mathcal{U}$ is an infinite universe) then*

$$(X \times Y)/\mathcal{L}(\mathcal{U}) = (X/\mathcal{L}(\mathcal{U})) \times (Y/\mathcal{L}(\mathcal{U})).$$

The proof will follow after two lemmas.

Definition 16.29. If $\mathcal{U}$ is a subuniverse of $\mathcal{V}$, we write $\mathcal{U}^\perp = \{v \in \mathcal{V} \mid \langle v, \mathcal{U}\rangle = \{0\}\}$. We say that $\mathcal{U}$ is *complemented* if $\mathcal{V} = \mathcal{U} \oplus \mathcal{U}^\perp$. It is not hard to see that any finite subuniverse is complemented (choose an orthonormal basis $\{u_i\}$ and observe that $v - \sum_i \langle v, u_i\rangle u_i$ lies in $\mathcal{U}^\perp$).

Lemma 16.30. *Given two maps $f, g \colon \mathcal{U} \to \mathcal{V}$ of infinite universes, there is an infinite subuniverse $\mathcal{W} \leq \mathcal{U}$ such that $f\mathcal{W}$, $g\mathcal{W}$ and $f\mathcal{W} + g\mathcal{W}$ are complemented in $\mathcal{V}$ and the universe $(f\mathcal{W} + g\mathcal{W})^\perp = (f\mathcal{W})^\perp \cap (g\mathcal{W})^\perp$ is infinite.*

Proof. We may assume that $\mathcal{V} = \mathbb{R}^\infty$, and let $\{e_i \mid i \geq 0\}$ be the standard basis. We choose an orthonormal sequence $w_0, w_1, \ldots$ in $\mathcal{U}$ as follows. Let w_0 be any unit vector. Given $w_0, \ldots, w_{n-1}$, let $d = d_{n-1}$ be the least integer such that $f(w_j)$ and $g(w_j)$ both lie in $\mathbb{R}^{d_{n-1}} < \mathbb{R}^\infty$ for all $j < n$ and $d_{n-1} > d_{n-2}$. Let $\mathcal{U}_n$ be the set of vectors $u \in \mathcal{U}$ such that

(a) $\langle u, w_i\rangle = 0$ for $i < n$.
(b) $\langle f(u), e_i\rangle = \langle g(u), e_i\rangle = 0$ for $i \leq d$.

This has finite codimension in $\mathcal{U}$ and thus is nonzero. Let w_n be any unit vector in $\mathcal{U}_n$.

We can now set $\mathcal{W} = \mathrm{span}\{w_i \mid i \geq 0\}$. If we write $\mathcal{V}_i = \mathrm{span}\{e_j \mid d_{i-1} \leq j < d_i\}$ then $\mathcal{V}_i$ is finite and $\mathcal{V} = \bigoplus_i \mathcal{V}_i$. It follows easily that if a subuniverse $\mathcal{X} \leq \mathcal{V}$ has the form $\mathcal{X} = \bigoplus_i \mathcal{X}_i$ with $\mathcal{X}_i \leq \mathcal{V}_i$, then $\mathcal{X}$ is complemented. Given that $fw_i \in \mathcal{V}_i$ and $f\mathcal{W} = \mathrm{span}\{fw_i \mid i \geq 0\}$, it follows that $f\mathcal{W}$ is complemented. Similarly, $g\mathcal{W}$ and $f\mathcal{W} + g\mathcal{W}$ are complemented. We have $e_{d_i} \in (f\mathcal{W})^\perp \cap (g\mathcal{W})^\perp$ for all i, so $(f\mathcal{W})^\perp \cap (g\mathcal{W})^\perp$ is infinite. □

Lemma 16.31. *If we let $\mathcal{L}(\mathcal{U})$ act on the right on $\mathcal{L}(\mathcal{U})^2$ by $(f,g)h = (fh, gh)$ then the action is quasi-transitive.*

Proof. Suppose that $f, g \colon \mathcal{U} \to \mathcal{U}$. Let $\sim$ be the equivalence relation on $\mathcal{L}(\mathcal{U})^2$ generated by the action of $\mathcal{L}(\mathcal{U})$, so we need to prove that $(f,g) \sim (1,1)$. Choose a subuniverse $\mathcal{W} \leq \mathcal{U}$ as in Lemma 16.30, and let $j \colon \mathcal{W} \to \mathcal{U}$ be the inclusion. As all infinite universes are isomorphic, we can choose an isomorphism $h \colon \mathcal{U} \to \mathcal{W}$. We then have $(f,g) \sim (fjh, gjh)$. After replacing f by fjh and g by gjh we may assume that $f\mathcal{U}$, $g\mathcal{U}$ and $f\mathcal{U} + g\mathcal{U}$ are complemented and that $\mathcal{V} = (f\mathcal{U} + g\mathcal{U})^{\perp} < \mathcal{U}$ is an infinite universe. We define a map $u \colon \mathcal{U} \to \mathcal{U}$ as follows. We may assume that $\dim((f\mathcal{U} + g\mathcal{U}) \ominus f\mathcal{U}) \leq \dim((f\mathcal{U} + g\mathcal{U}) \ominus g\mathcal{U})$ (otherwise exchange f and g). We start with the map $gf^{-1} \colon f\mathcal{U} \to g\mathcal{U}$. We take the direct sum with an arbitrary isometric embedding $(f\mathcal{U} + g\mathcal{U}) \ominus f\mathcal{U} \to (f\mathcal{U} + g\mathcal{U}) \ominus g\mathcal{U}$ to get an endomorphism of $f\mathcal{U} + g\mathcal{U}$. We then take the direct sum with the identity map of $(f\mathcal{U} + g\mathcal{U})^{\perp}$ to get an endomorphism u of $\mathcal{U}$. Note that $uf = g$. Thus $(f,g) = (1f, uf) \sim (1,u)$. Choose an isomorphism $k \colon \mathcal{U} \to (f\mathcal{U} + g\mathcal{U})^{\perp}$ and let $i \colon (f\mathcal{U} + g\mathcal{U})^{\perp} \to \mathcal{U}$ be the inclusion. Then $ui = i$ and thus $(1,u) \sim (ik, uik) = (ik, ik) \sim (1,1)$. □

Proof of Proposition 16.28. Write $\sim$ for the equivalence relation on $X \times Y$ generated by the action of $\mathcal{L}(\mathcal{U})$. Given $(x,y) \in X \times Y$ we define $S = \{(u,v) \in \mathcal{L}(\mathcal{U})^2 \mid (xu, yv) \sim (x,y)\}$. Clearly, for any $w \in \mathcal{L}(\mathcal{U})$ we have $(xuw, yvw) \sim (xu, yv)$, so $(uw, vw) \in S$ if and only if $(u,v) \in S$. It follows from Lemma 16.31 that $S = \mathcal{L}(\mathcal{U})^2$. This implies that $(X \times Y)/\mathcal{L}(\mathcal{U}) = (X \times Y)/\mathcal{L}(\mathcal{U})^2 = (X/\mathcal{L}(\mathcal{U})) \times (Y/\mathcal{L}(\mathcal{U}))$, as required. □

Proposition 16.32. *If $\mathcal{U}$ is an infinite complex universe and $\mathcal{V}$ is an infinite real universe then $\mathcal{L}_{\mathbb{C}}(\mathcal{U})$ acts quasi-transitively on $\mathcal{L}_{\mathbb{R}}(\mathcal{U}, \mathcal{V})$.*

Proof. As all complex universes are isomorphic, it is not hard to identify $\mathcal{L}_{\mathbb{R}}(\mathcal{U}, \mathcal{V})/\mathcal{L}_{\mathbb{C}}(\mathcal{U})$ with $\varinjlim_{\mathcal{W}} \mathcal{L}_{\mathbb{R}}(\mathcal{W}, \mathcal{V})$, where $\mathcal{W}$ runs over the category of complex universes. This is the set of pairs $(\mathcal{W}, g)$ where $\mathcal{W}$ is a complex universe, $g \colon \mathcal{W} \to \mathcal{V}$, and $(\mathcal{W}, g)$ is identified with $(\mathcal{U}, gh)$ whenever $h \colon \mathcal{U} \to \mathcal{W}$ is a complex linear isometry.

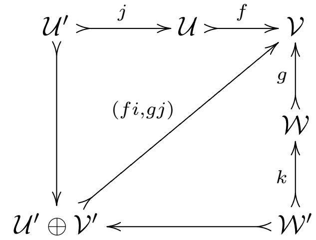
Consider two maps $\mathcal{U} \xrightarrow{f} \mathcal{V} \xleftarrow{g} \mathcal{W}$, where $\mathcal{U}$ and $\mathcal{W}$ are complex. We need to show that $(\mathcal{U}, f) \sim (\mathcal{W}, g)$. We first claim that there are infinite complex subuniverses $\mathcal{U}'$ and $\mathcal{W}'$ such that $f\mathcal{U}'$ is orthogonal to $g\mathcal{W}'$. To prove this, we choose orthonormal sequences $u_0, u_1, \dots$ and $w_0, w_1, \dots$ as follows. Given $u_0, \dots, u_{n-1}$ and $w_0, \dots, w_{n-1}$ we define

$$\mathcal{U}_n = \operatorname{span}_{\mathbb{C}}\{u_j \mid j < n\}^{\perp} \cap f^{-1}((g \operatorname{span}_{\mathbb{C}}\{w_j \mid j < n\})^{\perp}) \cap if^{-1}((g \operatorname{span}_{\mathbb{C}}\{w_j \mid j < n\})^{\perp}).$$

This is a complex subspace of finite codimension in $\mathcal{U}$, so it is nonzero. We take u_n to be any unit vector in $\mathcal{U}_n$. We then define

$$\mathcal{W}_n = \operatorname{span}_{\mathbb{C}}\{w_j \mid j < n\}^{\perp} \cap g^{-1}((f \operatorname{span}_{\mathbb{C}}\{u_j \mid j \leq n\})^{\perp}) \cap ig^{-1}((f \operatorname{span}_{\mathbb{C}}\{u_j \mid j \leq n\})^{\perp}),$$

and let w_n be any unit vector in $\mathcal{W}_n$. It is not hard to check that $\mathcal{U}' = \operatorname{span}_{\mathbb{C}}\{u_j \mid j \geq 0\}$ and $\mathcal{W}' = \operatorname{span}_{\mathbb{C}}\{w_j \mid j \geq 0\}$ are as advertised. Let $j \colon \mathcal{U}' \to \mathcal{U}$ and $k \colon \mathcal{W}' \to \mathcal{W}$ be the inclusions. As the images of fj and gk are orthogonal, we have a linear isometry $(fj, gk) \colon \mathcal{U}' \oplus \mathcal{W}' \to \mathcal{V}$. This fits into the following diagram:



This shows that

$$(\mathcal{U}, f) \sim (\mathcal{U}', fi) \sim (\mathcal{U}' \oplus \mathcal{V}', (fi, gj)) \sim (\mathcal{W}', gj) \sim (\mathcal{W}, g).$$

□

16.4. **Complements of subuniverses.** We next explore a little further the question of when a subuniverse $\mathcal{U} \leq \mathcal{V}$ is complemented. Amongst other things, we will argue that this is a very rare phenomenon; all most all universes are uncomplemented. It will be convenient to expand the terminology slightly: we say that a linear isometry $f: \mathcal{U} \to \mathcal{V}$ is *complemented* if $f\mathcal{U}$ is a complemented subuniverse of $\mathcal{V}$.

Proposition 16.33. *Let $\mathcal{U}$ be a subuniverse of $\mathcal{V}$, and let $\{u_k \mid k \in \mathbb{N}\}$ be an orthonormal basis of $\mathcal{U}$. Then $\mathcal{U}$ is complemented iff for all $v \in \mathcal{V}$ we have $\langle v, u_i\rangle = 0$ for $i \gg 0$.*

Proof. If $\mathcal{U}$ is complemented we can write $v = w + u$ for some $w \in \mathcal{U}^\perp$ and $u \in \mathcal{U}$, say $u = \sum_i x_i u_i$ with $x_i = 0$ for $i \gg 0$. Clearly $\langle v, u_i\rangle = x_i$ and the claim follows.

Conversely, suppose that $\langle v, u_i\rangle = 0$ for $i \gg 0$ for all $v \in \mathcal{V}$. We can then define $\pi(v) = \sum_i \langle v, u_i\rangle u_i$. It is easy to check that $\pi(v) \in \mathcal{U}$ and $v - \pi(v) \in \mathcal{U}^\perp$, and thus that $\mathcal{V} = \mathcal{U} \oplus \mathcal{U}^\perp$. □

Concrete examples?

For $k, l \geq 0$ we define $a_k, b_l \in \mathbb{R}^\infty$ by

$$a_k = e_{2k} + \sum_{j<k} e_{2j+1}$$

$$b_l = e_{2l+3} + e_{2l+2} - e_{2l+1}.$$

By looking at the leading terms we see that the a_k's and b_l's are linearly independent, and that e_1 does not lie in their span. We put $\mathcal{A} = \mathbb{R}\{a_k \mid k \geq 0\}$ and $\mathcal{B} = \mathbb{R}\{b_l \mid l \geq 0\}$.

Proposition 16.34. *We have $\mathcal{A} = \mathcal{B}^\perp$ and $\mathcal{B} = \mathcal{A}^\perp$ but $\mathcal{A} \oplus \mathcal{B} \neq \mathbb{R}^\infty$.*

Proof. First, we have

$$\begin{aligned}\langle a_k, b_l\rangle &= \langle e_{2k}, b_l\rangle + \sum_{j=0}^{k-1} \langle e_{2j+1}, b_l\rangle \\ &= I_{l+1=k} + \sum_{j=0}^{k-1} (I_{j=l+1} - I_{j=l}) \\ &= I_{l+1=k} + I_{l+1<k} - I_{l<k} = 0.\end{aligned}$$

Thus $\langle \mathcal{A}, \mathcal{B}\rangle = 0$. Next, suppose that $x \in \mathcal{A}^\perp$. Put $x' = \sum_l x_{2l+2} b_l \in \mathcal{B} \leq \mathcal{A}^\perp$, so the element $x'' := x' - x$ also lies in $A^\perp$. By construction we have $x''_{2l+2} = 0$ for $l \geq 0$, and also $x''_0 = 0$ because x'' is orthogonal to $a_0 = e_0$, so x'' is orthogonal to e_{2k} for all k. As $e_{2k+1} = a_{k+1} - a_k - e_{2k+2} + e_{2k}$ we find that $x''_{2k+1} = 0$ for all k, so $x'' = 0$, so $x \in \mathcal{B}$ as required.

Now suppose instead that $y \in \mathcal{B}^\perp$. Put $y' = \sum_k y_{2k} a_k \in \mathcal{A}$ and $y'' = y - y' \in \mathcal{B}^\perp$ so $y''_{2k} = 0$. Given this, the equation $\langle y'', b_l\rangle = 0$ reduces to $y''_{2l+1} = y''_{2l+3}$, so y''_{2j+1} is independent of j. As $y''_{2j+1} = 0$ for $j \gg 0$, this implies that $y'' = 0$. Thus $y \in \mathcal{A}$, as required.

We have already observed that $e_1 \notin \mathcal{A} \oplus \mathcal{B}$, so $\mathcal{A} \oplus \mathcal{B} \neq \mathbb{R}^\infty$. □

17. Orthogonal diagrams

Let $\mathcal{L}_0$ be the category of finite universes, and let $\mathcal{C}$ be the category of those functors from $\mathcal{L}_0$ to the category of sets that preserve orthogonal pullbacks. We call the objects of $\mathcal{C}$ *orthogonal diagrams*.

A *basic diagram* is a functor of the form $A \mapsto \mathcal{L}_0(\mathbb{R}^n, A)/G$, where G is a subgroup of $O(n)$.

Theorem 17.1. *Any orthogonal diagram is a disjoint union of basic diagrams.*

Corollary 17.2. *An orthogonal diagram preserves* all *pullbacks in $\mathcal{L}_0$.*

The rest of this document constitutes the proof.

Lemma 17.3. *Any functor $X \in \mathcal{C}$ sends all morphisms to monomorphisms.*

Proof. Any morphism is isomorphic to one of the form $A \to A \oplus B$, and by using the orthogonal pullback square $\{A, A \oplus B, A \oplus B, A \oplus B \oplus B\}$ we see that the map $X(A) \to X(A \oplus B)$ is injective. □

Suppose we have $X \in \mathcal{C}$ and $a \in X(A)$ and $b \in X(B)$. Put

$$E(a,b)(C) = \{(\alpha,\beta) \in \mathcal{L}_0(A,C) \times \mathcal{L}_0(B,C) \mid \alpha_*(a) = \beta_*(b)\}.$$

This gives an object $E(a,b) \in \mathcal{C}$.

Definition 17.4. Let $Q(A,B)$ be the set of positive semidefinite quadratic forms q on $A \oplus B$ such that $q(u) = \|u\|_A^2$ for $u \in A$ and $q(v) = \|v\|_B^2$ for $v \in B$.

Definition 17.5. Let $Q'(A,B)$ be the set of linear maps $\gamma\colon A \to B$ such $\|\gamma(u)\| \leq \|u\|$ for all $u \in A$. Given $\gamma \in Q'(A,B)$, we can define $q_\gamma \in Q(A,B)$ by

$$q_\gamma(u,v) = \|u\|_A^2 + \|v\|_B^2 + 2\langle\gamma(u)\rangle v_B.$$

One checks that this construction gives a bijection $Q(A,B) \simeq Q'(A,B)$.

Definition 17.6. Given any pair of maps $(\alpha,\beta) \in \mathcal{L}_0(A,C) \times \mathcal{L}_0(B,C)$, we define $q_{\alpha,\beta}(u,v) = \|\alpha(u)+\beta(v)\|^2$. One checks easily that $q_{\alpha,\beta} \in Q(A,B)$ and $q_{\alpha,\beta} = q_\gamma$ where $\gamma = \beta^*\alpha\colon A \to B$.

For any $q \in Q(A,B)$ we define $Z_q = \{(u,v) \mid q(u,v) = 0\}$ and $C_q = (A \oplus B)/Z_q$. The form q gives an inner product on C_q, the evident maps $\alpha_q\colon A \to C_q$ and $\beta_q\colon B \to C_q$ are isometries, and clearly $q_{\alpha_q,\beta_q} = q$.

Lemma 17.7. *There is a set $Q(a,b) \subseteq Q(A,B)$ (independent of C) such that $(\alpha,\beta) \in E(a,b)(C)$ iff $q_{\alpha,\beta} \in Q(a,b)$.*

Proof. We put

$$Q(a,b) = \{q \mid (\alpha_q,\beta_q) \in E(a,b)(C_q)\}.$$

If $\alpha\colon A \to C$ and $\beta\colon B \to C$ and $q = q_{\alpha,\beta}$, then $\alpha(u) + \beta(v) = 0$ for $(u,v) \in Z_q$ so there is a unique map $\phi \in \mathcal{L}_0(C_q,C)$ with $\phi\alpha_q = \alpha$ and $\phi\beta_q = \beta$. As $\phi_*\colon X(C_q) \to X(C)$ is injective, we see that $(\alpha,\beta) \in E(a,b)(C)$ if and only if $(\alpha_q,\beta_q) \in E(a,b)(C_q)$, if and only if $q \in Q(a,b)$. ☐

Definition 17.8. We write $Q'(a,b) = \{\gamma \in Q'(A,B) \mid q_\gamma \in Q(a,b)\}$.

Lemma 17.9. *Suppose that $\gamma\colon A \to B$ has $\|\gamma\| < 1$. Define isometric embeddings $\alpha_\theta\colon A \to A^2$ by $\alpha_\theta(u) = (\cos(2\theta)u, \sin(2\theta)u)$, and let $j\colon A^2 \to A^3 \oplus B$ be the evident inclusion. Then for sufficiently small θ there exists an isometric embedding $\beta_\theta\colon B \to A^2 \oplus B$ such that $\beta_\theta^* j\alpha_0 = \beta_\theta^* j\alpha_\theta = \gamma\colon A \to B$.*

Proof. Suppose that $\|\gamma\| \leq \cos(\theta)$; as $\|\gamma\| < 1$, this will hold for sufficiently small θ.

Let $e_1,\dots,e_n$ be an orthonormal basis of eigenvectors for the positive self-adjoint operator $\gamma^*\gamma$. Let t_i be the eigenvalue of e_i, and order the basis so that $t_1 \geq \ldots \geq t_n \geq 0$. In particular, if r is the rank of γ then this means that $t_1,\dots,t_r > 0$ and $t_i = 0$ for $i > r$. Next note that $t_i = \|\gamma(e_i)\|^2 \leq \|\gamma\|^2 \leq \cos^2(\theta)$, so we can write $t_i = \cos^2(\theta)\cos^2(\phi_i)$ for some $\phi_i \in [0,\pi/2]$. For $i \leq r$ we define $f_i = \gamma(e_i)/\sqrt{t_i}$; these vectors form an an orthonormal basis for $\mathrm{Im}(\gamma) = B \ominus \ker(\gamma^*)$, which can be extended to an orthonormal basis $\{f_1,\dots,f_m\}$ for B. Now define

$$g_i = \begin{cases} (\cos(\phi_i)\cos(\theta)e_i, \cos(\phi_i)\sin(\theta)e_i, \sin(\phi_i)e_i, 0) & \text{if } 1 \leq i \leq r \\ (0,0,0,f_i) & \text{if } r < i \leq m. \end{cases}$$

It is clear that the g_i are orthonormal, so we can define an isometric embedding $\beta_\theta\colon B \to A^3 \oplus B$ by $\beta_\theta(f_i) = g_i$. It is also clear that $\langle g_i\rangle\alpha_\psi(e_j) = 0$ for all ψ if $i \neq j$. Also, for $i \leq r$ we have

$$\langle g_i\rangle j\alpha_0(e_i) = \langle g_i\rangle(e_i,0,0,0) = \cos(\phi_i)\cos(\theta) = \langle f_i\rangle\gamma(e_i).$$

Using this and easy arguments for the cases $i > r$ or $k > r$, we see that

$$\langle f_i\rangle\beta_\theta^* j\alpha_0(e_k) = \langle g_i\rangle j\alpha_0(e_k) = \langle f_i\rangle\gamma(e_k)$$

for $i = 1,\dots,n$ and $k = 1,\dots,m$, so $\beta_\theta^* j\alpha_0 = \gamma$.

With a little more work, one checks that

$$\begin{aligned} \langle g_i\rangle j\alpha_\theta(e_i) &= \langle(\cos(2\theta),\sin(2\theta),0)\rangle(\cos(\theta)\cos(\phi_i),\sin(\theta)\cos(\phi_i),\sin(\phi_i)) \\ &= \cos(\phi_i)(\cos(2\theta)\cos(\theta) + \sin(2\theta)\sin(\theta)) \\ &= \cos(\phi_i)\cos(\theta) \\ &= \langle f_i\rangle\gamma(e_i), \end{aligned}$$

and it follows that $\beta_\theta^* j \alpha_\theta = \gamma$ also. $\square$

Proposition 17.10. *Suppose that $a \in X(A)$, $b \in X(B)$ and there exists $\gamma \in Q'(a,b)$ such that $\|\gamma\| < 1$. Then a lies in the image of the map $X(0) \to X(A)$.*

Proof. We let $\rho_\phi \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation through angle 2ϕ, and we also write ρ_ϕ for the map $\rho_\phi \otimes 1 \colon \mathbb{R}^2 \otimes A = A^2 \to A^2$. Let $\alpha_\theta \colon A \to A^2$ be as in the previous proof, so that $\rho_\phi \alpha_\theta = \alpha_{\phi+\theta}$. If θ is sufficently small then we can apply the lemma and we find that

$$j_* \alpha_{0*}(a) = \beta_{\theta *}(b) = j_* \alpha_{\theta *}(a) = j_* \rho_{\theta *} \alpha_{0*}(a).$$

As j_* is injective, we see that $\alpha_{0*}(a)$ is invariant under the group generated by ρ_θ. By taking $\theta = \pi/4N$ for sufficiently large N, we see that $(\alpha_{\pi/4})_*(a) = \alpha_{0*}(a)$. The claim now follows from the following orthogonal pullback square.

$$\begin{array}{ccc} 0 & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle \alpha_0} \\ A & \xrightarrow[\alpha_{\pi/4}]{} & A^2 \end{array}$$

$\square$

Definition 17.11. For any $q \in Q(A,B)$, we let A_1 and B_1 be the images of $Z := \{(a,b) \in A \oplus B \mid q(a,b) = 0\}$ under the projection maps from $A \oplus B$ to A and B. We also put $A_0 = A \ominus A_1$ and $B_0 = B \ominus B_1$.

Lemma 17.12. *Suppose that $q = q_\gamma$ for some $\gamma \colon A \to B$. Then $A_1 = \ker(\gamma^*\gamma - 1)$ and $B_1 = \ker(\gamma\gamma^* - 1)$. Moreover, we have $\gamma = \gamma_0 \oplus \gamma_1$, where $\gamma_1 \colon A_1 \to B_1$ is an isomorphism and $\gamma_0 \colon A_0 \to B_0$ has $\|\gamma_0\| < 1$.*

Proof. From the definitions, we have

$$q(u,v) = (\|u\|^2 - \|\gamma(u)\|^2) + \|\gamma(u) + v\|^2.$$

Both terms on the right hand side are nonnegative, so we can only have $q(u,v) = 0$ if $\|u\| = \|\gamma(u)\|$ and $\gamma(u) + v = 0$, or in other words $v = -\gamma(u)$ and $\|\gamma(u)\| = \|u\|$. Thus, $A_1 = \{u \in A \mid \|\gamma(u)\| = \|u\|\}$, and by a similar argument, $B_1 = \{v \in B \mid \|\gamma(v)\| = \|v\|\}$. Clearly, if $\gamma^*\gamma(u) = u$ then

$$\|\gamma(u)\|^2 = \langle \gamma(u) \rangle \gamma(u) = \langle u \rangle \gamma^*\gamma(u) = \|u\|^2$$

so $\|\gamma(u)\| = \|u\|$. For the converse, we first recall the well-known fact that $\|\gamma^*\| = \|\gamma\| \leq 1$, so $\|\gamma^*\gamma(u)\| \leq \|\gamma(u)\| = \|u\|$. Thus, we have

$$\begin{aligned} 0 &\geq \|\gamma^*\gamma(u)\|^2 - \|u\|^2 \\ &= \|\gamma^*\gamma(u)\|^2 - 2\|\gamma(u)\|^2 + \|u\|^2 \\ &= \|\gamma^*\gamma(u)\|^2 - 2\langle u \rangle \gamma^*\gamma(u) + \|u\|^2 \\ &= \|u - \gamma^*\gamma(u)\| \geq 0. \end{aligned}$$

This means that $\|u - \gamma^*\gamma(u)\|^2 = 0$, so $\gamma^*\gamma(u) = u$. Thus $A_1 = \ker(\gamma^*\gamma - 1)$, and $B_1 = \ker(\gamma\gamma^* - 1)$ by a dual argument. If $u \in A_1$ then $\gamma^*\gamma(u) = u$ so $\gamma\gamma^*\gamma(u) = \gamma(u)$, so $\gamma(u) \in B_1$. Similarly, we have $\gamma^* B_1 \leq A_1$, so $\langle \gamma(A_0) \rangle B_1 = \langle A_0 \rangle \gamma^* B_1 = \{0\}$, so $\gamma(A_0) \leq B_1^\perp = B_0$. It is easy to see that $\gamma \colon A_1 \to B_1$ is an isometry with inverse γ^*. The norm of the map $\gamma \colon A_0 \to A_0$ is the supremum of the values $\|\gamma(u)\|$ as u runs over the unit sphere of A_0. For such u we have $u \notin A_1$ so $\|\gamma(u)\| < \|u\| = 1$, and the unit sphere is compact so the supremum is the maximum and thus is strictly less than one. $\square$

Corollary 17.13. *Suppose that $a \in X(A)$, $b \in X(B)$ and $q \in Q(a,b)$. Then a lies in the image of the map $X(A_1) \to X(A)$ and b lies in the image of the map $X(B_1) \to X(B)$.*

Proof. Define $Y(C) = X(A_1 \oplus C)$, so $a \in Y(A_0) = X(A)$. Define $b' = (\gamma_1^* \oplus 1)_*(b) \in X(A_1 \oplus B_0) = Y(B_0)$. One checks that $\gamma_0 \in Q(a,b')$ and $\|\gamma_0\| < 1$, so we see from Proposition 17.10 that a lies in the image of the map $Y(0) \to Y(A_0)$, or in other words the map $X(A_1) \to X(A)$. The argument for b is similar. $\square$

Definition 17.14. An element $a \in X(A)$ is *minimal* if there is no proper subspace $A' < A$ such that a lies in the image of $X(A')$.

If $a \in X(A)$ is arbitrary, then it is clear that there exists $A' \leq A$ and a minimal element $a' \in X(A')$ that maps to a in $X(A)$. As $A' \simeq \mathbb{R}^n$ for some n, it follows in turn that there is a minimal element $a'' \in X(\mathbb{R}^n)$ and a map $\alpha\colon \mathbb{R}^n \to A$ such that $a = \alpha_*(a'')$.

Proposition 17.15. *If $a \in X(A)$ and $b \in X(B)$ are minimal and $\alpha_*(a) = \beta_*(b)$ then the map $\gamma := \beta^*\alpha\colon A \to B$ is an isometric isomorphism, $\alpha = \beta\gamma\colon A \to C$ and $b = \gamma_*(a)$.*

Proof. The map γ lies in $Q'(a, b)$, so Corollary 17.13 tells us that a lies in the image of $X(\ker(\gamma^*\gamma - 1))$. As a is minimal we must have $\ker(\gamma^*\gamma - 1) = A$, so $\gamma^*\gamma = 1$. A similar argument gives $\gamma\gamma^* = 1$, so γ is an isometric isomorphism. Now put $\delta = \alpha - \beta\gamma\colon A \to C$; we claim that $\delta^*\delta = 0$. As α, β and γ are isometries, we have $\alpha^*\alpha = \beta^*\beta = 1$ and $\alpha^*\beta\beta^*\alpha = \gamma^*\gamma = 1$. It follows that

$$\delta^*\delta = (\alpha^* - \alpha^*\beta\beta^*)(\alpha - \beta\beta^*\alpha) = \alpha^*\alpha - 2\alpha^*\beta\beta^*\alpha + \alpha^*\beta\beta^*\beta\beta^*\alpha = 1 - 2 + 1 = 0,$$

as required. This means that $\|\delta(a)\|^2 = \langle a\rangle\delta^*\delta(a) = 0$ for all a, so $\delta = 0$, so $\alpha = \beta\gamma$. This means that $\beta_*\gamma_*(a) = \alpha_*(a) = \beta_*(b)$ and β_* is injective, so $\gamma_*(a) = b$. ☐

Corollary 17.16. *If we let X'_n be the set of minimal elements in $X(\mathbb{R}^n)$, then there is a natural isomorphism*

$$X(C) = \coprod_n X'_n \times_{O(n)} \mathcal{L}^+(\mathbb{R}^n, C). \quad \square$$

The main theorem now follows, by choosing representatives for the orbits of $O(n)$ on X'_n.

18. The infinite case

We next give examples to show that these results do not extend to infinite universes.

First note that if $\mathcal{U} \leq \mathcal{V} \leq \mathcal{W}$ and $\mathcal{U}$ is complemented in $\mathcal{W}$ then any $w \in \mathcal{W}$ can be written as $u + x$ with $u \in \mathcal{U}$ and $x \perp \mathcal{U}$. If $w \in \mathcal{V}$ then $x = v - u \in \mathcal{V}$ also. Using this, we see that $\mathcal{U}$ is complemented in $\mathcal{V}$. To say this the other way around, if $\mathcal{U}$ is uncomplemented in $\mathcal{V}$ then it is uncomplemented in $\mathcal{W}$.

Now define

$$X(\mathcal{U}) = \{f \in \mathcal{L}(\mathbb{R}^\infty, \mathcal{U}) \mid f(\mathbb{R}^\infty) \text{ is uncomplemented in } \mathcal{U}\}.$$

One can also identify $X(\mathcal{U})$ with the space of orthonormal sequences (u_k) in $\mathcal{U}$ such that there exists $u \in \mathcal{U}$ with $\langle u\rangle u_k \neq 0$ for infinitely many k.

This gives a continuous functor from universes to unbased spaces **check for topological pathology here**. It is easy to see that X converts orthogonal pullback squares to pullbacks. However, if $\mathcal{U}$ is uncomplemented in $\mathcal{V}$, I claim that X does not preserve the following pullback:

$$\begin{array}{ccc} \mathcal{U} \oplus \mathcal{U} & \longrightarrow & \mathcal{U} \oplus \mathcal{V} \\ \downarrow & & \downarrow \\ \mathcal{V} \oplus \mathcal{U} & \longrightarrow & \mathcal{V} \oplus \mathcal{V} \end{array}$$

Indeed, if $f\colon \mathbb{R}^\infty \to \mathcal{U} \oplus \mathcal{U}$ is any isomorphism then $f \in X(\mathcal{U} \oplus \mathcal{V}) \cap X(\mathcal{V} \oplus \mathcal{U})$ but $f \notin X(\mathcal{U} \oplus \mathcal{U})$.

Next, I claim that if $\mathcal{U}_0 > \mathcal{U}_1 > \mathcal{U}_2 > \ldots$, then we need not have $X(\bigcap_n \mathcal{U}_n) = \bigcap_n X(\mathcal{U}_n)$. To see this, put $\mathcal{U}_0 = \mathbb{R}^\infty \oplus \mathbb{R}^\infty$, and define $\phi\colon \mathbb{R}^\infty \to \mathbb{R}$ by $\phi(x) = \sum_i x_i/2^i$. Define a quadratic form on $\mathcal{U}_0$ by $q(x, y) = \|x\|^2 + \|y\|^2 + \phi(x)\phi(y)$. The Cauchy-Schwartz inequality implies that $|\phi(x)|^2 \leq 4\|x\|^2/3$ and thus that $|\phi(x)\phi(y)| \leq 2\|x\|\|y\|$, with equality only when $x = 0$ or $y = 0$. From this it is not hard to deduce that q is positive definite, so it makes $\mathcal{U}_0$ into a universe. Let $\mathcal{U}_k$ be the subspace of pairs (x, y) where $y_i = 0$ for $i < k$. The inclusion of the first copy of $\mathbb{R}^\infty$ in $\mathcal{U}_0$ gives an element of $\bigcap_k X(\mathcal{U}_k)$ that does not lie in $X(\bigcap_k \mathcal{U}_k)$.

19. Complex spheres

Given $z, w \in \mathbb{C}^n$ we put $z.w = \sum_a z_a w_a \in \mathbb{C}$, so $\|z\|^2 = z.\overline{z}$. We then put $X^n = \{z \in \mathbb{C}^{n+1} \mid z.z = 1\}$, so $X^n \cap \mathbb{R}^{n+1} = S^n$.

Proposition 19.1. *The inclusion $i_n\colon S^n \to X^n$ is a homotopy equivalence.*

We treat the odd and even cases separately.

Lemma 19.2. *The inclusion $i = i_{2m-1} \colon S^{2m-1} \to X^{2m-1}$ is a homotopy equivalence.*

Proof. Put $Y = \{(u,v) \in \mathbb{C}^m \times \mathbb{C}^m \mid u.v = 1\}$, and define $f \colon X \to Y$ by $f(z) = (u,v)$, where $u_b = z_{2b} + iz_{2b+1}$ and $v_b = z_{2b} - iz_{2b+1}$. It is clear that this is a homeomorphism. If $(u,v) \in Y$ then $u \in \mathbb{C}^m \setminus \{0\}$ so we can define $p \colon Y \to S^{2m-1}$ by $p(u,v) = x$, where $x_{2b} = \mathrm{Re}(u_b/\|u\|)$ and $x_{2b+1} = \mathrm{Im}(u_b/\|u\|)$. One checks that $pfi = 1 \colon S^{2m-1} \to S^{2m-1}$. We also have $fip(u,v) = (u/\|u\|, \overline{u}/\|u\|)$. By considering the maps $h_t(u,v) = (u\|u\|^{-t}, \overline{u}\|u\|^{t-2})$ and $k_t(u,v) = (u, (1-t)v + t\overline{u}/\|u\|^2)$ we see that $fip \simeq 1_Y$. As f is a homeomorphism it follows that $ipf \simeq 1_X$, so pf is homotopy inverse to i. □

Lemma 19.3. *The inclusion $i = i_{2m} \colon S^{2m} \to X^{2m}$ is a homotopy equivalence.*

Proof. Here it is more convenient to fatten things up a litle. Put

$$\begin{aligned}
\widetilde{X} &= \{z \in \mathbb{C}^{2m+1} \mid z.z \in (0,\infty)\} \\
\widetilde{S} &= \widetilde{X} \cap \mathbb{R}^{2m+1} = \mathbb{R}^{2m+1} \setminus \{0\} \\
\widetilde{Y} &= \{(u,v,w) \in \mathbb{C}^m \times \mathbb{C}^m \times \mathbb{C} \mid u.v + w^2 \in (0,\infty)\}.
\end{aligned}$$

There are evident homeomorphisms $\widetilde{X} = (0,\infty) \times X^{2m}$ and $\widetilde{S} = (0,\infty) \times S^{2m}$, so it will suffice to show that the inclusion $i \colon \widetilde{S} \to \widetilde{X}$ is a homotopy equivalence. Define $f \colon \widetilde{X} \to \widetilde{Y}$ by $f(z) = (u,v,w)$, where $u_b = z_{2b} + iz_{2b+1}$ and $v_b = z_{2b} - iz_{2b+1}$ and $w = z_{2m}$. It is clear that this is a homeomorphism. If $(u,v,w) \in Y$ and $u = 0$ then $w^2 = 1$ so $\mathrm{Re}(w) \neq 0$. It follows that we can define $p \colon \widetilde{Y} \to \widetilde{S}$ by

$$p(u,v,w) = (\mathrm{Re}(u_0), \mathrm{Im}(u_0), \dots, \mathrm{Re}(u_{m-1}), \mathrm{Im}(u_{m-1}), \mathrm{Re}(w)).$$

One checks that $pfi = 1 \colon \widetilde{S} \to \widetilde{S}$. Now consider a point $(u,v,x+iy) \in \widetilde{Y}$, so the number $r = u.v + x^2 - y^2 + 2ixy$ lies in $(0,\infty)$. One checks that $fip(u,v,x+iy) = (u,\overline{u},x)$. Given $t \in [0,1]$, consider the point

$$h_t(u,v,x) = (u, (1-t)\overline{u} + tv, x + ity).$$

We have

$$\begin{aligned}
u.((1-t)\overline{u} + tv) + (x+ity)^2 &= (1-t)\|u\|^2 + tu.v + x^2 - t^2y^2 + 2itxy \\
&= (1-t)\|u\|^2 + t(r - x^2 + y^2 - 2ixy) + x^2 - t^2y^2 + 2itxy \\
&= (1-t)(\|u\|^2 + x^2 + ty^2) + tr.
\end{aligned}$$

As $r > 0$ this is positive unless $t = 0$ and $u = 0$ and $x = 0$, which would contradict the positivity of r. It follows that $h_t(u,v,x) \in \widetilde{Y}$, so we have a homotopy between $h_0 = fip$ and $h_1 = 1_{\widetilde{Y}}$. As $fip \simeq 1$ and f is a homeomorphism we have $ipf \simeq 1$, so pf is a homotopy inverse for i. □

Corollary 19.4. *The inclusion $O_n(\mathbb{R}) \to O_n(\mathbb{C})$ is a homotopy equivalence.*

Proof. We have a commutative diagram

$$\begin{array}{ccccc}
O_{n-1}(\mathbb{R}) & \longrightarrow & O_n(\mathbb{R}) & \longrightarrow & S^n \\
\downarrow & & \downarrow & & \downarrow \\
O_{n-1}(\mathbb{C}) & \xrightarrow[j]{} & O_n(\mathbb{C}) & \xrightarrow[q]{} & X^n
\end{array}$$

The right hand vertical is a homotopy equivalence, and we may assume inductively that the same is true of the left hand vertical. Both rows are fibre bundles so it follows that the middle vertical is also a homotopy equivalence. □

20. Other things

Configuration spaces. The basic building blocks for surgery. Steiner embedding spaces.

References

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